Series Representation of the Modified Bessel Function of the Second Kind $K_0(\beta x)$

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Abstract

A series representation of the modified Bessel function of the second kind $K_0(\beta x)$ in terms of simple elementary functions is obtained. Predictions corresponding to different orders in the expansion are compared with the original function together with an analysis of the corresponding errors. An alternative way of writing $K_0(\beta x)$ is given in terms of a confluent hypergeometric function.

Keywords: Bessel Function, Series Expansion, Approximation Theory

1. Introduction

We want to find out a series representation of the modified Bessel function of the second kind $K_0(\beta x)$ in terms of simple elementary functions. In Ref. [1] it is shown that, for $\nu > 0$, $K_\nu(\beta x)$ can be represented by an infinite series given by

$$K_\nu(\beta x) = e^{-\beta x} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Lambda(\nu, n, k)\, (\beta x)^{k-\nu} \tag{1}$$

with the coefficients

$$\Lambda(\nu, n, k) = \frac{(-1)^k \sqrt{\pi}\, \Gamma(2\nu)\, \Gamma\left(\frac{1}{2} + n - \nu\right)\, L(n,k)}{2^{\nu-k}\, \Gamma\left(\frac{1}{2} - \nu\right)\, \Gamma\left(\frac{1}{2} + n + \nu\right) n!} \tag{2}$$

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where $L(n,k)$ are the Lah numbers defined by

$$L(n,k) = \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \frac{n!}{k!} = \frac{(n-1)!}{(k-1)!(n-k)!} \frac{n!}{k!} \qquad \text{for} \qquad n,k > 0 \qquad (3)$$

with the conventions $L(0,0) = 1$, and $L(n,0) = 0$, for $n \geq 1$.

2. The case $\nu = 0$

In order to get the expansion we are looking for, we start from the well known recursion formula [2]

$$zK_{\nu-1}(z) - zK_{\nu+1}(z) = -2\nu K_\nu(z)$$

which, from Eqs. (1-3), allow us to express $K_0(\beta x)$ as a sum of two series, *i.e.*,

$$K_0(\beta x) = -\frac{2}{\beta x} K_1(\beta x) + K_2(\beta x) \qquad (4)$$

where

$$K_1(\beta x) = e^{-\beta x} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Lambda(1,n,k)\,(\beta x)^{k-1}$$

and

$$K_2(\beta x) = e^{-\beta x} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Lambda(2,n,k)\,(\beta x)^{k-2}$$

with the respective coefficients

$$\Lambda(1,n,k) = \frac{(-1)^k \sqrt{\pi}\,\Gamma(2)\,\Gamma\left(n-\frac{1}{2}\right)\,L(n,k)}{2^{1-k}\,\Gamma\left(-\frac{1}{2}\right)\,\Gamma\left(n+\frac{3}{2}\right)\,n!}$$

and

$$\Lambda(2,n,k) = \frac{(-1)^k \sqrt{\pi}\,\Gamma(4)\,\Gamma\left(n-\frac{3}{2}\right)\,L(n,k)}{2^{2-k}\,\Gamma\left(-\frac{3}{2}\right)\,\Gamma\left(n+\frac{5}{2}\right)\,n!}$$

The dependence of these coefficients on the Γ function can be removed by applying its recursion relations. Doing so, both coefficients can be written in a simpler formula as

$$\Lambda(1,n,k) = \frac{(-1)^{k+1}}{2^{2-k}} \frac{1}{\left(n^2-\frac{1}{4}\right)} \frac{L(n,k)}{n!}$$

and

$$\Lambda(2,n,k) = \frac{9}{2} \frac{(-1)^k}{2^{2-k}} \frac{1}{\left(n^2-\frac{9}{4}\right)\left(n^2-\frac{1}{4}\right)} \frac{L(n,k)}{n!}$$

Substituting these factor in Eq. (4) we get, after a straightforward calculation, the polynomial expansion formula for $K_0(\beta x)$, namely

$$K_0(\beta x) = e^{-\beta x} \sum_{n=0}^{\infty} \frac{n\, n!}{\left(n^2 - \frac{1}{4}\right)\left(n^2 - \frac{9}{4}\right)} \sum_{k=0}^{n} \frac{(-1)^k\, 2^{k-1}\, k\ (\beta x)^{k-2}}{(k!)^2\, (n-k)!} \tag{5}$$

Knowing that the confluent hypergeometric function (Kummer's function) of the first kind ${}_1F_1$ is given by

$${}_1F_1(1-n;2;2\beta x) = -\sum_{k=0}^{n} \frac{(n-1)!(-1)^k 2^{k-1} k(\beta x)^{k-1}}{(k!)^2(n-k)!}$$

the result of Eq. (5) can be expressed in terms of only one summation as

$$K_0(\beta x) = e^{-\beta x} \sum_{n=0}^{\infty} \frac{-n^2}{\left(n^2 - \frac{9}{4}\right)\left(n^2 - \frac{1}{4}\right)} \frac{{}_1F_1(1-n;2;2\beta x)}{\beta x} \tag{6}$$

3. Discussions

In order to facilitate the numerical calculations, we give in Table 1 the explicit values of the coefficients $\Lambda(1,n,k)$, for n and k varying from 0 to 10. For simplicity, it is assumed that $\beta = 1$. The coefficients $\Lambda(2,n,k)$ can be straightforwardly obtained from the values of Table 1 by using the equality

$$\Lambda(2,n,k) = -\frac{4.5}{(n^2 - 2.25)}\,\Lambda(1,n,k) \tag{7}$$

From Table 1 and from Eqs. (1) and (7), we can easily get the explicit formulae for a finite terms expansion of the Bessel functions for $\nu = 1$ and $\nu = 2$. Taking $\beta = 1$, the truncated summation up to $n = 8$, for example, for K_1 and K_2 are, respectively,

$$\begin{aligned} K_1(x) &= e^{-x}\left(\frac{16}{7} + \frac{1}{x} - \frac{467144x}{765765} + \frac{373696x^2}{765765} - \frac{37372x^3}{153153} + \frac{22688x^4}{328185} + \right. \\ &- \left. \frac{1168x^5}{109395} + \frac{32x^6}{38675} - \frac{2x^7}{80325}\right) \end{aligned} \tag{8}$$

and

$$K_2(x) = e^{-x}\left(\frac{784}{1615}+\frac{2}{x^2}+\frac{3232}{1615x}-\frac{448x}{4845}+\frac{5416744x^2}{190855665}-\frac{93376x^3}{14549535}+\right.$$
$$\left.+\frac{27424x^4}{31177575}-\frac{512x^5}{8083075}+\frac{4x^6}{2204475}\right) \tag{9}$$

From Eqs. (8) and (9) the well known curves for both Bessel functions are reproduced with accuracy.

We can now plot the function given by Eq. (5). In Fig. 1 we compare the curve for the expansion until $n = 8$ with the well known behavior of $K_0(x)$ without any approximation.

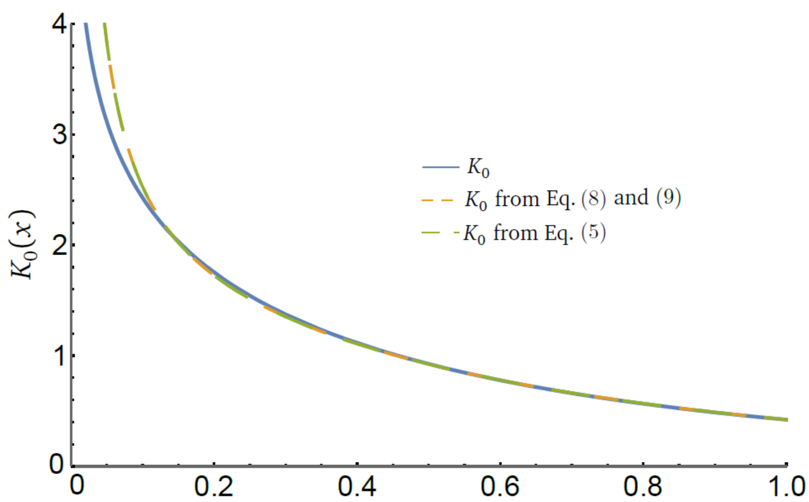

Figure 1: Comparison between the Bessel function K_0 and the result of its expansion up to $n = 8$.

Some particular values for $K_0(x)$ are given in Table 2, for the interval $0.1 \leq x \leq 5.0$ where the relative error $|\epsilon_r|$ is defined by

$$|\epsilon_r| = \frac{\left|K_0(x) - K_0(x)\big|_{\text{approx}}\right|}{K_0}$$

Notice that the maximum value of $|\epsilon_r|$, corresponding to the cases $n = 15$ and $n = 20$, is of the order of 1%.

For the expansion of $K_0(x)$ up to $n = 8$, we have a relative error of $\simeq 4\%$ for $x = 0.1$ (See Table 2). However, we get an error of 10% or more when $x \leq 0.0741$. For $n = 15$, the error is $\simeq 1\%$ or less, when $x \geq 0.0676$, and an error of 10% or more when $x \leq 0.0376$. Finally, for $n = 20$, we have an error of 1% or less when $x \geq 0.0505$ and an error of 10% or more when $x \leq 0.0274$.

In Fig.2 different approximations of $K_0(x)$ are compared.

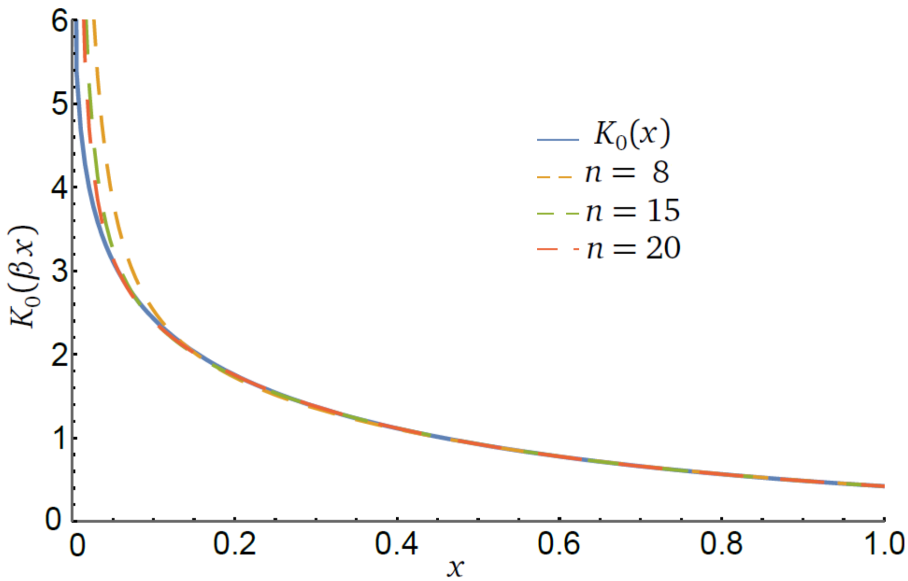

Figure 2: Comparison between the Bessel function $K_0(x)$ and its expansion (5) for different values of n.

In order to understand why the expansion for $K_1(x)$ and $K_2(x)$ have a great fit even for small values of x but $K_0(x)$ shows a much larger error for the same small x values we have to look carefully at Eq, (5). The problem is that the first term of Eq. (5) is the division $K_1(x)/x$. Therefore, for very small values of$x$, the error in the series predictions for $K_1(x)$ is amplified. In Table 3, some values of $K_1(x)$ and $K_2(x)$ are shown with the respective absolute error, for the interval $0.05 \leq x \leq 10$.

So, while the absolute errors in $K_1(0.05)$ and $K_2(0.05)$ are of the order of 10^{-2}, the error found for $K_0(0.05)$, given by Eq, (5), is $\simeq 0.35$, which is an order of magnitude greater than the error just in $K_1(0.05)$. The situation tends to become worse when $x \to 0$.

4. Conclusion

We got two equivalent expressions to represent the Bessel function $K_0(\beta x)$ in terms of simple functions. It is shown that these expressions describes $K_0(\beta x)$ in a very good approximation for $\beta x \geq 0.1$.

Acknowledgment

One of us (FS) was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brazil (CAPES), Finance Code 001.

References

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[2] G.N. Watson. *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, London, 1966.

Table 1: Values for $\Lambda(1, n, k)$, with $\beta = 1$

n \ k	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
0	1										
1	0	2/3									
2	0	2/15	-2/15								
3	0	2/35	-4/35	4/105							
4	0	2/63	-2/21	4/63	-2/189						
5	0	2/99	-8/99	8/99	-8/297	4/1485					
6	0	2/143	-10/143	40/429	-20/429	4/429	-4/6435				
7	0	2/195	-4/65	4/39	-8/117	4/195	-8/2925	8/61425			
8	0	2/255	-14/255	28/255	-14/153	28/765	-28/3825	8/11475	-2/80325		
9	0	2/323	-16/323	112/969	-112/969	56/969	-224/14535	32/14535	-16/101745	4/915705	
10	0	2/399	-6/133	16/133	-8/57	8/95	-8/285	32/5985	-8/13965	4/125685	-4/5655825

Table 2: Values for $K_0(\beta x)$ expansion given by Eq. (5) for different n values, with $\beta = 1$

x	$K_0(x)$	$K_0(x)$ expansion up to n					
		8	$\lvert\epsilon_r\rvert$ (%)	15	$\lvert\epsilon_r\rvert$ (%)	20	$\lvert\epsilon_r\rvert$ (%)
0.1	2.42707	2.5268	4.11	2.40169	1.046	2.39917	1.15
0.2	1.7527	1.72407	1.63	1.7402	0.71	1.75031	0.14
0.3	1.37246	1.35125	1.55	1.37292	0.03	1.37533	0.21
0.4	1.11453	1.10552	0.81	1.1174	0.26	1.11603	0.13
0.5	0.924419	0.922763	0.18	0.926341	0.21	0.924409	0.0011
0.6	0.777522	0.779281	0.23	0.778119	0.076	0.776932	0.076
0.7	0.66052	0.663358	0.43	0.66026	0.039	0.659982	0.081
0.8	0.565347	0.568067	0.48	0.564752	0.11	0.56509	0.045
0.9	0.48673	0.488824	0.43	0.48615	0.12	0.486736	0.0012
1.0	0.421024	0.422366	0.32	0.420628	0.094	0.421182	0.037
5.0	3.69×10^{-3}	3.66×10^{-3}	1.03	3.69×10^{-3}	0.10	3.69×10^{-3}	0.08

Table 3: Values for $K_1(x)$ and $K_2(x)$ obtained from Eqs. (8) and (9) up to $n = 8$. The error here is the difference between the value found and the expected value

x	$K_1(x)$	$K_2(x)$
0.05	19.892 $\pm$ 0.0176934	799.514 $\pm$ 0.0125116
0.1	9.84899 $\pm$ 0.00485623	199.507 $\pm$ 0.00260945
0.5	1.65683 $\pm$ 0.000392678	7.55012 $\pm$ 0.000066483
1.0	0.601256 $\pm$ 0.000650892	1.62492 $\pm$ 0.0000855086
5.0	0.00413672 $\pm$ 0.0000921043	0.0053286 $\pm$ 0.0000196572
10.0	0.000080361 $\pm$ 0.0000617122	0.000021682 $\pm$ 1.72217×10^{-7}