Inequivalent quantizations from gradings and $\mathbb{Z}_2 \times \mathbb{Z}_2$ parabosons

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Abstract

This paper introduces the parastatistics induced by $\mathbb{Z}_2 \times \mathbb{Z}_2$-*graded algebras*. It accommodates four kinds of particles: ordinary bosons and three types of parabosons which mutually anticommute when belonging to different type (so far, in the literature, only parastatistics induced by $\mathbb{Z}_2 \times \mathbb{Z}_2$-*graded superalgebras* and producing parafermions have been considered).

It is shown how to detect $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons in the multi-particle sector of a quantum model. The difference with respect to a system composed by ordinary bosons is spotted by measuring some selected observables on certain given eigenstates. The construction of the multi-particle states is made through the appropriate braided tensor product.

The application of $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$- gradings produces 9 inequivalent multi-particle Hilbert spaces of a 4×4 matrix oscillator. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosonic Hilbert space is one of them.

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1 Introduction

This paper introduces the parastatistics induced by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie *algebras* and gives the proof that the associated particles, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons, can be detected by performing a measurement in the multi-particle sector of a quantum model.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and Lie superalgebras were introduced by Rittenberg and Wyler in [1, 2]. The term "*color (super)algebra*" was used (see also [3]) to describe both cases.

The particles (bosons and fermions) of an ordinary theory can be associated with 1 bit of information (let's say 0 for bosons and 1 for fermions), while the particles of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded theory are described by 2 bits of information (00, 10, 11, 01).

The four types of particles in models based on $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie *superalgebras* are (see [4]) the ordinary bosons (00), two types (10 and 01) of parafermions and the exotic bosons (11); the parafermions of different type commute, while the exotic bosons anticommute with the parafermions.

The four types of particles in models based on $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie *algebras* are (see [5]) the ordinary bosons (00) and three types (10, 11 and 01) of parabosons; the parabosons of different type anticommute.

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded physics, with particles accommodated according to Lie superalgebras, is an obvious extension of ordinary physics. Indeed, ordinary bosons and fermions can be recovered from, respectively, the 00 and 10 sectors, while leaving empty the 11 and 01 sectors. Only recently, however, the open question that was lingering around was solved in [6], by showing that the *colored world* of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras produces quantum models which cannot be mimicked by black/white ordinary bosons and fermions alone.

Symbolically, this result can be expressed as

$$\mathbb{Z}_2^1 \cdot \boldsymbol{LSA} \quad \subset \quad \mathbb{Z}_2^2 \cdot \boldsymbol{LSA}, \tag{1}$$

meaning that the systems recovered from $\mathbb{Z}_2$-graded Lie superalgebras are a proper subset of those recovered from $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras.

The approach of [6] emphasizes the role of the braided tensor product, as defined in [7], in the construction of the multi-particle states. This approach is here extended to derive the physics of the parabosons obtained from $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras. A symbolical consequence of the present work can be expressed as

$$\mathbb{Z}_2^0 \cdot \boldsymbol{LA} \quad \subset \quad \mathbb{Z}_2^2 \cdot \boldsymbol{LA}, \tag{2}$$

meaning that the bosonic systems recovered from ordinary Lie algebras are a proper subset of those recovered from $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras.

In the first years after the introduction of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras their physical applications received some limited attention, see [8, 9, 10, 11]. More recently, a systematic investigation of their role as symmetries of dynamical systems started. Indeed, they appear [12, 13] as symmetries of the Lévy-Leblond equations for nonrelativistic spinors; furthermore, classical invariant worldline [4] and two-dimensional [14] sigma models have been constructed, invariant quantum mechanical models have been presented in [15, 16] and conformal quantum mechanics in [17]. The parastatistics induced by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras was introduced in [18, 19] and further investigated in [20, 21, 22, 23, 24, 25, 6].

Despite this activity on graded Lie superalgebras, the possibility of applications of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras has been ignored, probably because they are not extensions of ordinary [26] Lie superalgebras and do not include fermions. Indeed, a consistent number of present works are focusing on even larger (the $\mathbb{Z}_2^n$, for $n > 2$) graded extension of Lie superalgebras, see e.g. [27, 28, 29] and references therein for the mathematical literature.

On the other hand, as pointed out very recently in [5], several costructions (invariant models, the graded superspace of [30], etc.) which are available for graded superalgebras can be extended to $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras. This is the starting point of the present investigation.

The scheme of the paper is as follows. It is shown at first that the 4×4 matrix Hamiltonian discussed in [15, 16], besides being supersymmetric and invariant under the one-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Poincaré superalgebra, is also invariant under a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra. This offers the possibility to apply the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosonic statistics to its multi-particle sector.

The Hilbert space is constructed for the special case of the harmonic oscillator potential. It is shown that different statistics can be implemented by using $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$- gradings. As a consequence, a total number of 9 inequivalent multi-particle Hilbert spaces are encountered. This statement can also be rephrased as "9 inequivalent multi-particle quantizations". The analysis of [6] discussed just three of them (bosonic, supersymmetric and $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parafermionic variants). Among the extra quantizations presented in this paper a case corresponds to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons, while another case corresponds to a different implementation of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parafermions.

The multi-particle states are constructed by taking the raising operators as elements of a Universal Enveloping graded Lie (super)algebra and by applying the associated coproducts.

The proof that the 9 variants indeed produce inequivalent multi-particle models is given. In particular, observables discriminating the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosonic variant from the bosonic one are constructed.

More comments about the results and the future perspectives are given in the Conclusions.

The index of the paper is:

1 - Introduction,
2 - The 4×4 graded Hamiltonian,
3 - The construction of multi-particle Hilbert spaces,
4 - The 9 inequivalent 2-particle Hilbert spaces,
5 - Discriminating 2-particle observables,
6 - Conclusions,
Appendix **A** - Relevant formulas for graded (super)algebras,
Appendix **B** - Representations of 2-particle observables.

2 The 4×4 graded Hamiltonian

The 4×4 hermitian matrix Hamiltonian H, given by

$$H = \frac{1}{2}\begin{pmatrix} -\partial_x^2 + W^2(x) + W'(x) & 0 & 0 & 0 \\ 0 & -\partial_x^2 + W^2(x) + W'(x) & 0 & 0 \\ 0 & 0 & -\partial_x^2 + W^2(x) - W'(x) & 0 \\ 0 & 0 & 0 & -\partial_x^2 + W^2(x) - W'(x) \end{pmatrix}, \tag{3}$$

depends on the prepotential $W(x)$. We set in the above formula $W'(x) = \frac{d}{dx}W(x)$.
H is invariant, see [15, 16], under both supersymmetry and the one-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Poincaré superalgebra.

We point out here that H is also invariant under a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra. Let us introduce the hermitian first-order matrix operators Q_{10}, Q_{01} and constant matrix Z as

$$
\begin{aligned}
Q_{10} &= \frac{-i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \partial_x + W(x) & 0 \\ 0 & 0 & 0 & \partial_x + W(x) \\ \partial_x - W(x) & 0 & 0 & 0 \\ 0 & \partial_x - W(x) & 0 & 0 \end{pmatrix}, \\
Q_{01} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & \partial_x + W(x) \\ 0 & 0 & \partial_x + W(x) & 0 \\ 0 & -\partial_x + W(x) & 0 & 0 \\ -\partial_x + W(x) & 0 & 0 & 0 \end{pmatrix}, \\
Z &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.
\end{aligned}
\tag{4}
$$

The Hamiltonian H is invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded abelian Lie algebra $\mathfrak{a}$ defined by the following set of (all vanishing) 6 (anti)commutators:

$$
[H, Q_{10}] = [H, Q_{01}] = [H, Z] = 0, \qquad \{Q_{10}, Q_{01}\} = \{Z, Q_{10}\} = \{Z, Q_{01}\} = 0. \tag{5}
$$

The algebra $\mathfrak{a}$ is listed as "$A7$" in the Table **1** classification of minimal graded algebras presented in[5]. The grading assignment, according to the (A.8) decomposition, of the $\mathfrak{a}$ generators is

$$
H \in \mathfrak{a}_{00}, \quad Q_{10} \in \mathfrak{a}_{10}, \quad Q_{01} \in \mathfrak{a}_{01}, \quad Z \in \mathfrak{a}_{11}. \tag{6}
$$

The operators Q_{10}, Q_{01} are square roots of the Hamiltonian ($Q_{10}^2 = Q_{01}^2 = H$). It follows that, besides (5), H is invariant under the $\mathbb{Z}_2$-graded Lie superalgebra

$$
\{Q_{10}, Q_{10}\} = \{Q_{01}, Q_{01}\} = 2H, \qquad \{Q_{10}, Q_{01}\} = 0, \qquad [H, Q_{10}] = [H, Q_{01}] = 0, \tag{7}
$$

which defines H as a supersymmetric quantum mechanics [31] Hamiltonian.

We anticipate that these two graded invariant structures produce inequivalent quantum models in the multi-particle sectors. Essentially, this is due to the fact that the fermions present in (7) obey the Pauli exclusion principle; this is not the case for the parabosons that, as we will see, are obtained from (5).

If we specialize $W(x) = -x$, H becomes the Hamiltonian of the one-dimensional, 4×4 matrix oscillator. It will be denoted as H_{osc}; we have

$$
H_{osc} = \frac{1}{2} \begin{pmatrix} -\partial_x^2 + x^2 - 1 & 0 & 0 & 0 \\ 0 & -\partial_x^2 + x^2 - 1 & 0 & 0 \\ 0 & 0 & -\partial_x^2 + x^2 + 1 & 0 \\ 0 & 0 & 0 & -\partial_x^2 + x^2 + 1 \end{pmatrix}. \tag{8}
$$

The single-particle Hilbert space $\mathcal{H}$ of the H_{osc} Hamiltonian was constructed in [15, 6]. It is obtained by applying raising operators to a lowest weight vector state denoted as $|0; 00\rangle$.

The creation/annihilation oscillators $a, a^\dagger$, given by

$$
a = \frac{i}{\sqrt{2}}(\partial_x + x), \qquad a^\dagger = \frac{i}{\sqrt{2}}(\partial_x - x), \tag{9}
$$

satisfy the commutator

$$
[a, a^\dagger] = 1. \tag{10}
$$

The matrix raising (lowering) operators $f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger$ (f_{11}, f_{10}, f_{01}) can be introduced as

$$f_{11}^\dagger = \begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad f_{10}^\dagger = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad f_{01}^\dagger = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{pmatrix},$$
$$f_{11} = \begin{pmatrix} 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad f_{10} = \begin{pmatrix} 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad f_{01} = \begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}. \qquad (11)$$

The suffix is chosen in order to denote, in the $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings, the matrix decompositions expressed in (A.8).

In terms of these operators the Hamiltonian H_{osc} can be re-expressed as

$$H_{osc} \;=\; a^\dagger a \cdot \mathbb{I}_4 + f_{10}^\dagger f_{10} + f_{01}^\dagger f_{01} = a^\dagger a \cdot \mathbb{I}_4 + \Lambda, \quad \text{with} \quad \Lambda = diag(0,0,1,1). \qquad (12)$$

Here and in the following we denote a $m \times m$ identity matrix as $\mathbb{I}_m$.

The normalized lowest weight vector $|0;00\rangle$ satisfies the conditions

$$a|0;00\rangle = f_{11}|0;00\rangle = f_{10}|0;00\rangle = f_{01}|0;00\rangle = 0. \qquad (13)$$

We have

$$|0;00\rangle \;=\; \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2} \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}. \qquad (14)$$

The single-particle Hilbert space $\mathcal{H}$ is spanned by the orthonormal vectors
$|n;00\rangle, |n;11\rangle, |n;10\rangle, |n;01\rangle$ introduced through

$$\begin{array}{ll} |n;00\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0;00\rangle, & |n;10\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} f_{10}^\dagger|0;00\rangle, \\ |n;11\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} f_{11}^\dagger|0;00\rangle, & |n;01\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} f_{01}^\dagger|0;00\rangle. \end{array} \qquad (15)$$

At most a single power of $f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger$ enters the spanning vectors since we have, for any pair of such operators,

$$f_\sharp^\dagger f_\flat^\dagger \;=\; 0, \qquad \text{with} \quad \sharp, \flat \in \{11, 10, 01\}. \qquad (16)$$

Due to the commutators

$$[H_{osc}, a^\dagger] = a^\dagger, \quad [H_{osc}, f_{10}^\dagger] = f_{10}^\dagger, \quad [H_{osc}, f_{01}^\dagger] = f_{01}^\dagger, \quad [H_{osc}, f_{11}^\dagger] = 0, \qquad (17)$$

the (15) states are energy eigenstates whose eigenvalues are read from

$$\begin{array}{ll} H_{osc}|n;00\rangle = n|n;00\rangle, & H_{osc}|n;10\rangle = (n+1)|n;10\rangle, \\ H_{osc}|n;11\rangle = n|n;11\rangle, & H_{osc}|n;01\rangle = (n+1)|n;01\rangle. \end{array} \qquad (18)$$

One should note that the vacuum state is doubly degenerate:

$$H_{osc}|0;00\rangle = H_{osc}|0;11\rangle \;=\; 0. \qquad (19)$$

For later convenience we introduce the exchange matrices X_{11}, X_{10}, X_{01}. They are hermitian operators which mutually interchange the 11, 10 and 01 sectors. Their suffix indicates the (A.8) decomposition when a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading is applied. We have

$$X_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad X_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad X_{01} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

The matrices X_{11}, X_{10}, X_{01} are the building blocks in the construction of the observables which are presented in Section **5**.

3 The construction of multi-particle Hilbert spaces

The n-particle Hilbert space $\mathcal{H}^{(n)}$ of the (8) Hamiltonian H_{osc} is a subset of the tensor products of n single-particle Hilbert spaces $\mathcal{H}$:

$$\mathcal{H}^{(n)} \quad \subset \mathcal{H}^{\otimes n}. \tag{21}$$

$\mathcal{H}^{(n)}$ is a lowest weight vector space whose lowest weight vector $|0;00\rangle^{(n)}$ is a tensor product of the single-particle lowest weight vector $|0;00\rangle$ given in (14):

$$|0;00\rangle^{(n)} \quad = \quad |0;00\rangle^{\otimes n}. \tag{22}$$

The space coordinates entering the tensor products of the Hilbert spaces $\mathcal{H}^{(n)}$ are denoted as $x_1, x_2, \ldots, x_n$. In the 2-particle case we set, for simplicity, $x_1 = x$, $x_2 = y$. Therefore, the normalized lowest weight vector $|0;00\rangle^{(2)}$ is

$$|0;00\rangle^{(2)} \quad = \quad \pi^{-\frac{1}{2}} e^{-\frac{1}{2}(x^2+y^2)} \cdot v_1. \tag{23}$$

Here and in the following we denote as v_j, for $j = 1, 2, \ldots, 16$, the 16-component column vector with entry 1 in the j-th position and 0 otherwise.

The construction of the 2-particle Hilbert space assumes the raising operators $a^\dagger, f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger$ introduced in (9,11) to be elements of a graded algebra $\mathfrak{g}$ (the admissible gradings for $\mathfrak{g}$ are discussed in Section **4**). The graded algebra $\mathfrak{g}$ defines its Universal Enveloping Algebra $U \equiv \mathcal{U}(\mathfrak{g})$. As recalled in Appendix **A**, U is endowed with a Hopf algebra structure and in particular of an operation, the coproduct Δ, which satisfies (A.10,A.11,A.12).

The 2-particle states are recovered from applying the coproducts

$$\Delta\left((a^\dagger)^n)(f_{11}^\dagger)^{r_{11}}(f_{10}^\dagger)^{r_{10}}(f_{01}^\dagger)^{r_{01}}\right) \quad \in \quad U \otimes U \tag{24}$$

to the vector $|0;00\rangle^{(2)}$ which induces the lowest weight representation. Following the convention of Appendix **A**, a hat denotes the evaluation of the coproduct in the given representation. Therefore

$$\widehat{\Delta\left((a^\dagger)^n)(f_{11}^\dagger)^{r_{11}}(f_{10}^\dagger)^{r_{10}}(f_{01}^\dagger)^{r_{01}}\right)} \quad \in \quad End(\mathcal{H}^{(2)}). \tag{25}$$

The Hilbert space $\mathcal{H}^{(2)}$ is spanned by

$$|n; r_{11}, r_{10}, r_{01}\rangle^{(2)} \quad = \quad \widehat{\Delta\left((a^\dagger)^n)(f_{11}^\dagger)^{r_{11}}(f_{10}^\dagger)^{r_{10}}(f_{01}^\dagger)^{r_{01}}\right)} \cdot |0;00\rangle^{(2)}. \tag{26}$$

The identification $|0;0,0,0\rangle^{(2)} \equiv |0;00\rangle^{(2)}$ holds.

In (26) n is a non-negative integer ($n \in \mathbb{N}_0$); the restrictions on r_{11}, r_{10}, r_{01}, as discussed in Section **4**, depend on the grading.

As a useful example, it follows that the formula of the 2-particle creation operator $\widehat{\Delta(a^\dagger)}$ is

$$\widehat{\Delta(a^\dagger)} \quad = \quad \frac{i}{\sqrt{2}}(\partial_x - x + \partial_y - y). \tag{27}$$

By enlarging the graded algebra $\mathfrak{g}$ and its induced Universal Enveloping Algebra with the addition of observables such as H_{osc}, we can determine their actions on the multi-particle sectors. The 2-particle Hamiltonian $H_{osc}^{(2)}$ reads as

$$H_{osc}^{(2)} \quad = \quad \widehat{\Delta(H_{osc})} = H_{osc} \otimes \mathbb{I}_4 + \mathbb{I}_4 \times H_{osc}. \tag{28}$$

The construction of the $n+1$-particle Hilbert spaces, for $n > 1$, is made iteratively by replacing $\Delta \equiv \Delta^{(1)}$ with $\Delta^{(n)}$. Induced by the coassociativity (A.11) of the coproduct, $\Delta^{(n)}$ is defined as

$$\Delta^{(n)} \quad = \quad (id \otimes \Delta^{(1)})\Delta^{(n-1)}, \qquad (\Delta^{(1)} \equiv \Delta). \tag{29}$$

4 The 9 inequivalent 2-particle Hilbert spaces

The oscillator Hamiltonian H_{osc} given in (8) possesses nine inequivalent multi-particle quantizations. They are induced by the different gradings assigned to the raising operators $f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger$ introduced in (11) and under the assumption that the lowest weight vector is bosonic. Three of the quantizations (bosonic, supersymmetric and a version of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parafermions) were already discussed in [6]. The extra quantizations are divided into *standard* and *non-standard*. Among the standard ones we obtain the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons; the non-standard ones include an alternative quantization based on $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parafermions.

The construction goes as follows: in one case (the bosonic one) $f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger$ are assumed to be elements of an ordinary abelian Lie algebra; alternatively, they are assumed to be even/odd elements of a $\mathbb{Z}_2$-graded abelian Lie superalgebra, of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded abelian Lie superalgebra (parafermions) or of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded abelian Lie algebra (parabosons).

We proceed at first to discuss the 6 standard gradings.

4.1 The 6 standard gradings

In the $\mathbb{Z}_2$-grading assignment the 4×4 matrix Hamiltonian H_{osc} corresponds to a block-diagonal supermatrix of $(4-p|p)$ type, with $p = 0,1,2,3$. The $(4|0)$ case for $p = 0$ coincides with the ordinary bosonic matrix. The $p = 4$ case is excluded if we require the vacuum state to be even (bosonic).

The six assignments are:

1) $\{f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger\} \in 0, \quad \{\emptyset\} \in 1 \quad$ for $(4|0)$;
2) $\{f_{11}^\dagger, f_{10}^\dagger\} \in 0, \quad \{f_{01}^\dagger\} \in 1 \quad$ for $(3|1)$;
3) $\{f_{11}^\dagger\} \in 0, \quad \{f_{10}^\dagger, f_{01}^\dagger\} \in 1 \quad$ for $(2|2)$;
4) $\{\emptyset\} \in 0, \quad \{f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger\} \in 1 \quad$ for $(1|3)$;
5) $\{f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger\} \in \mathbb{Z}_2^2 \cdot LSA$;
6) $\{f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger\} \in \mathbb{Z}_2^2 \cdot LA$.

The corresponding vanishing (anti)commutators defining the graded abelian algebras $\mathfrak{a}_j$, where $j = 1, 2, \ldots, 6$, are

$$\begin{array}{ll}
\mathfrak{a}_1: & [f_{11}^\dagger, f_{10}^\dagger] = [f_{10}^\dagger, f_{01}^\dagger] = [f_{01}^\dagger, f_{11}^\dagger] = 0;\\
\mathfrak{a}_2: & [f_{11}^\dagger, f_{10}^\dagger] = [f_{10}^\dagger, f_{01}^\dagger] = [f_{01}^\dagger, f_{11}^\dagger] = \{f_{01}^\dagger, f_{01}^\dagger\} = 0;\\
\mathfrak{a}_3: & [f_{11}^\dagger, f_{10}^\dagger] = \{f_{10}^\dagger, f_{01}^\dagger\} = [f_{01}^\dagger, f_{11}^\dagger] = \{f_{10}^\dagger, f_{10}^\dagger\} = \{f_{01}^\dagger, f_{01}^\dagger\} = 0;\\
\mathfrak{a}_4: & \{f_{11}^\dagger, f_{10}^\dagger\} = \{f_{10}^\dagger, f_{01}^\dagger\} = \{f_{01}^\dagger, f_{11}^\dagger\} = \{f_{11}^\dagger, f_{11}^\dagger\} = \{f_{10}^\dagger, f_{10}^\dagger\} = \{f_{01}^\dagger, f_{01}^\dagger\} = 0;\\
\mathfrak{a}_5: & \{f_{11}^\dagger, f_{10}^\dagger\} = [f_{10}^\dagger, f_{01}^\dagger] = \{f_{01}^\dagger, f_{11}^\dagger\} = \{f_{10}^\dagger, f_{10}^\dagger\} = \{f_{01}^\dagger, f_{01}^\dagger\} = 0;\\
\mathfrak{a}_6: & \{f_{11}^\dagger, f_{10}^\dagger\} = \{f_{10}^\dagger, f_{01}^\dagger\} = \{f_{01}^\dagger, f_{11}^\dagger\} = 0.
\end{array} \tag{30}$$

The three cases already discussed in [6] correspond to the numbers 1 (the bosonic version of the theory), 3 (the supersymmetric version) and 5 (a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parafermionic version).

In the above construction we followed the standard block-diagonal matrix format of Lie superalgebras and, for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading, the (A.8) decomposition. Non-standard supermatrix formats are discussed in [32]. The procedure for the non-standard decompositions is presented in the subsection **4.2**.

Each algebra $\mathfrak{a}_j$ is extended to the graded algebra $\overline{\mathfrak{a}}_j = \{a^\dagger, f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger\}$ which contains the creation operator $a^\dagger$, introduced in (9), as extra generator. Depending on the case, $a^\dagger$ belongs to either the 0- or the 00-sector. Its commutators are vanishing ($[a^\dagger, f_\sharp^\dagger] = 0$ for $\sharp = 11, 10, 01$).

The multi-particle quantizations are recovered, as explained in Section **3**, from the coproducts defined on the corresponding Universal Eveloping Algebras $\mathcal{U}(\overline{\mathfrak{a}}_j)$. The multi-particle states are constructed according to formula (26). The signs entering the braided tensor products depend on the different grading assignments of each one of the above cases. They are given by (A.3,A.13).

The restrictions on the r_{11}, r_{10}, r_{01} exponents entering (26) are due to these respective signs. For instance, in the parafermionic quantization r_{10} takes the values $0, 1$; the values taken by r_{10} in the parabosonic case are $0, 1, 2$.

The 6 standard multi-particle quantizations, associated to the respective (30) gradings, are denoted as follows:

$$\begin{array}{llllll}
\mathbf{(4|0)}: & \mathfrak{a}_1, & \mathbf{(2|2)}: & \mathfrak{a}_3, & \mathbb{Z}_\mathbf{2}^\mathbf{2}\textbf{-PF}: & \mathfrak{a}_5,\\
\mathbf{(3|1)}: & \mathfrak{a}_2, & \mathbf{(1|3)}: & \mathfrak{a}_4, & \mathbb{Z}_\mathbf{2}^\mathbf{2}\textbf{-PB}: & \mathfrak{a}_6.
\end{array} \tag{31}$$

In the last column **PF** and **PB** stand for, respectively, parafermions and parabosons.

4.2 The 3 non-standard gradings

The non-standard cases are obtained by applying decompositions of the supermatrices which do not coincide with the ordinary block-diagonal decompositions; these non-standard formats are discussed in [32]. For the model under consideration these extra cases can be recovered from standard decompositions applied to a different diagonal Hamiltonian whose diagonal entries are permuted with respect to H_{osc}.

Before proceeding with the construction of the non-standard quantizations let us recall that the (11) raising operators $f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger$ create, see (17), particles of respective energy $0, 1, 1$.

In a $\mathbb{Z}_2$-grading, the standard decomposition of a vector $v^T = (B, B, F, F)$ with 2 bosons and 2 fermions can be replaced, for instance, by the decomposition $v^T = (B, F, B, F)$. In these two examples the entries of the fermionic supermatrices are respectively accommodated according to

$$\text{standard case:} \quad \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}, \qquad \text{non-standard case:} \quad \begin{pmatrix} 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \end{pmatrix}. \tag{32}$$

For 3 bosons and 1 fermion we can pass from $v^T=(B,B,B,F)$ to, e.g., $v^T=(B,F,B,B)$. In these new examples the entries of the fermionic supermatrices are respectively accommodated according to

$$\text{standard case:}\quad \begin{pmatrix} 0&0&0&*\\ 0&0&0&*\\ 0&0&0&*\\ *&*&*&0 \end{pmatrix},\qquad \text{non-standard case:}\quad \begin{pmatrix} 0&*&0&0\\ *&0&*&*\\ 0&*&0&0\\ 0&*&0&0 \end{pmatrix}. \tag{33}$$

The key issue to notice is that the raising operator $f_{11}^\dagger$ becomes fermionic in the non-standard decompositions above. This implies that the Pauli exclusion principle applies to the 0-energy particles created by $f_{11}^\dagger$. In the standard cases these particles are bosons. This affects the degeneracy of the energy levels of the multi-particle Hamiltonian producing inequivalent results.

Similarly, a non-standard decomposition of a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra is realized, e.g., by accommodating the $00,11,10,01$ sectors according to

$$\begin{aligned} M_{00} &= \begin{pmatrix} *&0&0&0\\ 0&*&0&0\\ 0&0&*&0\\ 0&0&0&* \end{pmatrix}, \qquad M_{10}=\begin{pmatrix} 0&*&0&0\\ *&0&0&0\\ 0&0&0&*\\ 0&0&*&0 \end{pmatrix},\\ M_{11} &= \begin{pmatrix} 0&0&*&0\\ 0&0&0&*\\ *&0&0&0\\ 0&*&0&0 \end{pmatrix}, \qquad M_{01}=\begin{pmatrix} 0&0&0&*\\ 0&0&*&0\\ 0&*&0&0\\ *&0&0&0 \end{pmatrix}. \end{aligned} \tag{34}$$

Contrary to the standard decomposition (A.8), in this case the 0-energy particles created by $f_{11}^\dagger$ are no longer exotic bosons, but parafermions.

On the other hand in the parabosonic case induced by the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie algebra, the non-standard decomposition above does not produce a inequivalent quantization with respect to the standard decomposition. This is so because, as already recalled, for parabosons the three sectors 11, 10 and 01 share the same properties and can be mutually interchanged.

A careful inspection shows that in three cases the non-standard decompositions for the Hamiltonian H_{osc} are not equivalent to the standard ones. Nevertheless, in all three cases these decompositions can be recovered from their corresponding standard ones after changing the Hamiltonian $H_{osc}=a^\dagger a\cdot\mathbb{I}_4+\Lambda$, with $\Lambda=diag(0,0,1,1)$, into the permuted Hamiltonian $\overline{H}_{osc}$ given by

$$\overline{H}_{osc} \;=\; a^\dagger a\cdot\mathbb{I}_4+\overline{\Lambda},\quad\text{with}\quad \overline{\Lambda}=diag(0,1,1,0). \tag{35}$$

These three non-standard multi-particle quantizations are denoted as $\mathbf{(3|1)}_{\boldsymbol{ns}}$, $\mathbf{(2|2)}_{\boldsymbol{ns}}$, $\mathbb{Z}_\mathbf{2}^\mathbf{2}$-$\mathbf{PF}_{\boldsymbol{ns}}$. Their corresponding graded algebras are

$$\begin{aligned} \mathbf{(3|1)}_{\boldsymbol{ns}}: &\quad \mathfrak{a}_2 \quad\text{for}\quad H_{osc}\mapsto\overline{H}_{osc},\\ \mathbf{(2|2)}_{\boldsymbol{ns}}: &\quad \mathfrak{a}_3 \quad\text{for}\quad H_{osc}\mapsto\overline{H}_{osc},\\ \mathbb{Z}_\mathbf{2}^\mathbf{2}\text{-}\mathbf{PF}_{\boldsymbol{ns}}: &\quad \mathfrak{a}_5 \quad\text{for}\quad H_{osc}\mapsto\overline{H}_{osc}. \end{aligned} \tag{36}$$

4.3 The 2-particle Hilbert spaces

The orthonormal vectors spanning the 2-particle Hilbert spaces, from (23,26,27), have the form

$$|m;I\rangle \;=\; \frac{1}{\sqrt{m!}}\left(\frac{i}{\sqrt{2}}(\partial_x+\partial_y-x-y)\right)^m\cdot(\pi^{-\frac{1}{2}}e^{-\frac{1}{2}(x^2+y^2)})\otimes V_I, \tag{37}$$

where V_I are 16-component constant orthonormal vectors which can be expressed in the v_j basis (we recall that v_j has entry 1 in the j-th position and 0 otherwise).

The 2-particle Hilbert spaces induced by the 6 standard gradings will be denoted as $\mathcal{H}_k^{(2)}$; the suffix $k = 1, 2, \ldots, 6$ denotes the respective (30) graded algebras. The finite dimensional Hilbert spaces $\overline{\mathcal{H}}_k^{(2)} \subset \mathcal{H}_k^{(2)}$ are spanned by the V_I vectors by taking $m = 0$ (the gaussian factor can be dropped for convenience). The spanning vectors V_I entering the six standard quantizations (31) are read from the following table:

	$(\mathbf{4}\vert\mathbf{0})$	$(\mathbf{3}\vert\mathbf{1})$	$(\mathbf{2}\vert\mathbf{2})$	$(\mathbf{1}\vert\mathbf{3})$	$\mathbb{Z}_2^2$-**PF**	$\mathbb{Z}_2^2$-**PB**
$V_1 = v_1$	X	X	X	X	X	X
$V_2 = v_6$	X	X	X		X	X
$V_3 = v_{11}$	X	X				X
$V_4 = v_{16}$	X					X
$V_5 = \frac{1}{\sqrt{2}}(v_2 + v_5)$	X	X	X	X	X	X
$V_6 = \frac{1}{\sqrt{2}}(v_3 + v_9)$	X	X	X	X	X	X
$V_7 = \frac{1}{\sqrt{2}}(v_4 + v_{13})$	X	X	X	X	X	X
$V_8 = \frac{1}{\sqrt{2}}(v_7 + v_{10})$	X	X	X			
$V_9 = \frac{1}{\sqrt{2}}(v_7 - v_{10})$				X	X	X
$V_{10} = \frac{1}{\sqrt{2}}(v_8 + v_{14})$	X	X	X			
$V_{11} = \frac{1}{\sqrt{2}}(v_8 - v_{14})$				X	X	X
$V_{12} = \frac{1}{\sqrt{2}}(v_{12} + v_{15})$	X	X			X	
$V_{13} = \frac{1}{\sqrt{2}}(v_{12} - v_{15})$			X	X		X

Table 1: Spanning vectors of the standard finite dimensional 2-particle Hilbert spaces of the 4×4 matrix oscillator. The first four columns correspond to supermatrices: $(\mathbf{4}|\mathbf{0})$, i.e. the bosonic case, $(\mathbf{3}|\mathbf{1})$, $(\mathbf{2}|\mathbf{2})$, i.e. the supersymmetric case, and $(\mathbf{1}|\mathbf{3})$. The last two columns present the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Hilbert spaces for parafermions ($\mathbb{Z}_2^2$-**PF**) and parabosons ($\mathbb{Z}_2^2$-**PB**). The "X" denotes the presence of the vector.

One can observe, in certain cases, the absence of the vectors V_2, V_3, V_4. It is a consequence of the Pauli exclusion principle for (para)fermions; this principle is encoded [6, 7] in the language of the coproduct.

The finite-dimensional Hilbert spaces $\overline{\mathcal{H}}_k^{(2)}$ have dimensions d_k given by

$$d_1 = d_6 = 10, \qquad d_2 = 9, \qquad d_3 = d_5 = 8, \qquad d_4 = 7. \tag{38}$$

The six 2-particle Hilbert spaces $\mathcal{H}_k^{(2)}$ recovered from the standard decompositions are therefore spanned by the vectors $|m; I\rangle$, with $m = 0, 1, 2, \ldots$, while I is restricted according to

$$\begin{array}{lllllll}
\mathcal{H}_1^{(2)}: & |m;I\rangle & \text{for} & m \in \mathbb{N}_0 & \text{and} & I = 1,2,3,4,5,6,7,8,10,12; \\
\mathcal{H}_2^{(2)}: & |m;I\rangle & \text{for} & m \in \mathbb{N}_0 & \text{and} & I = 1,2,3,5,6,7,8,10,12; \\
\mathcal{H}_3^{(2)}: & |m;I\rangle & \text{for} & m \in \mathbb{N}_0 & \text{and} & I = 1,2,5,6,7,8,10,13; \\
\mathcal{H}_4^{(2)}: & |m;I\rangle & \text{for} & m \in \mathbb{N}_0 & \text{and} & I = 1,5,6,7,9,11,13; \\
\mathcal{H}_5^{(2)}: & |m;I\rangle & \text{for} & m \in \mathbb{N}_0 & \text{and} & I = 1,2,5,6,7,9,11,12; \\
\mathcal{H}_6^{(2)}: & |m;I\rangle & \text{for} & m \in \mathbb{N}_0 & \text{and} & I = 1,2,3,4,5,6,7,9,11,13.
\end{array} \tag{39}$$

One should note, see (36), that the three 2-particle Hilbert spaces recovered from the non-standard decompositions coincide with the associated standard Hilbert spaces. The difference is encoded in the modified Hamiltonian, $H_{osc}^{(2)}\mapsto \overline{H}_{osc}^{(2)}$, with the latter given in (35). Therefore, we have

$$\mathcal{H}_2^{(2)} \quad \text{for} \quad \mathbf{(3|1)}_{\boldsymbol{ns}}, \qquad \mathcal{H}_3^{(2)} \quad \text{for} \quad \mathbf{(2|2)}_{\boldsymbol{ns}}, \qquad \mathcal{H}_5^{(2)} \quad \text{for} \quad \mathbb{Z}_{\mathbf{2}}^{\mathbf{2}}\textbf{-PF}_{\boldsymbol{ns}}. \tag{40}$$

Any vector $|m;I\rangle$ is an energy eigenstate.

For the standard quantizations the energy eigenvalues $E_{m,I}$ are read from

$$\begin{array}{c} H_{osc}^{(2)}|m;I\rangle = E_{m,I}|m;I\rangle, \qquad \text{with} \qquad E_{m,I}=m+S_I \\ (S_1=S_2=S_5=0, \quad S_6=S_7=S_8=S_9=S_{10}=S_{11}=1, \quad S_3=S_4=S_{12}=S_{13}=2). \end{array} \tag{41}$$

For the non-standard quantizations the energy eigenvalues $\overline{E}_{m,I}$ are read from

$$\begin{array}{c} \overline{H}_{osc}^{(2)}|m;I\rangle = \overline{E}_{m,I}|m;I\rangle, \qquad \text{with} \qquad \overline{E}_{m,I}=m+\overline{S}_I \\ (\overline{S}_1=\overline{S}_7=0, \quad \overline{S}_5=\overline{S}_6=\overline{S}_{10}=\overline{S}_{11}=\overline{S}_{12}=\overline{S}_{13}=1, \quad \overline{S}_2=\overline{S}_3=\overline{S}_8=\overline{S}_9=2). \end{array} \tag{42}$$

For all quantizations (standard and non-standard) the spectrum of the energy eigenvalues E_n is given by the non-negative integers 0,1,2,...:

$$E_n \quad = \quad n\in\mathbb{N}_0. \tag{43}$$

We now discuss the degeneracy of the energy levels and the inequivalence of the multi-particle quantizations.

4.4 Degeneracy of the energy levels

The degeneracy of a energy level depends on the given quantization and is obtained from (41,42). The results are summarized in the table below which presents the nine cases (1 to 6 corresponding to the standard decompositions, 7, 8 and 9 to the non-standard ones). For any given quantization the degeneracy of its energy levels $n=2,3,4,\ldots$ is the same. We have

	$E=0$	$E=1$	$E=n\geq 2$
1^*- $\mathbf{(4|0)}$	3	7	10
2 - $\mathbf{(3|1)}$	3	7	9
$3^\dagger$- $\mathbf{(2|2)}$	3	7	8
4 - $\mathbf{(1|3)}$	2	6	7
$5^\dagger$- $\mathbb{Z}_{\mathbf{2}}^{\mathbf{2}}$**-PF**	3	7	8
6^*- $\mathbb{Z}_{\mathbf{2}}^{\mathbf{2}}$**-PB**	3	7	10
7 - $\mathbf{(3|1)}_{\boldsymbol{ns}}$	2	6	9
$8^\ddagger$ - $\mathbf{(2|2)}_{\boldsymbol{ns}}$	2	6	8
$9^\ddagger$ - $\mathbb{Z}_{\mathbf{2}}^{\mathbf{2}}\textbf{-PF}_{\boldsymbol{ns}}$	2	6	8

Table 2: The numbers give the degeneracy of the 2-particle energy eigenvalues for each one of the nine quantizations of the 4×4 quantum oscillator. Different numbers indicate inequivalent

quantizations. The inequivalence of the quantizations 1 versus 6, 3 versus 5 and 8 versus 9 cannot be read from this table; it requires a subtler analysis of other observables.

The final proof of the inequivalence of the nine quantizations is given in Section **5** with the construction of the observables discriminating the cases 1 versus 6 and 8 versus 9. The observables discriminating the cases 3 versus 5 are found in [6].

Remark: due to the coassociativity of the coproduct, see (A.11), nine inequivalent M-particle graded Hilbert spaces are recovered for any integer number $M>1$. The formulas are straightforward generalizations of the 2-particle construction. In [6] inequivalent 3-particle Hilbert spaces were presented for the supersymmetric and (standard) parafermionic gradings.

5 Discriminating 2-particle observables

In (39) we presented the 2-particle Hilbert spaces $\mathcal{H}_k^{(2)}$ (for $k=1,2,\ldots,6$) which were used to derive the nine (standard and non-standard) quantizations entering Table **2**. We present here the proof that these nine quantizations are all inequivalent.
Since the construction of the observables which discriminate the parafermionic case $\mathbb{Z}_\mathbf{2}^\mathbf{2}$**-PF** from the supersymmetric case $\mathbf{(2|2)}$ was given in [6], what is left here is to present:
i) observables which discriminate the parabosonic case $\mathbb{Z}_\mathbf{2}^\mathbf{2}$**-PB** from the bosonic case $\mathbf{(4|0)}$,
ii) at least one observable which discriminates the non-standard cases $\mathbb{Z}_\mathbf{2}^\mathbf{2}\textbf{-PF}_{\boldsymbol{ns}}$ versus $\mathbf{(2|2)}_{\boldsymbol{ns}}$.
Let's proceed.

5.1 Discriminating $\mathbb{Z}_2\times\mathbb{Z}_2$-graded parabosons from bosons

The 2-particle observables discriminating parabosons from bosons should satisfy the following requirements:

i) they should apply to both bosonic and parabosonic Hilbert spaces,
ii) they should be hermitian and
iii) they should belong to the 00-graded sector of the parabosonic theory in order to have real (00-graded) eigenvalues.

The following set of 2-particle observables, constructed in terms of the exchange operators X_{11}, X_{10}, X_{01} introduced in (20), satisfy the above three criteria.
We have

$$X_s = X_{10}\otimes X_{10},\quad X_t = X_{01}\otimes X_{01},\quad X_u = X_{11}\otimes X_{11},\quad X_* = X_s+X_t+X_u \tag{44}$$

and

$$\begin{aligned}
Y_s &= (\mathbb{I}_4\otimes X_{11}+X_{11}\otimes\mathbb{I}_4)(\mathbb{I}_4\otimes X_{10}+X_{10}\otimes\mathbb{I}_4)(\mathbb{I}_4\otimes X_{01}+X_{01}\otimes\mathbb{I}_4)+\\
&\quad (\mathbb{I}_4\otimes X_{01}+X_{01}\otimes\mathbb{I}_4)(\mathbb{I}_4\otimes X_{10}+X_{10}\otimes\mathbb{I}_4)(\mathbb{I}_4\otimes X_{11}+X_{11}\otimes\mathbb{I}_4),\\
Y_t &= (\mathbb{I}_4\otimes X_{10}+X_{10}\otimes\mathbb{I}_4)(\mathbb{I}_4\otimes X_{01}+X_{01}\otimes\mathbb{I}_4)(\mathbb{I}_4\otimes X_{11}+X_{11}\otimes\mathbb{I}_4)+\\
&\quad (\mathbb{I}_4\otimes X_{11}+X_{11}\otimes\mathbb{I}_4)(\mathbb{I}_4\otimes X_{01}+X_{01}\otimes\mathbb{I}_4)(\mathbb{I}_4\otimes X_{10}+X_{10}\otimes\mathbb{I}_4),\\
Y_u &= (\mathbb{I}_4\otimes X_{01}+X_{01}\otimes\mathbb{I}_4)(\mathbb{I}_4\otimes X_{11}+X_{11}\otimes\mathbb{I}_4)(\mathbb{I}_4\otimes X_{10}+X_{10}\otimes\mathbb{I}_4)+\\
&\quad (\mathbb{I}_4\otimes X_{10}+X_{10}\otimes\mathbb{I}_4)(\mathbb{I}_4\otimes X_{11}+X_{11}\otimes\mathbb{I}_4)(\mathbb{I}_4\otimes X_{01}+X_{01}\otimes\mathbb{I}_4),\\
Y_* &= Y_s+Y_t+Y_u.
\end{aligned} \tag{45}$$

Under the $\mathbf{S}_3$ permutations which interchange the parabosonic sectors $11, 10, 01$, the operators X_s, X_t (Y_s, Y_t) are mapped into X_u (Y_u), while X_* and Y_* are $\mathbf{S}_3$-invariant. Without loss of generality we can therefore consider the four operators X_u, X_*, Y_u, Y_*. Their 16×16 matrix representations are given in Appendix **B**.

For the purpose of making easier the comparison of the bosonic versus parabosonic Hilbert spaces it is convenient to rename the respective vectors V_I entering Table **1**.

They will be expressed in terms of a sign ε ($\varepsilon = +1$ for bosons, $\varepsilon = -1$ for parabosons); the corresponding finite-dimensional Hilbert spaces will be denoted as $H_\varepsilon^{(2)}$. We set

$$\begin{array}{lll} U_{00,A} = v_1, & & \\ U_{00,B} = v_6, & U_{11} = \frac{1}{\sqrt{2}}(v_2 + v_5), & W_{11,\varepsilon} = \frac{1}{\sqrt{2}}(v_{12} + \varepsilon v_{15}), \\ U_{00,C} = v_{11}, & U_{10} = \frac{1}{\sqrt{2}}(v_3 + v_9), & W_{10,\varepsilon} = \frac{1}{\sqrt{2}}(v_8 + \varepsilon v_{14}), \\ U_{00,D} = v_{16}, & U_{01} = \frac{1}{\sqrt{2}}(v_4 + v_{13}), & W_{01,\varepsilon} = \frac{1}{\sqrt{2}}(v_7 + \varepsilon v_{10}). \end{array} \tag{46}$$

The suffix denotes the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading of the vector in the parabosonic case.

The difference between the two Hilbert spaces should be spotted by measuring the subspaces spanned by $W_{11,\varepsilon}, W_{10,\varepsilon}, W_{01,\varepsilon}$ and should appear as ε-dependent eigenvalues.

The eigenvectors of X_u with nonvanishing eigenvalues are $U_\pm$ and $W_{11,\varepsilon}$:

$$X_u U_\pm = \pm U_\pm \quad \text{for} \quad U_\pm = U_{00,C} \pm U_{00,D}, \qquad X_u W_{11,\varepsilon} = \varepsilon W_{11,\varepsilon}. \tag{47}$$

The eigenvectors of X_* with their respective nonvanishing eigenvalues are

$$\begin{array}{ll} X_*(U_{00,B} - U_{00,C}) = -(U_{00,B} - U_{00,C}), & X_* W_{11,\varepsilon} = \varepsilon W_{11,\varepsilon} \\ X_*(U_{00,C} - U_{00,D}) = -(U_{00,C} - U_{00,D}), & X_* W_{10,\varepsilon} = \varepsilon W_{10,\varepsilon} \\ X_*(U_{00,D} - U_{00,B}) = -(U_{00,D} - U_{00,B}), & X_* W_{01,\varepsilon} = \varepsilon W_{01,\varepsilon}. \end{array} \tag{48}$$

The presence of the ε eigenvalues in (47,48) proves that, by performing X_u, X_* measurements, one can determine whether a system under consideration is composed by ordinary bosons or by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons.

A basis of eigenvectors with respective eigenvalues for Y_* is given by

$$\begin{array}{rcl} Y_* U_{00,A} &=& 0, \\ Y_*(U_{00,B} + U_{00,C} + U_{00,D}) &=& (12 + 4\varepsilon)(U_{00,B} + U_{00,C} + U_{00,D}), \\ Y_*(U_{00,B} - U_{00,C}) &=& -2\varepsilon(U_{00,B} - U_{00,C}), \\ Y_*(U_{00,B} - U_{00,D}) &=& -2\varepsilon(U_{00,B} - U_{00,D}), \\ Y_* U_{11} &=& 2U_{11}, \\ Y_* U_{10} &=& 2U_{10}, \\ Y_* U_{01} &=& 2U_{01}, \\ Y_* W_{11,\varepsilon} &=& (6 + 4\varepsilon) W_{11,\varepsilon}, \\ Y_* W_{10,\varepsilon} &=& (6 + 4\varepsilon) W_{10,\varepsilon}, \\ Y_* W_{01,\varepsilon} &=& (6 + 4\varepsilon) W_{01,\varepsilon}. \end{array} \tag{49}$$

Unlike X_*, the operator Y_* is ε-dependent, see formula (B.2), due to the braiding properties of the tensor products in (45). This observation explains the presence of the ε sign in the eigenvalues obtained from the $U_{00,\bullet}$ vectors.

The Y_u eigenvectors and eigenvalues are

$$\begin{array}{rcl}
Y_u U_{00,A} &=& 0,\\
Y_u U_{11} &=& 2U_{11},\\
Y_u U_{10} &=& 0,\\
Y_u U_{01} &=& 0,\\
Y_u W_{11,\varepsilon} &=& 2W_{11,\varepsilon},\\
Y_u W_{10,\varepsilon} &=& (2+2\varepsilon)W_{10,\varepsilon},\\
Y_u W_{01,\varepsilon} &=& (2+2\varepsilon)W_{01,\varepsilon},\\
Y_u(U_{00,C}-U_{00,D}) &=& -2\varepsilon(U_{00,C}-U_{00,D})
\end{array} \tag{50}$$

and, for $\varepsilon = 1$,

$$\begin{array}{rcl}
Y_u(U_{00,B}+\frac{1}{2}U_{00,C}+\frac{1}{2}U_{00,D}) &=& 6(U_{00,B}+\frac{1}{2}U_{00,C}+\frac{1}{2}U_{00,D}),\\
Y_u(U_{00,B}-U_{00,C}-U_{00,D}) &=& 0,
\end{array} \tag{51}$$

while, for $\varepsilon = -1$, one has

$$Y_u\left(U_{00,B}+\frac{1}{4}(-3\pm\sqrt{17})(U_{00,C}+U_{00,D})\right) = (1\pm\sqrt{17})\left(U_{00,B}+\frac{1}{4}(-3\pm\sqrt{17})(U_{00,C}+U_{00,D})\right). \tag{52}$$

5.2 Discriminating two non-standard quantizations

The matrix operator X_u is an observable for both Hilbert spaces giving the non-standard cases $\mathbb{Z}_2^2$**-PF**$_{ns}$ and $\mathbf{(2|2)}_{ns}$. We rename the vectors entering the finite-dimensional Hilbert spaces as

$$\begin{array}{ccc}
 & \overline{V}_1 = v_1, & \overline{V}_2 = v_6,\\
\overline{V}_3 = \frac{1}{\sqrt{2}}(v_2+v_5), & \overline{V}_4 = \frac{1}{\sqrt{2}}(v_3+v_9), & \overline{V}_5 = \frac{1}{\sqrt{2}}(v_4+v_{13}),\\
\overline{V}_{6,\varepsilon} = \frac{1}{\sqrt{2}}(v_7+\delta v_{10}), & \overline{V}_{7,\delta} = \frac{1}{\sqrt{2}}(v_8+\delta v_{14}), & \overline{V}_{8,\delta} = \frac{1}{\sqrt{2}}(v_{12}-\delta v_{15}).
\end{array} \tag{53}$$

The sign $\delta = \pm 1$ corresponds to the $\mathbb{Z}_2^2$**-PF**$_{ns}$ case for $\delta = -1$ and to the $\mathbf{(2|2)}_{ns}$ case for $\delta = 1$.

The X_u eigenvalues are read from

$$\begin{array}{rcl}
X_u\overline{V}_J &=& 0 \quad (\text{for} \quad J = 1,2,3,4,5),\\
X_u\overline{V}_{6,\delta} &=& 0,\\
X_u\overline{V}_{7,\delta} &=& 0,\\
X_u\overline{V}_{8,\delta} &=& \delta\overline{V}_{8,\delta}.
\end{array} \tag{54}$$

Due to the presence of the δ eigenvalue in the last equation, a measurement of X_u allows to discriminate the two non-standard cases.

This concludes the proof of the inequivalence of the nine quantizations presented in Table **2**.

6 Conclusions

This paper presents the 9 inequivalent multi-particle quantizations of the 4×4 matrix oscillator given in (8). Each quantization is recovered from different $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$- gradings (and associated statistics) which are consistently imposed on the component particles. 6 of the quantizations are obtained from the standard block-decompositions of supermatrices, 3 of them from the non-standard ones.

This analysis completes the multi-particle quantizations discussed in [6] for just three cases (bosonic, supersymmetric and the standard version of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parafermions).

The extra quantizations presented in the paper include, in particular, a non-standard version of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parafermions and the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosonic statistics induced by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie *algebras*; unlike the parafermionic statistics induced by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie *superalgebras*, see [20, 21, 22, 23, 24, 25, 6], this parastatistics has not been previously considered in the literature.

Furthermore, we showed that suitable measurements of observables allow to distinguish if a multi-particle system is composed by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons or by ordinary bosons. This result gives to the notion of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons a legitimate status in physics, proving that it is not just a mathematical artifact void of physically measurable consequences.

A next step, in this line of research, would involve the construction of phenomenological $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosonic models which could be put to experimental test. A possible scenario could apply to emergent particles in condensed matter. Concerning model-building, a general framework to construct $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parafermionic models was presented in [4] for the classical case and [16] for the quantum case. As pointed out in [5], an extension of the method allows to derive $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosonic models.

On a separate line, the inequivalent multi-particle quantizations induced by gradings shed some light on open issues regarding the quantization, as discussed in [33] both from a historical and an actual perspective.

Appendix A: Relevant formulas for graded (super)algebras

In order to make the paper self-consistent we collect the relevant formulas concerning

$$\begin{array}{rl} i) & \mathbb{Z}_2\text{-graded Lie superalgebras,} \\ ii) & \mathbb{Z}_2 \times \mathbb{Z}_2\text{-graded Lie superalgebras and} \\ iii) & \mathbb{Z}_2 \times \mathbb{Z}_2\text{-graded Lie algebras.} \end{array} \tag{A.1}$$

As recalled in the text they induce inequivalent multi-particle quantizations of the 4×4 matrix harmonic oscillator (8). Following [5] and [6] we use a compact notation to describe, at once, the three cases. Ordinary (bosonic) Lie algebras can be assumed to be $\mathbb{Z}_2$-graded Lie superalgebras with empty odd (fermionic) sector. This allows, e. g., to identify the bosonic case listed in Table **1** with $(4|0)$ supermatrices.

Depending on the case under consideration, a graded algebra $\mathfrak{g}$ is decomposed into

$$\begin{array}{rcl} i) & : & \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \\ ii) \text{ and } iii) & : & \mathfrak{g} = \mathfrak{g}_{00} \oplus \mathfrak{g}_{01} \oplus \mathfrak{g}_{10} \oplus \mathfrak{g}_{11}. \end{array} \tag{A.2}$$

The even (0) and odd (1) generators in $i)$ are bosonic (fermionic). The four sectors of $ii)$ and $iii)$ are described by 2 bits. The grading of a generator in $i)$ is given by $\vec{\alpha} \equiv \alpha \in \{0, 1\}$. The grading of a generator in $ii)$ and $iii)$ is given by the pair $\vec{\alpha} = (\alpha_1, \alpha_2)$, with $\alpha_{1,2} \in \{0, 1\}$.

Three respective inner products, with addition mod 2, are defined:

$$\begin{array}{rcrcl} i) & : & \vec{\alpha}\cdot\vec{\beta} & := & \alpha\beta\in\{0,1\},\\ ii) & : & \vec{\alpha}\cdot\vec{\beta} & := & \alpha_1\beta_1+\alpha_2\beta_2\in\{0,1\},\\ iii) & : & \vec{\alpha}\cdot\vec{\beta} & := & \alpha_1\beta_2-\alpha_2\beta_1\in\{0,1\}. \end{array} \quad (A.3)$$

The graded algebra $\mathfrak{g}$ is endowed with the operation $(\cdot,\cdot):\mathfrak{g}\times\mathfrak{g}\rightarrow\mathfrak{g}$.

Let $a,b,c\in\mathfrak{g}$ be three generators whose respective gradings are $\vec{\alpha},\vec{\beta},\vec{\gamma}$. The bracket $(\cdot,\cdot)$ is defined as

$$(a,b) \;:=\; ab-(-1)^{\vec{\alpha}\cdot\vec{\beta}}ba, \quad (A.4)$$

resulting in either commutators or anticommutators.

The operation satisfies the graded Jacobi identities

$$(-1)^{\vec{\gamma}\cdot\vec{\alpha}}(a,(b,c))+(-1)^{\vec{\alpha}\cdot\vec{\beta}}(b,(c,a))+(-1)^{\vec{\beta}\cdot\vec{\gamma}}(c,(a,b)) \;=\; 0. \quad (A.5)$$

The grading $\deg[(a,b)]$ of the generator (a,b) is the mod 2 sum

$$\deg[(a,b)] \;=\; \vec{\alpha}+\vec{\beta}. \quad (A.6)$$

Remark: in the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded superalgebra case $ii)$ the only sectors which are on equal footing and can be interchanged are 10 and 01. In the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded algebra case $iii)$ the three sectors $11,10,01$ are on equal footing and can be interchanged. In Sections **4** and **5** we made use of this observation.

A graded algebra $\mathfrak{g}$ is represented on a graded vector space $\mathcal{V}$ such that

$$i):\quad \mathcal{V}=\mathcal{V}_0\oplus\mathcal{V}_1;\qquad ii)\text{ and }iii):\quad \mathcal{V}=\mathcal{V}_{00}\oplus\mathcal{V}_{01}\oplus\mathcal{V}_{10}\oplus\mathcal{V}_{11}. \quad (A.7)$$

The grading of a vector $v\in\mathcal{V}$ is denoted with $\vec{\nu}$. Depending on the case, it is either $\vec{\nu}\equiv\nu\in\{0,1\}$ or $\vec{\nu}=(\nu_1,\nu_2)$ with $\nu_{1,2}\in\{0,1\}$. A generator $a\in\mathfrak{g}$ (of grading $\vec{\alpha}$) is represented by the operator $\widehat{a}\in End(\mathcal{V})$. The compatibility of the gradings requires that the grading $\vec{\nu}'$ of the transformed vector $v'=av\in\mathcal{V}$ is $\vec{\nu}'=\vec{\alpha}+\vec{\nu}$. The sum is taken mod 2.

Without loss of generality, see the discussion in Section **4**, we can assume the $\mathbb{Z}_2$-graded matrices to be split into block-diagonal even and odd sectors. For the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded (super)algebras, also without loss of generality, the 4×4 graded matrices can be decomposed according to

$$\begin{array}{ll} M_{00}=\begin{pmatrix} m_1&0&0&0\\0&m_2&0&0\\0&0&m_3&0\\0&0&0&m_4\end{pmatrix}\in\mathfrak{g}_{00}, & M_{11}=\begin{pmatrix} 0&m_5&0&0\\m_6&0&0&0\\0&0&0&m_7\\0&0&m_8&0\end{pmatrix}\in\mathfrak{g}_{11},\\ M_{10}=\begin{pmatrix} 0&0&m_9&0\\0&0&0&m_{10}\\m_{11}&0&0&0\\0&m_{12}&0&0\end{pmatrix}\in\mathfrak{g}_{10}, & M_{01}=\begin{pmatrix} 0&0&0&m_{13}\\0&0&m_{14}&0\\0&m_{15}&0&0\\m_{16}&0&0&0\end{pmatrix}\in\mathfrak{g}_{01}, \end{array} \quad (A.8)$$

where the entries $m_1,m_2,\ldots,m_{16}$ are either constant numbers or, as in (4), operators.

By assuming this convention, the graded vector space $\mathcal{V}$ is decomposed according to

$$v_{00} = \begin{pmatrix} v \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{V}_{00}, \quad v_{11} = \begin{pmatrix} 0 \\ v \\ 0 \\ 0 \end{pmatrix} \in \mathcal{V}_{11}, \quad v_{10} = \begin{pmatrix} 0 \\ 0 \\ v \\ 0 \end{pmatrix} \in \mathcal{V}_{10}, \quad v_{01} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix} \in \mathcal{V}_{01}. \tag{A.9}$$

The construction of the multi-particle states is based, see [6, 7], on the notions of coproduct and braided tensor product as applied in the context of Hopf algebras. For all three cases *i*), *ii*), *iii*) in (A.1) the Universal Enveloping Algebra $U \equiv \mathcal{U}(\mathfrak{g})$ of a graded algebra $\mathfrak{g}$ is endowed with a Hopf algebra structure. Definition and properties of Hopf algebras can be found in [7]. We limit here to recall the properties that we have used in the main text.

The coproduct Δ is a map

$$\Delta \quad : \quad U \rightarrow U \otimes U \tag{A.10}$$

which satisfies the coassociativity property

$$(\Delta \otimes id)\Delta(U) \quad = \quad (id \otimes \Delta)\Delta(U). \tag{A.11}$$

The action $\Delta(u)$ of the coproduct on a generic element $u \in U$ can be recovered from the action on the identity $\mathbf{1} \in \mathcal{U}(\mathfrak{g})$, the action on a primitive element $g \in \mathfrak{g}$ and from the comultiplication. We have

$$\begin{aligned} \Delta(\mathbf{1}) &= \mathbf{1} \otimes \mathbf{1}, \\ \Delta(g) &= \mathbf{1} \otimes g + g \otimes \mathbf{1}, \\ \Delta(u_1 u_2) &= \Delta(u_1) \cdot \Delta(u_2). \end{aligned} \tag{A.12}$$

Concerning the braided tensor product, let $a, b, c, d \in \mathfrak{g}$. We assume, as before, the grading of b, c to be respectively given by $\vec{\beta}, \vec{\gamma}$. The braiding of two tensor spaces is expressed by the formula

$$(a \otimes b) \cdot (c \otimes d) \quad = \quad (-1)^{\vec{\beta} \cdot \vec{\gamma}} ac \otimes bd. \tag{A.13}$$

For the $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$- gradings the braiding corresponds to the $(-1)^{\vec{\beta} \cdot \vec{\gamma}}$ sign. Its expression, depending on case *i*), *ii*) or *iii*), is given in formula (A.3).

Appendix B: Representations of 2-particle observables

We present here for completeness the 16×16 constant hermitian matrices which realize the 2-particle observables X_u, X_*, Y_u, Y_* introduced in Section **5**, formulas (44) and (45). These observables allow to discriminate whether the system under consideration is composed by ordinary bosons or by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons.

We have

$$
\begin{aligned}
X_u &= \left(\begin{array}{cccc|cccc|cccc|cccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0
\end{array}\right), \\
X_* &= \left(\begin{array}{cccc|cccc|cccc|cccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&\mathbf{1}\\
0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&\mathbf{1}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&\mathbf{1}&0&0&0&0&\mathbf{1}&0&0&0&0&0
\end{array}\right)
\end{aligned}
\tag{B.1}
$$

and

$$
Y_u = \left(\begin{array}{cccc|cccc|cccc|cccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&\mathbf{2}&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&\mathbf{2}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\mathbf{4}&0&0&0&0&\mathbf{2}&0&0&0&0&\mathbf{2}\\
0&0&0&0&0&0&\mathbf{2}&0&0&\mathbf{2}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\mathbf{2}&0&0&0&0&0&\mathbf{2}&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\mathbf{2}&0&0&\mathbf{2}&0&0&0&0&0&0\\
0&0&0&0&0&\mathbf{2}&0&0&0&0&0&0&0&0&0&\mathbf{2\varepsilon}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{2\varepsilon}&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\mathbf{2}&0&0&0&0&0&\mathbf{2}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&\mathbf{2\varepsilon}&0&0&0&0\\
0&0&0&0&0&\mathbf{2}&0&0&0&0&\mathbf{2\varepsilon}&0&0&0&0&0
\end{array}\right),
$$

$$
Y_* = \left(\begin{array}{cccc|cccc|cccc|cccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&\mathbf{2}&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbf{2}&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&\mathbf{2}&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&\mathbf{2}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\mathbf{4}&0&0&0&0&\mathbf{4+2\varepsilon}&0&0&0&0&\mathbf{4+2\varepsilon}\\
0&0&0&0&0&0&\mathbf{4}&0&0&\mathbf{4+2\varepsilon}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\mathbf{4}&0&0&0&0&0&\mathbf{4+2\varepsilon}&0&0\\
\hline
0&0&0&0&0&0&0&0&\mathbf{2}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\mathbf{4+2\varepsilon}&0&0&\mathbf{4}&0&0&0&0&0&0\\
0&0&0&0&0&\mathbf{4+2\varepsilon}&0&0&0&0&\mathbf{4}&0&0&0&0&\mathbf{4+2\varepsilon}\\
0&0&0&0&0&0&0&0&0&0&0&\mathbf{4}&0&0&\mathbf{4+2\varepsilon}&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{2}&0&0&0\\
0&0&0&0&0&0&0&\mathbf{4+2\varepsilon}&0&0&0&0&0&\mathbf{4}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&\mathbf{4+2\varepsilon}&0&0&\mathbf{4}&0\\
0&0&0&0&0&\mathbf{4+2\varepsilon}&0&0&0&0&\mathbf{4+2\varepsilon}&0&0&0&0&\mathbf{4}
\end{array}\right).
\qquad \text{(B.2)}
$$

The $\boldsymbol{\varepsilon}$ sign entering the (B.2) matrices takes the value $\boldsymbol{\varepsilon = +1}$ in the bosonic case and $\boldsymbol{\varepsilon = -1}$ in the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosonic case. Unlike X_u, X_*, the operators Y_u, Y_* are $\boldsymbol{\varepsilon}$-dependent.

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