$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded mechanics: the quantization

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Abstract

In the previous paper arXiv:2003.06470 we introduced the notion of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded classical mechanics and presented a general framework to construct, in the Lagrangian setting, the worldline sigma models invariant under a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra. In this work we discuss at first the classical Hamiltonian formulation of some of these models and later present their canonical quantization.

As the simplest application of the construction we recover the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded quantum Hamiltonian introduced by Bruce and Duplij in arXiv:1904.06975. We prove that this is the first example of a large class of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded quantum models. We derive in particular interacting multiparticle quantum Hamiltonians given by Hermitian, matrix, differential operators. The interacting terms appear as non-diagonal entries in the matrices.

The construction of the Noether charges, both classical and quantum, is presented. A comprehensive discussion of the different $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetries possessed by the quantum Hamiltonians is given.

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1 Introduction

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded (super)algebras were introduced by Rittenberg and Wyler in [1, 2], while earlier related structures were investigated in [3]. In the Rittenberg-Wyler works, which were inspired by the construction of ordinary superalgebras, some possible physical applications to elementary particle physics were suggested. Since then these new graded structures attracted the attention of mathematicians [4] with a steady flow of papers devoted to their classifications [5, 6], representations [7–9], generalizations [10]. At the beginning, on the physical side, $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded structures received limited attention (see, however, the works [11–15] and, in connection with de Sitter supergravity, [16, 17]) since the main focus was reserved to relativistic theories which respect the spin-statistics connection.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras naturally lead to the broad field of parastatistics (for the mathematical aspects of the connection with parastatistics, see [18, 19]). It is therefore quite natural to expect that they could play a role in low-dimensional (where anyons can enter the game) and/or non-relativistic physics. This recognition is responsible for the recent surge of interest to physical applications of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras, with several works investigating this problem from different sides. In [20, 21] it was shown, quite unexpectedly, that $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras are symmetries of the well-known Lévy-Leblond equations. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded analogues of supersymmetric and superconformal quantum mechanics were introduced in [22–25]. Graded structures with commuting fermions appear in dual double field theory and mixed symmetry tensors [26–28]. In the meanwhile, mathematical properties continue to be investigated [29–32].

This state of the art motivated us to launch a systematic investigation of the features of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded mechanics. In the first paper in this direction [33] we presented the general framework for the construction of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded invariant, classical, worldline sigma models in the Lagrangian setting. We mimicked the construction of supermechanics, that is the classical supersymmetric mechanics, extending the tools (the D-module representations of supermultiplets and their application to the derivation of the invariant actions) developed in [34–36].

In this paper we proceed (adapting to our case the approach to the canonical quantization of supermechanical models discussed in [37]) to define the classical Hamiltonian formalism and the canonical quantization for the subclass of models in [33] derived from fundamental $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded multiplets with one propagating boson. This subclass of models, besides being simpler to quantize, already contains what is the most relevant feature, the physics of interacting multiplets (which is tantamount to the construction of multiparticle wavefunctions). It is essential for the aforementioned parastatistics properties of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras and the physical significance of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry (more on that will be commented in the Conclusions).

As recalled in [33], the time-dependent $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded classical fields are divided into ordinary bosons, exotic bosons and two classes of fermions (fermions belonging to different classes mutually commute instead of anticommuting). These fields (anti)commute according to the table (A.4) presented in the Appendix. The construction of the Hamiltonian framework requires some delicate steps. The Poisson brackets which respect the ordering of the fields have to be carefully introduced. Furthermore, in analogy with the construction of supersymmetric mechanics, see [37], the existence of second-class constraints for the fermionic fields leads to the introduction of Dirac brackets. The canonical quantization is achieved by eliminating the auxiliary fields via algebraic equations of motion and by a suitable choice of the classical canonical variables, the so-called "constant kinetic basis". The quantum theories so derived present hermitian, second-order differential, matrix Hamiltonians.

The simplest application of our approach concerns a single $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded multiplet. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded quantum mechanics introduced in [23] is recovered as the quantization of this classical model. The analysis of the Noether charges allows understanding the different $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetries possessed by this Hamiltonian. It is shown, in particular, that both the one-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supertranslation algebra and the Beckers-Debergh algebra introduced in [38] (the definition of these algebras is recalled in the Appendix, see formulas (A.7) and (A.8), respectively) are obtained by taking into account different matrix representations for the (anti)commutators induced by the Dirac brackets.

A key point is that the single-multiplet quantum Hamiltonian is just the first ($n = 1$) example of a large class of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded quantum Hamiltonians derived from the classical construction for n interacting multiplets of type $(1, 2, 1)_{[00]}$ (these multiplets produce n propagating ordinary bosons, n auxiliary exotic bosons and n fermions in each one of the two classes). The $n > 1$ quantum Hamiltonians have interesting features that cannot be observed for $n = 1$. The interacting terms among the multiplets appear as non-diagonal entries in their hermitian matrices. The 8×8 differential matrix Hamiltonian for $n = 2$ and the 16×16 differential matrix Hamiltonian for $n = 3$ are respectively presented in formulas (146) and (157-161). They are given in terms of the unconstrained functions $f(x, y)$ for $n = 2$ and $f(x, y, z)$ for $n = 3$.

The scheme of the paper is the following. In Section **2**, after recalling the Lagrangian formulation of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded classical invariant models under consideration, we derive their classical Noether charges. In Section **3** we present the Hamiltonian formulation of these classical models; the "constant kinetic basis", which paves the way for the canonical quantization, is introduced. The canonical quantization and the derivation of the conserved quantum Noether charges is presented in Section **4**. The different graded symmetries of the simplest quantum Hamiltonian are discussed in Section **5**. In Section **6** we present invariant, $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded, interacting multiparticle quantum Hamiltonians. In the Conclusions we discuss the relevance of the results obtained in the paper and point out various directions of future works. For completeness, the relevant features of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras and of their graded representations are recalled in the Appendix.

2 The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded classical Lagrangian mechanics

We revisit at first the simplest cases of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded classical mechanics in the Lagrangian formulation. Later, at the end of the Section, we present the computation of the classical Noether charges. For our purposes the simplest worldline models are, see [33], the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded classical invariant actions of the $(1, 2, 1)_{[00]}$ and $(1, 2, 1)_{[11]}$ multiplets. Both multiplets present one propagating bosonic field, two propagating fermionic fields and one auxiliary bosonic field. In the first multiplet the auxiliary field is the exotic boson, while in the second multiplet the auxiliary field is the ordinary boson (see [33] for details). The four time-dependent fields of respective $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading "[00], [11], [10], [01]" are accommodated into the multiplet $(x(t), z(t), \psi(t), \xi(t))^T$. In this paper it is more convenient to use the real time t, instead of the Euclidean time τ employed in [33]. Accordingly, the D-module representation acting on the $(1, 2, 1)_{[00]}$ multiplet is defined

by the operators

$$\widehat{H}=\begin{pmatrix} i\partial_t & 0 & 0 & 0\\ 0 & i\partial_t & 0 & 0\\ 0 & 0 & i\partial_t & 0\\ 0 & 0 & 0 & i\partial_t \end{pmatrix},\qquad \widehat{Z}=\begin{pmatrix} 0 & 1 & 0 & 0\\ \partial_t^2 & 0 & 0 & 0\\ 0 & 0 & 0 & -i\partial_t\\ 0 & 0 & i\partial_t & 0 \end{pmatrix},$$
$$\widehat{Q}_{10}=\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & i\partial_t\\ i\partial_t & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix},\qquad \widehat{Q}_{01}=\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -i\partial_t & 0\\ 0 & -1 & 0 & 0\\ i\partial_t & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

They close the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded supertranslation algebra (A.7) defined by the (anti)commutators

$$\{\widehat{Q}_{10},\widehat{Q}_{10}\}=\{\widehat{Q}_{01},\widehat{Q}_{01}\}=2\widehat{H},\quad [\widehat{Q}_{10},\widehat{Q}_{01}]=-2\widehat{Z},\quad [\widehat{H},\widehat{Q}_{10}]=[\widehat{H},\widehat{Q}_{01}]=[\widehat{H},\widehat{Z}]=0. \tag{2}$$

The D-module representation acting on the $(1,2,1)_{[11]}$ multiplet is defined by the operators

$$\breve{H}=\begin{pmatrix} i\partial_t & 0 & 0 & 0\\ 0 & i\partial_t & 0 & 0\\ 0 & 0 & i\partial_t & 0\\ 0 & 0 & 0 & i\partial_t \end{pmatrix},\qquad \breve{Z}=\begin{pmatrix} 0 & -\partial_t^2 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -i\partial_t\\ 0 & 0 & i\partial_t & 0 \end{pmatrix},$$
$$\breve{Q}_{10}=\begin{pmatrix} 0 & 0 & i\partial_t & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & i\partial_t & 0 & 0 \end{pmatrix},\qquad \breve{Q}_{01}=\begin{pmatrix} 0 & 0 & 0 & i\partial_t\\ 0 & 0 & -1 & 0\\ 0 & -i\partial_t & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

They close the (2) algebra (in the notation, the " $\breve{}$ " symbol replaces " $\widehat{}$ ").

The field transformations are respectively read from (1) and (3). We have

$$\widehat{Q}_{10}:\begin{pmatrix} x\\ z\\ \psi\\ \xi \end{pmatrix}\mapsto\begin{pmatrix} \psi\\ i\dot{\xi}\\ i\dot{x}\\ z \end{pmatrix},\qquad \widehat{Q}_{01}:\begin{pmatrix} x\\ z\\ \psi\\ \xi \end{pmatrix}\mapsto\begin{pmatrix} \xi\\ -i\dot{\psi}\\ -z\\ i\dot{x} \end{pmatrix},\qquad \widehat{Z}:\begin{pmatrix} x\\ z\\ \psi\\ \xi \end{pmatrix}\mapsto\begin{pmatrix} z\\ \ddot{x}\\ -i\dot{\xi}\\ i\dot{\psi} \end{pmatrix} \tag{4}$$

and

$$\breve{Q}_{10}:\begin{pmatrix} x\\ z\\ \psi\\ \xi \end{pmatrix}\mapsto\begin{pmatrix} i\dot{\psi}\\ \xi\\ x\\ i\dot{z} \end{pmatrix},\qquad \breve{Q}_{01}:\begin{pmatrix} x\\ z\\ \psi\\ \xi \end{pmatrix}\mapsto\begin{pmatrix} i\dot{\xi}\\ -\psi\\ -i\dot{z}\\ x \end{pmatrix},\qquad \breve{Z}:\begin{pmatrix} x\\ z\\ \psi\\ \xi \end{pmatrix}\mapsto\begin{pmatrix} -\ddot{z}\\ -x\\ -i\dot{\xi}\\ i\dot{\psi} \end{pmatrix}. \tag{5}$$

The operator $\widehat{H}=\breve{H}$ maps the fields into their time derivatives multiplied by i.

In the construction of the classical actions the [10]-graded and the [01]-graded component fields are assumed to be Grassmann. It is a consequence of the more general (A.2) prescription for the (anti)commutators of the graded component fields. The action of the operators (1) and (3) on the graded component fields is assumed to satisfy the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Leibniz rule.

For the $(1,2,1)_{[00]}$ multiplet, the classical action $\mathcal{S}=\int dt\mathcal{L}$, invariant under the (4) transformations and (2) algebra, is given [33] by the Lagrangian

$$\mathcal{L}=\mathcal{L}_\sigma+\mathcal{L}_{lin},$$
$$\text{where}\qquad \mathcal{L}_\sigma=\tfrac{1}{2}\phi(x)(\dot{x}^2-z^2+i\psi\dot{\psi}-i\xi\dot{\xi})-\tfrac{1}{2}\phi_x(x)z\psi\xi\qquad\text{and}\qquad \mathcal{L}_{int}=\mu z. \tag{6}$$

The coupling constant μ has $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading $\deg(\mu) = [11]$. We have denoted $\phi_x = \frac{d\phi(x)}{dx}$.

The Lagrangian term $\mathcal{L}_\sigma$ is written, in manifestly invariant form, as

$$\mathcal{L}_\sigma = -\frac{1}{2}\widehat{Z}\widehat{Q}_{10}\widehat{Q}_{01}g(x) + \frac{1}{2}\frac{d}{dt}(g_x\dot{x}), \qquad \phi(x) := g_{xx}(x). \tag{7}$$

The Euler-Lagrange equation for a component field q, which can be expressed, taking into account how fields are ordered, either as

$$\frac{d}{dt}(\overrightarrow{\partial}_{\dot{q}}\mathcal{L}) - \overrightarrow{\partial}_q\mathcal{L} = 0 \tag{8}$$

or as

$$\frac{d}{dt}(\mathcal{L}\overleftarrow{\partial}_{\dot{q}}) - \mathcal{L}\overleftarrow{\partial}_q = 0, \tag{9}$$

produces in both cases the same set of equations:

$$\begin{aligned}
\phi\ddot{x} &= -\frac{1}{2}\phi_x(\dot{x}^2 + z^2 - i\psi\dot{\psi} + i\xi\dot{\xi}) - \frac{1}{2}\phi_{xx}z\psi\xi, \\
2\phi z &= -\phi_x\psi\xi + 2\mu, \\
i\phi\dot{\psi} &= -\frac{1}{2}\phi_x(i\dot{x}\psi + z\xi), \\
i\phi\dot{\xi} &= -\frac{1}{2}\phi_x(i\dot{x}\xi - z\psi).
\end{aligned} \tag{10}$$

For the $(1,2,1)_{[11]}$ multiplet the invariant classical action $\mathcal{S} = \int dt\overline{\mathcal{L}}$ is obtained, see [33], from the Lagrangian

$$\begin{gathered}
\overline{\mathcal{L}} = \overline{\mathcal{L}}_\sigma + \overline{\mathcal{L}}_{lin}, \\
\text{where} \qquad \overline{\mathcal{L}}_\sigma = \tfrac{1}{2}\Phi(z)(\dot{z}^2 - x^2 + i\psi\dot{\psi} - i\xi\dot{\xi}) + \tfrac{1}{2}\Phi_z(z)x\psi\xi \qquad \text{and} \qquad \overline{\mathcal{L}}_{int} = \overline{\mu}x.
\end{gathered} \tag{11}$$

As notable differences with the previous case $\Phi(z)$ is an even function of z and the coupling constant $\overline{\mu}$ is real, its $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading being given by $\deg(\overline{\mu}) = [00]$.

The Lagrangian term $\overline{\mathcal{L}}_\sigma$ is written, in manifestly invariant form, as

$$\overline{\mathcal{L}}_\sigma = -\frac{1}{2}\breve{Z}\breve{Q}_{10}\breve{Q}_{01}f(z) + \frac{1}{2}\frac{d}{dt}(f_z(z)\dot{z}), \qquad \Phi(z) := f_{zz}(z), \tag{12}$$

with $f(z)$ an even function of z.

The Euler-Lagrange equations of motion now read

$$\begin{aligned}
\Phi\ddot{z} &= -\frac{1}{2}\Phi_z(\dot{z}^2 + x^2 - i\psi\dot{\psi} + i\xi\dot{\xi}) + \frac{1}{2}\Phi_{zz}x\psi\xi, \\
2\Phi x &= \Phi_z\psi\xi + 2\overline{\mu}, \\
i\Phi\dot{\psi} &= \frac{1}{2}\Phi_z(-i\dot{z}\psi + x\xi), \\
i\Phi\dot{\xi} &= -\frac{1}{2}\Phi_z(i\dot{z}\xi + x\psi).
\end{aligned} \tag{13}$$

2.1 The Noether charges

We recall the general construction of the Noether charges. An action $S = \int \mathcal{L}(q_i(t), \dot{q}_i(t))dt$ is invariant under the variation $q_i(t) \rightarrow q_i(t) + \delta q_i(t)$ provided that there exists a $\Lambda(t)$ such that

$$\delta S = \int \dot{\Lambda}(t)dt. \tag{14}$$

For our $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded fields the variation δS is computed as

$$\delta S = \int dt\big(\delta q_i \overrightarrow{\partial}_{q_i}\mathcal{L} + \delta\dot{q}_i \overrightarrow{\partial}_{\delta\dot{q}_i}\mathcal{L}\big) = \int dt\left[\delta q_i\Big(\overrightarrow{\partial}_{q_i}\mathcal{L} - \frac{d}{dt}(\overrightarrow{\partial}_{\dot{q}_i}\mathcal{L})\Big) + \frac{d}{dt}\big(\delta q_i \overrightarrow{\partial}_{\dot{q}_i}\mathcal{L}\big)\right]. \tag{15}$$

By using the equations of motion the invariance of S produces the identity

$$\int dt \frac{d}{dt}\big(\Lambda - \delta q_i \overrightarrow{\partial}_{\dot{q}_i}\mathcal{L}\big) = 0 \tag{16}$$

which implies the existence of a conserved charge Q given by

$$Q = \Lambda - \delta q_i \overrightarrow{\partial}_{\dot{q}_i}\mathcal{L}. \tag{17}$$

Applying formula (17) to the Lagrangian (6) under a (4) transformation we easily obtain Λ. Since no confusion arises, for convenience we use the same notation for the generator of the transformation and its corresponding conserved charge.
The Noether charges for the invariant action of the $(1,2,1)_{[00]}$ multiplet are therefore given as

$$\begin{aligned}
\widehat{H} &= \frac{1}{2}\phi(\dot{x}^2 + z^2) + \frac{1}{2}\phi_x z\psi\xi - \mu z,\\
\widehat{Q}_{10} &= \phi\dot{x}\psi + i\mu\xi,\\
\widehat{Q}_{01} &= \phi\dot{x}\xi - i\mu\psi,\\
\widehat{Z} &= \big(\phi z + \frac{1}{2}\phi_x\psi\xi - \mu\big)\dot{x}.
\end{aligned} \tag{18}$$

The following intermediate results were used to compute (18):

	Λ	$\delta_{q_i}\overrightarrow{\partial}_{\dot{q}_i}\mathcal{L}$
$\widehat{H}$:	$\mathcal{L}$	$\phi\Big(\dot{x}^2 + \frac{i}{2}\psi\dot{\psi} - \frac{i}{2}\xi\dot{\xi}\Big)$
$\widehat{Q}_{10}$:	$\frac{1}{2}\phi(iz\xi + \dot{x}\psi) - i\mu\xi$	$\frac{1}{2}\phi(3\dot{x}\psi + iz\xi)$
$\widehat{Q}_{01}$:	$\frac{1}{2}\phi(iz\psi - \dot{x}\xi) + i\mu\psi$	$\frac{1}{2}\phi(\dot{x}\xi + iz\psi)$
$\widehat{Z}$:	$-\frac{1}{2}\Big(\phi_x\dot{x}\psi\xi + \phi\frac{d}{dt}(\psi\xi)\Big) + \mu\dot{x}$	$\phi\Big(\dot{x}z - \frac{1}{2}\frac{d}{dt}(\psi\xi)\Big)$

(19)

The Noether charges $\widehat{Q}_{10}$, $\widehat{Q}_{01}$ in (18) do not depend on the auxiliary field z.

By using the algebraic relation for z in the (10) equations of motion, we can eliminate z from all formulae obtained above. We get

$$z = -\frac{\phi_x}{2\phi}\psi\xi + \frac{\mu}{\phi}. \tag{20}$$

With this position the Lagrangian (6) now reads

$$\mathcal{L} = \frac{1}{2}\phi(\dot{x}^2 + i\psi\dot{\psi} - i\xi\dot{\xi}) - \frac{\mu\phi_x}{2\phi}\psi\xi + \frac{\mu^2}{2\phi}. \tag{21}$$

The Euler-Lagrange equations (10) become

$$\begin{aligned}
\phi\ddot{x} &= -\frac{1}{2}\phi_x(\dot{x}^2 - i\psi\dot{\psi} + i\xi\dot{\xi}) - \frac{\mu}{2}\Big(\frac{\phi_x}{\phi}\Big)_x\psi\xi - \frac{\mu^2}{2}\frac{\phi_x}{\phi^2},\\
i\phi\dot{\psi} &= -\frac{1}{2}\phi_x\Big(i\dot{x}\psi + \frac{\mu}{\phi}\xi\Big),\\
i\phi\dot{\xi} &= -\frac{1}{2}\phi_x\Big(i\dot{x}\xi - \frac{\mu}{\phi}\psi\Big).
\end{aligned} \tag{22}$$

From the second and third equations we obtain

$$i\phi\psi\dot{\psi} = -i\phi\xi\dot{\xi} = \mu\frac{\phi_x}{2\phi}\psi\xi. \tag{23}$$

These relations simplify the first equation in (22); at the end we get the equations of motion

$$\begin{aligned}
\phi\ddot{x} &= -\frac{1}{2}\phi_x\dot{x}^2 + \mu\Big(\frac{\phi_x^2}{\phi^2} - \frac{\phi_{xx}}{2\phi}\Big)\psi\xi - \frac{\mu^2}{2}\frac{\phi_x}{\phi^2},\\
i\phi\dot{\psi} &= -\frac{1}{2}\phi_x\Big(i\dot{x}\psi + \frac{\mu}{\phi}\xi\Big),\\
i\phi\dot{\xi} &= -\frac{1}{2}\phi_x\Big(i\dot{x}\xi - \frac{\mu}{\phi}\psi\Big).
\end{aligned} \tag{24}$$

The Noether charges now read as

$$\begin{aligned}
\widehat{H} &= \frac{1}{2}\phi\dot{x}^2 + \mu\frac{\phi_x}{2\phi}\psi\xi - \frac{\mu^2}{2\phi},\\
\widehat{Q}_{10} &= \phi\dot{x}\psi + i\mu\xi,\\
\widehat{Q}_{01} &= \phi\dot{x}\xi - i\mu\psi,\\
\widehat{Z} &= 0.
\end{aligned} \tag{25}$$

One should note that the Noether charge $\widehat{Z}$ now vanishes.

The (25) Noether charges are conserved under the (24) equations of motion:

$$\frac{d}{dt}\widehat{H} = \frac{d}{dt}\widehat{Q}_{10} = \frac{d}{dt}\widehat{Q}_{01} = 0. \tag{26}$$

The same procedure can be repeated to compute the Noether charges of the $(1,2,1)_{[11]}$ model defined by the Lagrangian (11). The results corresponding to formula (18) in this case are

$$\begin{aligned}
\breve{H} &= \frac{1}{2}\Phi(\dot{z}^2 + x^2) - \frac{1}{2}\Phi_z x\psi\xi - \overline{\mu}x,\\
\breve{Q}_{10} &= \Phi\dot{z}\xi + i\overline{\mu}\psi,\\
\breve{Q}_{01} &= \Phi\dot{z}\psi - i\overline{\mu}\xi,\\
\breve{Z} &= \big(\Phi x - \frac{1}{2}\Phi_z\psi\xi - \overline{\mu}\big)\dot{z}.
\end{aligned} \tag{27}$$

3 The Hamiltonian formulation of the classical models

We discuss here the Hamiltonian formulation of the classical models introduced in Section **2**. In order to apply the canonical quantization of the models in Section **4**, following the procedure of [37] in connection with supersymmetric mechanics, we reexpress at first the component fields entering the given multiplet in terms of a new basis called the "constant kinetic basis". We then introduce the Hamiltonian dynamics, defined by Poisson and Dirac brackets, in this basis. To avoid unnecessary doubling of the text we extensively discuss the $(1,2,1)_{[00]}$ model, while presenting only the main results concerning the $(1,2,1)_{[11]}$ model.

3.1 Classical Lagrangians in the constant kinetic basis

For the $(1,2,1)_{[00]}$ model the "constant kinetic basis" which eliminates ϕ from the kinetic term of (6) is reached through the positions

$$y = y(x), \qquad u = \sqrt{\phi(x)}\, z, \qquad \overline{\psi} = \sqrt{\phi(x)}\, \psi, \qquad \overline{\xi} = \sqrt{\phi(x)}\, \xi, \tag{28}$$

where y satisfies

$$y_x = \sqrt{\phi(x)}, \qquad \text{which implies} \qquad \phi_x(y) = \phi_y(y) y_x = \sqrt{\phi(y)}\, \phi_y(y). \tag{29}$$

We rewrite the Lagrangian, equations of motions and Noether charges of Section **2** in terms of the new set of fields (28).

The Lagrangian (6) now reads

$$\mathcal{L} = \frac{1}{2}(\dot{y}^2 - u^2 + i\overline{\psi}\,\dot{\overline{\psi}} - i\overline{\xi}\,\dot{\overline{\xi}}) - \frac{1}{2}\frac{\phi_y}{\phi}\, u\overline{\psi}\,\overline{\xi} + \frac{\mu}{\sqrt{\phi}}\, u. \tag{30}$$

In terms of the new function $W(y)$, introduced through

$$W(y) := \frac{1}{\sqrt{\phi(y)}}, \tag{31}$$

we get

$$\mathcal{L} = \frac{1}{2}(\dot{y}^2 - u^2 + i\overline{\psi\dot{\psi}} - i\overline{\xi\dot{\xi}}) + (\ln W)_y\, u\overline{\psi\xi} + \mu W u. \tag{32}$$

The Euler-Lagrange equations (10) are written as

$$\begin{aligned} \ddot{y} &= (\ln W)_{yy}\, u\overline{\psi\xi} + \mu W_y\, u, \qquad & u &= (\ln W)_y \overline{\psi\xi} + \mu W, \\ i\dot{\overline{\psi}} &= (\ln W)_y\, u\overline{\xi}, & i\dot{\overline{\xi}} &= -(\ln W)_y\, u\overline{\psi}. \end{aligned} \tag{33}$$

The Noether charges (18) now read

$$\begin{aligned} \widehat{H} &= \frac{1}{2}(\dot{y}^2 + u^2) - (\ln W)_y\, u\overline{\psi\xi} - \mu W u, \\ \widehat{Q}_{10} &= \dot{y}\overline{\psi} + i\mu W\overline{\xi}, \\ \widehat{Q}_{01} &= \dot{y}\overline{\xi} - i\mu W\overline{\psi}, \\ \widehat{Z} &= \dot{y}(u - (\ln W)_y\overline{\psi\xi} - \mu W). \end{aligned} \tag{34}$$

When eliminating the auxiliary field u we get the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\dot{y}^2 + i\overline{\psi\dot{\psi}} - i\overline{\xi\dot{\xi}}) + \mu W_y\overline{\psi\xi} + \frac{1}{2}\mu^2 W^2, \tag{35}$$

the equations of motion

$$\begin{aligned} \ddot{y} &= \mu W_{yy}\overline{\psi\xi} + \mu^2 W_y W, \\ i\dot{\overline{\psi}} &= \mu W_y \overline{\xi}, \\ i\dot{\overline{\xi}} &= -\mu W_y \overline{\psi} \end{aligned} \tag{36}$$

and the Noether charges

$$\begin{aligned} \widehat{H} &= \frac{1}{2}\dot{y}^2 - \mu W_y \overline{\psi\xi} - \frac{1}{2}\mu^2 W^2, \\ \widehat{Q}_{10} &= \dot{y}\overline{\psi} + i\mu W\overline{\xi}, \\ \widehat{Q}_{01} &= \dot{y}\overline{\xi} - i\mu W\overline{\psi}, \\ \widehat{Z} &= 0. \end{aligned} \tag{37}$$

For the $(1,2,1)_{[11]}$ model the constant kinetic basis is given by

$$\tilde{x} = \sqrt{\Phi(z)}x, \quad \tilde{z} = C(z), \quad \tilde{\psi} = \sqrt{\Phi(z)}\psi, \quad \tilde{\xi} = \sqrt{\Phi(z)}\xi, \tag{38}$$

where $C(z)$ is a function with $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading [11]. It satisfies the relation

$$C_z(z) = \sqrt{\Phi(z)}, \qquad \text{so that} \qquad \dot{\tilde{z}}^2 = \dot{z}^2\Phi(z), \qquad \Phi_z(z) = \sqrt{\Phi(\tilde{z})}\,\Phi_{\tilde{z}}(\tilde{z}). \tag{39}$$

The consistency requires that $\sqrt{\Phi(z)}$ is an even, [00]-graded function.

After defining $\overline{W}(z)$ to be

$$\overline{W}(z) := \frac{\overline{\mu}}{\sqrt{\Phi(z)}}, \tag{40}$$

by repeating the same steps as before the Lagrangian (11) is expressed in the new basis as

$$\overline{\mathcal{L}} = \frac{1}{2}(\dot{\tilde{z}}^2 - \tilde{x}^2 + i\tilde{\psi}\dot{\tilde{\psi}} - i\tilde{\xi}\dot{\tilde{\xi}}) - (\ln \overline{W}(\tilde{z}))_{\tilde{z}}\tilde{x}\tilde{\psi}\tilde{\xi} + \overline{\mu}\overline{W}(\tilde{z})\tilde{x}. \tag{41}$$

After eliminating the auxiliary field $\tilde{x}$, the Lagrangian becomes

$$\overline{\mathcal{L}} = \frac{1}{2}(\dot{\tilde{z}}^2 + i\tilde{\psi}\dot{\tilde{\psi}} - i\tilde{\xi}\dot{\tilde{\xi}}) - \overline{W}_{\tilde{z}}(\tilde{z})\tilde{\psi}\tilde{\xi} + \frac{1}{2}\overline{W}(\tilde{z})^2, \tag{42}$$

while its associated Noether charges are

$$\begin{aligned} \breve{H} &= \frac{1}{2}\dot{\tilde{z}}^2 + \overline{W}_{\tilde{z}}(\tilde{z})\tilde{\psi}\tilde{\xi} - \frac{1}{2}\overline{W}(\tilde{z})^2, \\ \breve{Q}_{10} &= \dot{\tilde{z}}\tilde{\xi} + i\overline{W}(\tilde{z})\tilde{\psi}, \\ \breve{Q}_{01} &= \dot{\tilde{z}}\tilde{\psi} - i\overline{W}(\tilde{z})\tilde{\xi}, \\ \breve{Z} &= 0. \end{aligned} \tag{43}$$

3.2 The Hamiltonian mechanics of the $(1,2,1)_{[00]}$ model

We introduce now the Hamiltonian formulation of the $(1,2,1)_{[00]}$ model.

The conjugate momenta computed from the Lagrangian (35) are introduced through

$$p_y = \mathcal{L}\overleftarrow{\partial}_{\dot{y}} = \dot{y}, \qquad p_{\overline{\psi}} = \mathcal{L}\overleftarrow{\partial}_{\dot{\overline{\psi}}} = \frac{i}{2}\overline{\psi}, \qquad p_{\overline{\xi}} = \mathcal{L}\overleftarrow{\partial}_{\dot{\overline{\xi}}} = -\frac{i}{2}\overline{\xi}. \tag{44}$$

The Hamiltonian H is defined as

$$H := p_{q_i}\dot{q}_i - \mathcal{L} = \frac{1}{2}p_y^2 - \mu W_y\overline{\psi\xi} - \frac{1}{2}\mu^2 W^2. \tag{45}$$

One should note that the Hamiltonian is identical to the Noether charge $\widehat{H}$ given in (37).

In terms of the momenta (44) the Noether charges are

$$\widehat{H} = \frac{1}{2}p_y^2 - \mu W_y\overline{\psi\xi} - \frac{1}{2}\mu^2 W^2, \quad \widehat{Q}_{10} = p_y\overline{\psi} + i\mu W\overline{\xi}, \qquad \widehat{Q}_{01} = p_y\overline{\xi} - i\mu W\overline{\psi}. \tag{46}$$

We have two constraints, given by

$$f_1 = p_{\overline{\psi}} - \frac{i}{2}\overline{\psi}, \qquad f_2 = p_{\overline{\xi}} + \frac{i}{2}\overline{\xi}, \tag{47}$$

whose $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading are $\deg(f_1) = [10]$, $\deg(f_2) = [01]$.

The Poisson brackets are conveniently introduced through

$$\{A, B\}_{\rm P} = A\widehat{\Gamma}B - (-1)^{\mathbf{a}\cdot\mathbf{b}}B\widehat{\Gamma}A, \qquad \widehat{\Gamma} = \overleftarrow{\partial}_y\overrightarrow{\partial}_{p_y} + \overleftarrow{\partial}_{\overline{\psi}}\overrightarrow{\partial}_{p_{\overline{\psi}}} + \overleftarrow{\partial}_{\overline{\xi}}\overrightarrow{\partial}_{p_{\overline{\xi}}}, \tag{48}$$

where $\deg(A) = \mathbf{a}$, $\deg(B) = \mathbf{b}$.

One may easily see that the constraints (47) are of second class:

$$\{f_1, f_1\}_{\rm P} = -\{f_2, f_2\}_{\rm P} = -i, \qquad \{f_1, f_2\}_{\rm P} = 0. \tag{49}$$

The computation uses the identities

$$f_1\begin{pmatrix}\overleftarrow{\partial}_y\\ \overleftarrow{\partial}_\psi\\ \overleftarrow{\partial}_\xi\end{pmatrix} = \begin{pmatrix}0\\ -i/2\\ 0\end{pmatrix}, \qquad f_2\begin{pmatrix}\overleftarrow{\partial}_y\\ \overleftarrow{\partial}_\psi\\ \overleftarrow{\partial}_\xi\end{pmatrix} = \begin{pmatrix}0\\ 0\\ i/2\end{pmatrix},$$
$$\begin{pmatrix}\overrightarrow{\partial}_{p_y}\\ \overrightarrow{\partial}_{p_\psi}\\ \overrightarrow{\partial}_{p_\xi}\end{pmatrix}f_1 = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}, \qquad \begin{pmatrix}\overrightarrow{\partial}_{p_y}\\ \overrightarrow{\partial}_{p_\psi}\\ \overrightarrow{\partial}_{p_\xi}\end{pmatrix}f_2 = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}. \tag{50}$$

As easily seen from the definition (48), the nonvanishing Poisson brackets of the canonical variables are

$$\{y, p_y\}_{\rm P} = \{\psi, p_\psi\}_{\rm P} = \{\xi, p_\xi\}_{\rm P} = 1. \tag{51}$$

The Poisson brackets of the Noether charges (46) are computed by using the relations

$$\widehat{Q}_{01}\begin{pmatrix}\overleftarrow{\partial}_y\\ \overleftarrow{\partial}_{\overline{\psi}}\\ \overleftarrow{\partial}_{\overline{\xi}}\end{pmatrix} = \begin{pmatrix}-i\mu W_y\overline{\psi}\\ -i\mu W\\ p_y\end{pmatrix}, \qquad \widehat{Q}_{10}\begin{pmatrix}\overleftarrow{\partial}_y\\ \overleftarrow{\partial}_{\overline{\psi}}\\ \overleftarrow{\partial}_{\overline{\xi}}\end{pmatrix} = \begin{pmatrix}i\mu W_y\overline{\xi}\\ p_y\\ i\mu W\end{pmatrix},$$
$$\widehat{H}\begin{pmatrix}\overleftarrow{\partial}_y\\ \overleftarrow{\partial}_{\overline{\psi}}\\ \overleftarrow{\partial}_{\overline{\xi}}\end{pmatrix} = \begin{pmatrix}-\mu W_{yy}\overline{\psi\xi} - \mu^2 W_y W\\ -\mu W_y\overline{\xi}\\ -\mu W_y\overline{\psi}\end{pmatrix} \tag{52}$$

and

$$\begin{pmatrix}\overrightarrow{\partial}_{p_y}\\ \overrightarrow{\partial}_{p_{\overline{\psi}}}\\ \overrightarrow{\partial}_{p_{\overline{\xi}}}\end{pmatrix}\widehat{Q}_{01} = \begin{pmatrix}\overline{\xi}\\ 0\\ 0\end{pmatrix}, \qquad \begin{pmatrix}\overrightarrow{\partial}_{p_y}\\ \overrightarrow{\partial}_{p_{\overline{\psi}}}\\ \overrightarrow{\partial}_{p_{\overline{\xi}}}\end{pmatrix}\widehat{Q}_{10} = \begin{pmatrix}\overline{\psi}\\ 0\\ 0\end{pmatrix}, \qquad \begin{pmatrix}\overrightarrow{\partial}_{p_y}\\ \overrightarrow{\partial}_{p_{\overline{\psi}}}\\ \overrightarrow{\partial}_{p_{\overline{\xi}}}\end{pmatrix}\widehat{H} = \begin{pmatrix}p_y\\ 0\\ 0\end{pmatrix}. \tag{53}$$

One proves that the Poisson bracket of the pair $\widehat{Q}_{01}, \widehat{Q}_{10}$ vanishes in accordance with the $\widehat{Z} = 0$ relation (25).

All Poisson brackets among the Noether charges (46) are

$$\begin{aligned}
&\{\widehat{Q}_{01}, \widehat{Q}_{01}\}_{\mathrm{P}} = -\{\widehat{Q}_{10}, \widehat{Q}_{10}\}_{\mathrm{P}} = -2i\mu W_y \overline{\psi\xi}, \\
&\{\widehat{Q}_{01}, \widehat{Q}_{10}\}_{\mathrm{P}} = 0, \\
&\{\widehat{Q}_{01}, \widehat{H}\}_{\mathrm{P}} = -i\mu W_y \overline{\psi} p_y + \mu^2 W_y W \overline{\xi}, \\
&\{\widehat{Q}_{10}, \widehat{H}\}_{\mathrm{P}} = i\mu W_y \overline{\xi} p_y + \mu^2 W_y W \overline{\psi}.
\end{aligned} \tag{54}$$

At the level of the Poisson bracket, the Noether charges (46) do not recover the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supertranslation algebra (2).

Due to the presence of the second class constraints (47), the Dirac brackets are defined as

$$\begin{aligned}
\{A, B\}_{\mathrm{D}} &= \{A, B\}_{\mathrm{P}} - \{A, f_i\}_{\mathrm{P}} (\Delta^{-1})_{ij} \{f_j, B\}_{\mathrm{P}} = \\
&= \{A, B\}_{\mathrm{P}} - i\{A, f_1\}_{\mathrm{P}} \{f_1, B\}_{\mathrm{P}} + i\{A, f_2\}_{\mathrm{P}} \{f_2, B\}_{\mathrm{P}},
\end{aligned} \tag{55}$$

where

$$\Delta := \begin{pmatrix} \{f_1, f_1\}_{\mathrm{P}} & \{f_1, f_2\}_{\mathrm{P}} \\ \{f_2, f_1\}_{\mathrm{P}} & \{f_2, f_2\}_{\mathrm{P}} \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad \Delta^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{56}$$

The Dirac brackets of the Noether charges (46) satisfy, with $\widehat{Z} = 0$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supertranslation algebra (2):

$$\begin{aligned}
&\{\widehat{Q}_{01}, \widehat{Q}_{01}\}_{\mathrm{D}} = 2i\widehat{H}, \qquad \{\widehat{Q}_{10}, \widehat{Q}_{10}\}_{\mathrm{D}} = -2i\widehat{H}, \\
&\{\widehat{Q}_{01}, \widehat{Q}_{10}\}_{\mathrm{D}} = \{\widehat{Q}_{01}, \widehat{H}\}_{\mathrm{D}} = \{\widehat{Q}_{10}, \widehat{H}\}_{\mathrm{D}} = 0.
\end{aligned} \tag{57}$$

This set of equation is verified by using the identities

$$\begin{aligned}
&\{\widehat{Q}_{01}, f_1\}_{\mathrm{P}} = -i\mu W, \qquad \{\widehat{Q}_{01}, f_2\}_{\mathrm{P}} = p_y, \qquad \{\widehat{Q}_{10}, f_1\}_{\mathrm{P}} = p_y, \\
&\{\widehat{Q}_{10}, f_2\}_{\mathrm{P}} = i\mu W, \qquad \{f_1, \widehat{H}\}_{\mathrm{P}} = \mu W_y \overline{\xi}, \qquad \{f_2, \widehat{H}\}_{\mathrm{P}} = \mu W_y \overline{\psi}.
\end{aligned} \tag{58}$$

The canonical equations of motion obtained by the Dirac brackets with the Hamiltonian $H = \widehat{H}$ of (46) are identical to the Euler-Lagrange equations (36):

$$\begin{aligned}
&\dot{y} = \{y, \widehat{H}\}_{\mathrm{D}} = p_y, \qquad \dot{p_y} = \{p_y, \widehat{H}\}_{\mathrm{D}} = \mu W_{yy} \overline{\psi\xi} + \mu^2 W_y W, \\
&\dot{\overline{\psi}} = \{\psi, \widehat{H}\}_{\mathrm{D}} = -i\mu W_y \overline{\xi}, \qquad \dot{\overline{\xi}} = \{\overline{\xi}, \widehat{H}\}_{\mathrm{D}} = i\mu W_y \overline{\psi}.
\end{aligned} \tag{59}$$

The nonvanishing Dirac bracket for the canonical variables are

$$\{y, p_y\}_{\mathrm{D}} = 1, \qquad \{\overline{\psi}, \overline{\psi}\}_{\mathrm{D}} = -i, \qquad \{\overline{\xi}, \overline{\xi}\}_{\mathrm{D}} = i. \tag{60}$$

This is seen as follows: let $\deg(A) = \mathbf{a} = [a_1, a_2]$; then one immediately checks that

$$\{A, f_1\}_{\mathrm{P}} = A \overleftarrow{\partial}_{\overline{\psi}} + (-1)^{a_1} \frac{i}{2} \overrightarrow{\partial}_{p_{\overline{\psi}}} A \tag{61}$$

implies that f_1 has nonvanishing Poisson brackets only with $\overline{\psi}$ and $p_{\overline{\psi}}$,

$$\{\overline{\psi}, f_1\}_{\mathrm{P}} = 1, \qquad \{p_{\overline{\psi}}, f_1\}_{\mathrm{P}} = -\frac{i}{2}. \tag{62}$$

Similarly,

$$\{A, f_2\}_{\rm P} = A\overleftarrow{\partial}_{\overline{\xi}} - (-1)^{a_2}\frac{i}{2}\overrightarrow{\partial}_{p_{\overline{\xi}}}A \tag{63}$$

implies that f_2 has nonvanishing Poisson brackets only with $\overline{\xi}$ and $p_{\overline{\xi}}$:

$$\{\overline{\xi}, f_2\}_{\rm P} = 1, \qquad \{p_{\overline{\xi}}, f_2\}_{\rm P} = \frac{i}{2}. \tag{64}$$

Combining these nonvanishing Poisson brackets with the ones given in (51), it is not difficult to see that, in addition to (60), further nonvanishing Dirac brackets for the canonical variables are

$$\{\overline{\psi}, p_{\overline{\psi}}\}_{\rm D} = \{\overline{\xi}, p_{\overline{\xi}}\}_{\rm D} = \frac{1}{2}. \tag{65}$$

They are reduced to the ones in (60) by using the expression of the momenta (44).

3.3 The Hamiltonian mechanics of the $(1,2,1)_{[11]}$ model

We collect here the relevant formulas of the Hamiltonian formulation of the $(1,2,1)_{[11]}$ model. They are derived from the Lagrangian mechanics (42) by using the same procedure discussed in the previous subsection.

The conjugate momenta are

$$p_{\tilde{z}} = \overline{\mathcal{L}}\overleftarrow{\partial}_{\dot{\tilde{z}}} = \dot{\tilde{z}}, \quad p_{\tilde{\psi}} = \overline{\mathcal{L}}\overleftarrow{\partial}_{\dot{\tilde{\psi}}} = \frac{i}{2}\tilde{\psi}, \quad p_{\tilde{\xi}} = \overline{\mathcal{L}}\overleftarrow{\partial}_{\dot{\tilde{\xi}}} = -\frac{i}{2}\tilde{\xi}. \tag{66}$$

The Hamiltonian $H = \breve{H}$ is

$$H = \breve{H} = \frac{1}{2}p_{\tilde{z}}^2 + \overline{W}_{\tilde{z}}(\tilde{z})\tilde{\psi}\tilde{\xi} - \frac{1}{2}\overline{W}(\tilde{z})^2. \tag{67}$$

It coincides with the Nother charge (43). The other nonvanishing Noether charges are

$$\breve{Q}_{10} = p_{\tilde{z}}\tilde{\xi} + i\overline{W}(\tilde{z})\tilde{\psi}, \qquad \breve{Q}_{01} = p_{\tilde{z}}\tilde{\psi} - i\overline{W}(\tilde{z})\tilde{\xi}. \tag{68}$$

We present the Dirac brackets for the canonical variables. They are given by

$$\{\tilde{z}, p_{\tilde{z}}\}_{\rm D} = 1, \qquad \{\tilde{\psi}, \tilde{\psi}\}_{\rm D} = -i, \qquad \{\tilde{\xi}, \tilde{\xi}\}_{\rm D} = i. \tag{69}$$

The nonvanishing Dirac brackets of the Noether charges are

$$\{\breve{Q}_{10}, \breve{Q}_{10}\}_{\rm D} = -2i\breve{H}, \qquad \{\breve{Q}_{01}, \breve{Q}_{01}\}_{\rm D} = 2i\breve{H}. \tag{70}$$

The canonical equations of motion defined by the Dirac brackets are

$$\begin{aligned}
\dot{\tilde{z}} &= \{\tilde{z}, \breve{H}\}_{\rm D} = p_{\tilde{z}}, \\
\dot{p}_{\tilde{z}} &= \{p_{\tilde{z}}, \breve{H}\}_{\rm D} = -\overline{W}_{\tilde{z}\tilde{z}}\tilde{\psi}\tilde{\xi} + \overline{W}_{\tilde{z}}\overline{W}, \\
\dot{\tilde{\psi}} &= \{\tilde{\psi}, \breve{H}\}_{\rm D} = i\overline{W}_{\tilde{z}}\tilde{\xi}, \\
\dot{\tilde{\xi}} &= \{\tilde{\xi}, \breve{H}\}_{\rm D} = -i\overline{W}_{\tilde{z}}\tilde{\psi}.
\end{aligned} \tag{71}$$

They are identical to the Euler-Lagrange equations obtained from (42).

4 The canonical quantization

We present the canonical quantization of the $(1,2,1)_{[00]}$ and $(1,2,1)_{[11]}$ classical models.

As is customary, the quantization is obtained by replacing the Dirac brackets in (60) and (69) with (anti)commutators. It is obtained through the mapping

$$\{.,.\}_{\rm D} \quad \mapsto \quad -i[.,.\}, \tag{72}$$

where the "$[.,.\}$" symbol introduced in (A.2) denotes the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie brackets.

4.1 The quantization of the $(1,2,1)_{[00]}$ model

For the $(1,2,1)_{[00]}$ model the canonical Dirac brackets (60) are replaced by the (anti)commutators

$$[y, p_y] = i, \qquad \{\overline{\psi}, \overline{\psi}\} = 1, \qquad \{\overline{\xi}, \overline{\xi}\} = -1, \qquad [\overline{\psi}, \overline{\xi}] = 0. \tag{73}$$

In the Heisenberg picture the (anti)commutators (73) are computed at equal time, let's say at $t = 0$. A consistency condition further requires the $deg(\mu) = [11]$ coupling constant μ entering (45) to anticommute with $\overline{\psi}$, $\overline{\xi}$, so that

$$\{\mu, \overline{\psi}\} = \{\mu, \overline{\xi}\} = 0. \tag{74}$$

It follows from (73) and (74) that the quantization of the Noether charges (46) satisfy the relations:

$$\begin{aligned} &\{\widehat{Q}_{10}, \widehat{Q}_{10}\} = 2\widehat{H}, \qquad \{\widehat{Q}_{01}, \widehat{Q}_{01}\} = -2\widehat{H}, \\ &[\widehat{Q}_{01}, \widehat{Q}_{10}] = [\widehat{Q}_{01}, \widehat{H}] = [\widehat{Q}_{10}, \widehat{H}] = 0. \end{aligned} \tag{75}$$

Before going ahead we introduce the representations of the relations (73) and (74).

The quantum operators $\overline{\psi}, \overline{\xi}, \mu$ can be represented by 4×4 matrices satisfying (A.9), so that

$$\overline{\psi} \in \mathcal{G}_{10}, \quad \overline{\xi} \in \mathcal{G}_{01}, \quad \mu \in \mathcal{G}_{11} \tag{76}$$

and

$$\{\overline{\psi}, \overline{\psi}\} = -\{\overline{\xi}, \overline{\xi}\} = \mathbb{I}_4, \quad [\overline{\psi}, \overline{\xi}] = 0, \quad \{\overline{\psi}, \mu\} = \{\overline{\xi}, \mu\} = 0. \tag{77}$$

We are looking for real matrix solutions to the above system of equations.

This means that in particular $\overline{\psi}$ should be given by a linear combination

$$\overline{\psi} = \lambda_1 \cdot Y \otimes I + \lambda_2 \cdot Y \otimes X$$

of the split-quaternion matrices I, X, Y, A introduced in (A.10). Similar relations hold for $\overline{\xi}$ and μ. The requirement that $\overline{\psi}^2$ is proportional to the identity implies that $\lambda_{1,2}$ cannot be both different from 0. Therefore, up to an overall sign, there are only two possible solutions for $\overline{\psi}$ (a similar argument applies to $\overline{\xi}$ as well). The system of equations (77) is solved (up to an overall sign for each one of the three matrices) by the two sets of triples:

$$\overline{\psi}_A = \tfrac{1}{\sqrt{2}} Y \otimes I, \quad \overline{\xi}_A = \tfrac{1}{\sqrt{2}} Y \otimes A, \quad \overline{\mu}_A = X \otimes A \tag{78}$$

and

$$\overline{\psi}_B = \tfrac{1}{\sqrt{2}} Y \otimes X, \quad \overline{\xi}_B = \tfrac{1}{\sqrt{2}} A \otimes Y, \quad \overline{\mu}_B = -I \otimes A. \tag{79}$$

It is convenient to introduce the triples of Hermitian matrices

$$\psi_A = \overline{\psi}_A, \quad \xi_A = i\overline{\xi}_A, \quad \mu_A = i\overline{\mu}_A \qquad \text{and} \quad \psi_B = \overline{\psi}_B, \quad \xi_B = i\overline{\xi}_B, \quad \mu_B = i\overline{\mu}_B, \tag{80}$$

so that

$$\psi_{A,B} = \psi^\dagger_{A,B}, \quad \xi_{A,B} = \xi^\dagger_{A,B}, \quad \mu_{A,B} = \mu^\dagger_{A,B}. \tag{81}$$

In terms of these representations and up to an overall normalization factor, the quantized Noether charges $\widehat{Q}_{10}, \widehat{Q}_{01}$ are associated to the hermitian conserved charges Q^A_{10}, Q^A_{01} and Q^B_{10}, Q^B_{01}, given by

$$\begin{aligned} Q^A_{10} &= -i(\psi_A \cdot \partial_y + W(y)\mu_A\xi_A), \qquad Q^B_{10} = -i(\psi_B \cdot \partial_y + W(y)\mu_B\xi_B), \\ Q^A_{01} &= -i(\xi_A \cdot \partial_y + W(y)\mu_A\psi_A), \qquad Q^B_{10} = -i(\xi_B \cdot \partial_y + W(y)\mu_B\psi_B). \end{aligned} \tag{82}$$

By construction the hermiticity conditions hold:

$$(Q^A_{10})^\dagger = Q^A_{10}, \quad (Q^B_{10})^\dagger = Q^B_{10}, \quad (Q^A_{01})^\dagger = Q^A_{01}, \quad (Q^B_{01})^\dagger = Q^B_{01}. \tag{83}$$

These four supercharges are

$$\begin{aligned} Q^A_{10} &= -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \partial_y + W(y) & 0 \\ 0 & 0 & 0 & \partial_y + W(y) \\ \partial_y - W(y) & 0 & 0 & 0 \\ 0 & \partial_y - W(y) & 0 & 0 \end{pmatrix}, \\ Q^B_{10} &= -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \partial_y + W(y) & 0 \\ 0 & 0 & 0 & -\partial_y - W(y) \\ \partial_y - W(y) & 0 & 0 & 0 \\ 0 & -\partial_y + W(y) & 0 & 0 \end{pmatrix}, \\ Q^A_{01} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & \partial_y + W(y) \\ 0 & 0 & -\partial_y - W(y) & 0 \\ 0 & \partial_y - W(y) & 0 & 0 \\ -\partial_y + W(y) & 0 & 0 & 0 \end{pmatrix}, \\ Q^B_{01} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & \partial_y + W(y) \\ 0 & 0 & \partial_y + W(y) & 0 \\ 0 & -\partial_y + W(y) & 0 & 0 \\ -\partial_y + W(y) & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{84}$$

The supercharges are the square roots of the Hamiltonian H, since

$$\{Q^A_{10}, Q^A_{10}\} = \{Q^B_{10}, Q^B_{10}\} = \{Q^A_{01}, Q^A_{01}\} = \{Q^B_{01}, Q^B_{01}\} = 2H, \tag{85}$$

where

$$H = H^\dagger \tag{86}$$

and

$$H = \frac{1}{2} \begin{pmatrix} -\partial_y^2 + W^2 + W' & 0 & 0 & 0 \\ 0 & -\partial_y^2 + W^2 + W' & 0 & 0 \\ 0 & 0 & -\partial_y^2 + W^2 - W' & 0 \\ 0 & 0 & 0 & -\partial_y^2 + W^2 - W' \end{pmatrix}. \tag{87}$$

In the above formula $W \equiv W(y)$ and $W' \equiv \frac{d}{dy}W(y)$.

One should note that, unlike the supercharges (82), the Hamiltonian H does not depend on which choice of three matrices, either $\psi_A,\ \xi_A,\ \mu_A$ or $\psi_B,\ \xi_B,\ \mu_B$, is made.

The Hamiltonian (87) reproduces, up to a normalization convention and the reordering of the diagonal elements by a similarity transformation, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded quantum Hamiltonian introduced in formula (3.1.1) of [23]. As promised, we obtained this quantum model from the canonical quantization of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded classical model based on the $(1,2,1)_{[00]}$ multiplet.

We point out that the four matrices $\psi_A, \psi_B, \xi_A, \xi_B$ from (80) do not commute with the Hamiltonian. This is in agreement with the fact that they correspond to the quantization of the classical dynamical variables $\overline{\psi}$, $\overline{\xi}$ and that their Heisenberg evolution is expected. The matrices μ_A, μ_B from (80) are, in their respective representations, the quantum counterparts of the classical coupling constant μ. It is rewarding that they commute with the Hamiltonian:

$$[H, \mu_A] = [H, \mu_B] = 0. \tag{88}$$

This is an extra consistency check of the correctness of the proposed quantization prescription. All quantized coupling constants introduced in the following commute with the respective Hamiltonians.

4.2 The quantization of the $(1,2,1)_{[11]}$ model

For the $(1,2,1)_{[11]}$ model the canonical Dirac brackets (69) are replaced by the (anti)commutators

$$[\tilde{z}, p_{\tilde{z}}] = i, \qquad \{\tilde{\psi}, \tilde{\psi}\} = 1, \qquad \{\tilde{\xi}, \tilde{\xi}\} = -1, \qquad [\tilde{\psi}, \tilde{\xi}] = 0. \tag{89}$$

The coupling constant $\tilde{\mu}$ has now $\deg(\tilde{\mu}) = [00]$, so that

$$[\tilde{\psi}, \tilde{\mu}] = [\tilde{\xi}, \tilde{\mu}] = 0. \tag{90}$$

On the other hand, since $\deg \tilde{z} = [11]$, we have the vanishing anticommutators

$$\{\tilde{z}, \tilde{\psi}\} = \{\tilde{z}, \tilde{\xi}\} = \{p_{\tilde{z}}, \tilde{\psi}\} = \{p_{\tilde{z}}, \tilde{\xi}\} = 0. \tag{91}$$

One can therefore set

$$\tilde{z} \ = \ y\tilde{\rho}, \tag{92}$$

where y is a standard coordinate and $\tilde{\rho}$ is a matrix anticommuting with $\tilde{\psi}, \tilde{\xi}$.

In analogy with the $(1,2,1)_{[00]}$ model, two sets of triples of hermitian 4×4 matrices can be defined. They can be expressed, in terms of the matrices defined in (A.10), as

$$\tilde{\psi}_A = \tfrac{1}{\sqrt{2}} Y \otimes I, \quad \tilde{\xi}_A = \tfrac{i}{\sqrt{2}} Y \otimes A, \quad \tilde{\rho}_A = iX \otimes A \tag{93}$$

and

$$\tilde{\psi}_B = \tfrac{1}{\sqrt{2}} Y \otimes X, \quad \tilde{\xi}_B = \tfrac{i}{\sqrt{2}} A \otimes Y, \quad \tilde{\rho}_B = -iI \otimes A. \tag{94}$$

The hermitian supercharges $\widetilde{Q}_{10}^{A,B}$, $\widetilde{Q}_{01}^{A,B}$ are obtained as the quantized version of the Noether charges $\breve{Q}_{10}$, $\breve{Q}_{01}$ given in (68) by using the representations (93,94), respectively. We have

$$\begin{aligned}
\widetilde{Q}_{10}^{A} &= \tilde{\rho}_A \tilde{\xi}_A \cdot \partial_y + \tilde{\psi}_A \cdot \overline{W}(y), \\
\widetilde{Q}_{10}^{B} &= \tilde{\rho}_B \tilde{\xi}_B \cdot \partial_y + \tilde{\psi}_B \cdot \overline{W}(y), \\
\widetilde{Q}_{01}^{A} &= \tilde{\rho}_A \tilde{\psi}_A \cdot \partial_y + \tilde{\xi}_A \cdot \overline{W}(y), \\
\widetilde{Q}_{01}^{B} &= \tilde{\rho}_B \tilde{\psi}_B \cdot \partial_y + \tilde{\xi}_B \cdot \overline{W}(y).
\end{aligned} \tag{95}$$

The supercharges satisfy

$$(\widetilde{Q}^A_{10})^\dagger = \widetilde{Q}^A_{10}, \quad (\widetilde{Q}^B_{10})^\dagger = \widetilde{Q}^B_{10}, \quad (\widetilde{Q}^A_{01})^\dagger = \widetilde{Q}^A_{01}, \quad (\widetilde{Q}^B_{01})^\dagger = \widetilde{Q}^B_{01}. \tag{96}$$

The four supercharges are square roots of the hermitian quantum Hamiltonian $\widetilde{H}$, recovered from the anticommutators

$$\{\widetilde{Q}^A_{10}, \widetilde{Q}^A_{10}\} = \{\widetilde{Q}^B_{10}, \widetilde{Q}^B_{10}\} = \{\widetilde{Q}^A_{01}, \widetilde{Q}^A_{01}\} = \{\widetilde{Q}^B_{01}, \widetilde{Q}^B_{01}\} = 2\widetilde{H}, \qquad \widetilde{H}^\dagger = \widetilde{H}. \tag{97}$$

The quantum Hamiltonian $\widetilde{H}$ coincides with the quantum Hamiltonian H given in (87) when setting $\overline{W}(y) = W(y)$. This implies that the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded models $(1,2,1)_{[00]}$ and $(1,2,1)_{[11]}$, despite being classically different, produce after canonical quantization the same quantum theory.

5 Symmetries of the single-multiplet quantum Hamiltonian

We present here the different graded symmetries of the quantum Hamiltonian H given in (87). Since it coincides with the Hamiltonian derived from the quantization of the $(1,2,1)_{[11]}$ model, it is sufficient to discuss the symmetry operators obtained from the quantum $(1,2,1)_{[00]}$ theory. We recall that, while the Hamiltonian does not depend on the chosen triple of hermitian matrices (80), the Noether supercharges introduced in (82), on the other hand, depend on the given choice. By construction, each one of the four operators $Q^A_{10}, Q^B_{10}, Q^A_{01}, Q^B_{01}$ in formulas (82) and (84) is a conserved symmetry operator. Therefore it makes sense to introduce, for each such pair of operators in the $\mathcal{G}_{10}$ and $\mathcal{G}_{01}$ sectors, the induced $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra.

The results are the following:

i) two copies of the Beckers-Debergh algebra.

By taking a pair of operators defined by the same triple of hermitian matrices, either "A" or "B", we obtain two separate realizations of the Beckers-Debergh algebra, namely the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra with vanishing $\mathcal{G}_{11}$ sector, see (A.8).

The two copies of Beckers-Debergh algebras are respectively given by the two sets of three operators, $\mathfrak{S}_A$ and $\mathfrak{S}_B$,

$$\mathfrak{S}_A = \{Q^A_{10}, Q^A_{01}, H\} \quad \text{and} \quad \mathfrak{S}_B = \{Q^B_{10}, Q^B_{01}, H\}, \tag{98}$$

which contain the common Hamiltonian H, with $H \in \mathcal{G}_{00}$.

In both cases the (A.8) (anti)commutators are satisfied since, besides the anticommutators in (85), namely $\{Q^A_{10}, Q^A_{10}\} = \{Q^B_{10}, Q^B_{10}\} = 2H$, all commutators are vanishing:

$$[H, Q^A_{10}] = [H, Q^A_{01}] = [Q^A_{10}, Q^A_{01}] = 0 \quad \text{and} \quad [H, Q^B_{10}] = [H, Q^B_{01}] = [Q^B_{10}, Q^B_{01}] = 0; \tag{99}$$

ii) two copies of the one-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supertranslation algebra.

By taking a"mixed" pair of operators constructed from the two different triples ("A" and "B") of the hermitian matrices (80), we obtain two, conveniently normalized, separate realizations of the one-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supertranslation algebra (A.7). In both cases the $\mathcal{G}_{11}$ sector is nonvanishing.

The two copies of the supertranslation algebra are respectively spanned by the two sets of four operators $\mathfrak{S}_1$ and $\mathfrak{S}_2$, given by

$$\mathfrak{S}_1 = \{Q^B_{10}, Q^A_{01}, H, Z\} \quad \text{and} \quad \mathfrak{S}_2 = \{Q^A_{10}, Q^B_{01}, H, \overline{Z}\}. \tag{100}$$

Their respective $\mathcal{G}_{11}$ sectors are spanned by the hermitian operators Z and $\overline{Z}$,

$$Z = Z^\dagger, \qquad \overline{Z} = \overline{Z}^\dagger. \tag{101}$$

We have

$$Z = \begin{pmatrix} 0 & -\partial_y^2 + W^2 + W' & 0 & 0 \\ -\partial_y^2 + W^2 + W' & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_y^2 + W^2 - W' \\ 0 & 0 & -\partial_y^2 + W^2 - W' & 0 \end{pmatrix} \tag{102}$$

and

$$\overline{Z} = \begin{pmatrix} 0 & \partial_y^2 - W^2 - W' & 0 & 0 \\ \partial_y^2 - W^2 - W' & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_y^2 + W^2 - W' \\ 0 & 0 & -\partial_y^2 + W^2 - W' & 0 \end{pmatrix}. \tag{103}$$

The nonvanishing (anti)commutators of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra $\mathfrak{S}_1$ are

$$\{Q^B_{10}, Q^B_{10}\} = \{Q^A_{01}, Q^A_{01}\} = 2H, \qquad [Q^B_{10}, Q^A_{01}] = iZ. \tag{104}$$

The nonvanishing (anti)commutators of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra $\mathfrak{S}_2$ are

$$\{Q^A_{10}, Q^A_{10}\} = \{Q^B_{01}, Q^B_{01}\} = 2H, \qquad [Q^A_{10}, Q^B_{01}] = i\overline{Z}; \tag{105}$$

iii) the $\mathbb{Z}_2$-graded superalgebra of the $\mathcal{N} = 4$ supersymmetric quantum mechanics.

Besides the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry algebras, the Hamiltonian H possesses a $\mathbb{Z}_2$-graded symmetry, making it an example of $\mathcal{N} = 4$ supersymmetric quantum mechanics [39] satisfying

$$\{Q_i, Q_j\} = 2\delta_{ij} H, \qquad [H, Q_i] = 0, \qquad \text{for} \qquad i, j = 1, 2, 3, 4. \tag{106}$$

The two sets of mixed Noether supercharges operators, Q^B_{10}, Q^A_{01} and Q^A_{10}, Q^B_{01}, define two copies of superalgebras of the $\mathcal{N} = 2$ supersymmetric quantum mechanics, since their respective anti-commutators are both vanishing

$$\{Q^B_{10}, Q^A_{01}\} = 0 \qquad \text{and} \qquad \{Q^A_{10}, Q^B_{01}\} = 0. \tag{107}$$

In order to get the $\mathcal{N} = 4$ superalgebra (106), two extra supersymmetry operators, which do not coincide with the Noether supercharges (84), have to be added. A convenient presentation

of the 4 operators Q_i satisfying (106) is given by

$$\begin{aligned} Q_1 &= Q_{10}^B = -\frac{i}{\sqrt{2}}(Y\otimes X\cdot\partial_y + A\otimes X\cdot W(y)), \\ Q_2 &= Q_{01}^A = \frac{1}{\sqrt{2}}(Y\otimes A\cdot\partial_y + A\otimes A\cdot W(y)), \\ Q_3 &= -\frac{i}{\sqrt{2}}(Y\otimes Y\cdot\partial_y + A\otimes Y\cdot W(y)), \\ Q_4 &= \frac{1}{\sqrt{2}}(A\otimes I\cdot\partial_y + Y\otimes I\cdot W(y)). \end{aligned} \tag{108}$$

The matrices A, X, Y, I have been introduced in (A.10).

6 $\mathbb{Z}_2\times\mathbb{Z}_2$-graded interacting multiparticle Hamiltonians

We now further apply our scheme to derive a new class of $\mathbb{Z}_2\times\mathbb{Z}_2$-graded quantum Hamiltonians. Specifically, we present the quantum systems obtained by quantizing the classical actions of several, interacting, $(1,2,1)_{[00]}$ multiplets. The construction of these classical models was presented in [33] (see subsection **5.3** of that paper). In particular, the invariant action $\mathcal{S}_{2P}=\int dt\mathcal{L}_{2P}$ of the 2-particle case is expressed, in real time t, in terms of the Lagrangian

$$\begin{aligned} \mathcal{L}_{2P} &= \frac{1}{2}[g_{11}(\dot{x}_1^2 - z_1^2 + i\psi_1\dot{\psi}_1 - i\xi_1\dot{\xi}_1) + g_{22}(\dot{x}_2^2 - z_2^2 + i\psi_2\dot{\psi}_2 - i\xi_2\dot{\xi}_2) + \\ &\quad g_{12}(2\dot{x}_1\dot{x}_2 - 2z_1z_2 + i\psi_1\dot{\psi}_2 + i\psi_2\dot{\psi}_1 - i\xi_1\dot{\xi}_2 - i\xi_2\dot{\xi}_1) - g_{111}z_1\psi_1\xi_1 - g_{222}z_2\psi_2\xi_2 + \\ &\quad g_{112}(-z_2\psi_1\xi_1 - z_1(\psi_1\xi_2+\psi_2\xi_1)) + g_{221}(-z_1\psi_2\xi_2 - z_2(\psi_1\xi_2+\psi_2\xi_1)) + \\ &\quad \lambda_1\mu z_1 + \lambda_2\mu z_2]. \end{aligned} \tag{109}$$

The component fields of the two multiplets (respectively denoted as x_1,ψ_1,ξ_1,z_1 and x_2,ψ_2,ξ_2,z_2) transform independently under the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded transformations (4). The fields $x_1(t), x_2(t)$ describe the propagating bosons, while $z_1(t), z_2(t)$ describe the auxiliary bosons. The Lagrangian depends on the prepotential function $g(x_1,x_2)$. The functions $g_{ij}(x_1,x_2)$ are interpreted as the metric of the two-dimensional target manifold. The metric is constrained to satisfy the equation

$$g_{ij}(x_1,x_2) = \partial_{x_i}\partial_{x_j}g(x_1,x_2), \qquad \text{for} \quad i,j=1,2, \tag{110}$$

in terms of the prepotential function $g(x_1,x_2)$.

The condition

$$g_{12}(x_1,x_2) \neq 0 \tag{111}$$

is necessary in order to have interacting multiplets.

Finally, the linear terms in z_1, z_2 in the last line of the right hand side of (109) depend on the [11]-graded coupling constant μ, while $\lambda_1,\lambda_2\in\mathbb{R}$ are arbitrary real parameters.

The construction of invariant classical actions for $n>2$ interacting $(1,2,1)_{[00]}$ multiplets is a straightforward extension of the $n=2$ procedure.

6.1 Constant kinetic basis and classical Hamiltonian formulation

The quantization of the (109) action requires repeating the steps discussed in Section **3** and **4** concerning the quantization of the single multiplet Lagrangian (6). Here, we limit ourselves to discuss the main relevant differences with respect to this case.

The passage to the "constant kinetic basis" is obtained by introducing two smooth and differentiable functions $u(x_1, x_2)$ and $v(x_1, x_2)$; at least locally the transformations

$$x_1 \mapsto u(x_1, x_2), \qquad x_2 \mapsto v(x_1, x_2), \tag{112}$$

are assumed to be invertible.

The time derivatives are

$$\dot{u} = u_1\dot{x}_1 + u_2\dot{x}_2, \qquad \dot{v} = v_1\dot{x}_1 + v_2\dot{x}_2. \tag{113}$$

They are chosen so that the constant kinetic term

$$K \;=\; \frac{1}{2}(\dot{u}^2 + \dot{v}^2) \tag{114}$$

reproduces the kinetic term for the (109) propagating bosons, given by

$$K \;=\; \frac{1}{2}(g_{11}\dot{x}_1^2 + g_{22}\dot{x}_2^2 + 2g_{12}\dot{x}_1\dot{x}_2). \tag{115}$$

This is obtained with the identifications

$$\begin{aligned} g_{11} &= u_1^2 + v_1^2, \\ g_{22} &= u_2^2 + v_2^2, \\ g_{12} &= u_1u_2 + v_1v_2. \end{aligned} \tag{116}$$

For the above g_{ij} metric the Hessian $G = det(g_{ij})$ is

$$G \;=\; g_{11}g_{22} - g_{12}^2 = (u_1v_2 - u_2v_1)^2. \tag{117}$$

Being derived in terms of the prepotential $g(x_1, x_2)$ and satisfying the (110) equation, the metric g_{ij} satisfies the constraints

$$\begin{aligned} g_{11,2} &= g_{12,1}, \\ g_{22,1} &= g_{12,2}. \end{aligned} \tag{118}$$

Under the identifications (116) these constraints imply two nonlinear equations for the partial derivatives of u and v. They are, respectively,

$$\begin{aligned} C_1 &\equiv u_1u_{12} - u_2u_{11} + v_1v_{12} - v_2v_{11} = 0, \\ C_2 &\equiv u_1u_{22} - u_2u_{12} + v_1v_{22} - v_2v_{12} = 0. \end{aligned} \tag{119}$$

It is worth mentioning that the (119) constraints for u, v admit nontrivial solutions.

As an example, the cubic polynomials

$$u(x_1, x_2) = x_1(1 + \alpha x_1^2 + 3\alpha x_2^2), \qquad v(x_1, x_2) = x_2(1 + 3\alpha x_1^2 + \alpha x_2^2), \tag{120}$$

satisfy (119) and induce, for any real $\alpha \in \mathbb{R}$, the metric $g_{ij} = \partial_i \partial_j g(x_1, x_2)$ obtained from the prepotential

$$\begin{aligned} g(x_1, x_2) &= \frac{1}{2}(x_1^2 + x_2^2) + \frac{\alpha}{2}(x_1^4 + x_2^4) + \frac{3}{10}\alpha^2(x_1^6 + x_2^6) + \\ & 3\alpha x_1^2 x_2^2 + \frac{9}{2}\alpha^2 x_1^2 x_2^2 (x_1^2 + x_2^2). \end{aligned} \tag{121}$$

The two-dimensional constant Euclidean metric is recovered at $\alpha = 0$.

For the [10]-graded odd fields ψ_1, ψ_2 the change of variables

$$\begin{pmatrix} \psi_u \\ \psi_v \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{122}$$

guarantees that the fermionic constant kinetic term K_f,

$$\begin{aligned} K_f &= \frac{i}{2}(\psi_u \dot{\psi}_u + \psi_v \dot{\psi}_v) = \\ &= \frac{i}{2}(g_{11}\psi_1\dot{\psi}_1 + g_{22}\psi_2\dot{\psi}_2 + g_{12}(\psi_1\dot{\psi}_2 + \psi_2\dot{\psi}_2) + (C_1\dot{x}_1 + C_2\dot{x}_2)\psi_1\psi_2), \end{aligned} \tag{123}$$

reproduces the fermionic kinetic term for ψ_1, ψ_2 in (109) once provided that the nonlinear constraints $C_1 = C_2 = 0$ from formulas (119) are satisfied.

An analogous change of variables is made by replacing in equation (122) the fields ψ_1, ψ_2 and ψ_u, ψ_v with the [01]-graded fields ξ_1, ξ_2 and ξ_u, ξ_v, respectively.

From now on everything proceeds as in the single multiplet case. After solving the algebraic equations of motion for the auxiliary fields z_1, z_2, we then introduce canonical variables, Poisson brackets, Dirac brackets and the Hamiltonian formalism in terms of the new component fields of the constant kinetic basis.

In particular in this basis the analogs of formulas (46) now read, for the two-particle Noether charges $\widehat{Q}_{2P;10}$, $\widehat{Q}_{2P;01}$, as

$$\begin{aligned} \widehat{Q}_{2P;10} &= p_u\psi_u + p_v\psi_v + i\mu(W_u\xi_u + W_v\xi_v), \\ \widehat{Q}_{2P;01} &= p_u\xi_u + p_v\xi_v - i\mu(W_u\psi_u + W_v\psi_v), \end{aligned} \tag{124}$$

where p_u, p_v are the conjugate momenta $p_u \equiv \dot{u}$, $p_v \equiv \dot{v}$.

Instead of a single field W as in (34), we have now two fields, W_u, W_v. They are derived from the λ_1, λ_2 terms in the Lagrangian (109) and satisfy

$$W_u\psi_u + W_v\psi_v = \lambda_1\psi_1 + \lambda_2\psi_2, \qquad W_u\xi_u + W_v\psi_v\xi_v = \lambda_1\xi_1 + \lambda_2\xi_2, \tag{125}$$

so that the analog of formula (31) is provided by

$$W_u = \frac{1}{\widetilde{G}}(\lambda_1 v_2 - \lambda_2 v_1), \qquad W_v = \frac{1}{\widetilde{G}}(-\lambda_1 u_2 + \lambda_2 u_1). \tag{126}$$

One should note that the suffices u, v denote the two different fields W_u, W_v and are not a symbol of derivation.

In the above expressions

$$\widetilde{G} = u_1 v_2 - u_2 v_1 \tag{127}$$

is the determinant of the transformation matrix entering the right hand side of (122). Due to (116), we have that the Hessian G given in (117) is

$$G \;=\; \widetilde{G}^2. \tag{128}$$

In the $g_{12} = 0$ non-interacting case the functions u, v can be chosen as $u = u(x_1)$, $v = v(x_2)$. Therefore $u_2 = v_1 = 0$ and the expressions for W_u, W_v are simpler. Under this assumption we get:

$$u_2 = v_1 = 0 \quad \text{imply} \quad W_u = \tfrac{\lambda_1}{u_1}, \quad W_v = \tfrac{\lambda_2}{v_2} \quad \text{and} \quad \partial_u W_v = \partial_v W_u = 0. \tag{129}$$

The weaker condition $\partial_u W_v = \partial_v W_u$ for the interacting case is a result, as discussed below in subsection **6.3**, of the quantization procedure requiring a matrix representation of the variables $\psi_u, \psi_v, \xi_u, \xi_v$.

The non-vanishing Dirac brackets of the conjugate variables are

$$\{u, p_u\}_D = \{v, p_v\}_D = 1, \quad \{\psi_u, \psi_u\}_D = \{\psi_v, \psi_v\}_D = -i, \quad \{\xi_u, \xi_u\}_D = \{\xi_v, \xi_v\}_D = i. \tag{130}$$

The classical Hamiltonian $\widehat{H}_{2P}$ can be read, in analogy with formula (57), through the Dirac brackets

$$\{\widehat{Q}_{2P;01}, \widehat{Q}_{2P;01}\}_D = -\{\widehat{Q}_{2P;10}, \widehat{Q}_{2P;10}\} = 2i\widehat{H}_{2P}. \tag{131}$$

The extension of this construction to the case of $n > 2$ interacting multiplets $(1, 2, 1)_{[00]}$ is straightforward. One obtains n pairs of conjugated variables for the propagating bosons, n [10]-graded fields ψ_i and n [01]-graded fields ξ_i.

6.2 Matrix representations of the quantum (anti)commutators

The construction of the quantum theory requires matrices which solve the (anti)commutators which extend the set of single-particle relations (77) induced by the Dirac brackets. In the passage from a single to n multiplets we have n matrices denoted as $\overline{\psi}_i$, n matrices denoted as $\overline{\xi}_i$, with $\overline{\psi}_i \in \mathcal{G}_{10}$, $\overline{\xi}_i \in \mathcal{G}_{01}$ $(i = 1, 2, \ldots, n)$ and the single matrix $\mu \in \mathcal{G}_{11}$. They satisfy the (anti)commutators

$$\{\overline{\psi}_i, \overline{\psi}_j\} = -\{\overline{\xi}_i, \overline{\xi}_j\} = \delta_{ij} \cdot \mathbb{I}, \quad [\overline{\psi}_i, \overline{\xi}_j] = 0, \quad \{\overline{\psi}_i, \mu\} = \{\overline{\xi}_i, \mu\} = 0 \qquad (i, j = 1, \ldots n), \tag{132}$$

where $\mathbb{I}$ is the identity matrix of proper size. This system is minimally solved by $2^{n+1} \times 2^{n+1}$ real matrices. The matrix sectors $\mathcal{G}_{10}, \mathcal{G}_{01}, \mathcal{G}_{11}$ can be read from formula (A.9) with the entries given by $2^{n-1} \times 2^{n-1}$ blocks. We present the $n = 2, 3$ solutions.

We have at first to take into account the gradings $\overline{\psi}_i \in \mathcal{G}_{10}$, $\overline{\xi}_i \in \mathcal{G}_{01}$, $\mu \in \mathcal{G}_{11}$ and the (anti)symmetry of these real matrices. Therefore, up to a normalization factor, for $n = 2$ the matrices should be respectively picked up from the sets

$$\begin{aligned}
\overline{\psi}_i \;&:\; Y\otimes I\otimes I,\ Y\otimes X\otimes I,\ Y\otimes I\otimes X,\ Y\otimes X\otimes X,\\
&\quad A\otimes I\otimes A,\ A\otimes X\otimes A,\ Y\otimes I\otimes Y,\ Y\otimes X\otimes Y;\\
\overline{\xi}_i \;&:\; Y\otimes A\otimes I,\ Y\otimes A\otimes X,\ Y\otimes A\otimes Y,\ A\otimes A\otimes A,\\
&\quad A\otimes Y\otimes I,\ A\otimes Y\otimes X,\ A\otimes Y\otimes Y,\ Y\otimes Y\otimes A;\\
\mu \;&:\; X\otimes A\otimes X,\ X\otimes A\otimes Y,\ X\otimes A\otimes I,\ X\otimes Y\otimes A,\\
&\quad I\otimes A\otimes X,\ I\otimes A\otimes Y,\ I\otimes A\otimes I,\ I\otimes Y\otimes A,
\end{aligned} \tag{133}$$

where A, X, Y, I are the 2×2 matrices given in (A.10).

Consistent $n = 2$ solutions of the (anti)commutator relations (132) are found by setting, e.g.:

$$\overline{\psi}_1 = \tfrac{1}{\sqrt{2}} Y \otimes X \otimes X, \quad \overline{\psi}_2 = \tfrac{1}{\sqrt{2}} Y \otimes X \otimes Y, \quad \text{then} \quad \overline{\xi}_1 = \tfrac{1}{\sqrt{2}} A \otimes Y \otimes I, \quad \overline{\xi}_2 = \tfrac{1}{\sqrt{2}} Y \otimes Y \otimes A,$$
$$\text{while} \quad \mu \quad \text{is either} \quad \mu = X \otimes Y \otimes A \quad \text{or} \quad \mu = I \otimes A \otimes I \tag{134}$$

and, alternatively,

$$\overline{\psi}_1 = \tfrac{1}{\sqrt{2}} Y \otimes I \otimes X, \quad \overline{\psi}_2 = \tfrac{1}{\sqrt{2}} Y \otimes I \otimes Y, \quad \text{then} \quad \overline{\xi}_1 = \tfrac{1}{\sqrt{2}} Y \otimes A \otimes I, \quad \overline{\xi}_2 = \tfrac{1}{\sqrt{2}} A \otimes A \otimes A,$$
$$\text{while} \quad \mu \quad \text{is either} \quad \mu = X \otimes A \otimes I \quad \text{or} \quad \mu = I \otimes Y \otimes A. \tag{135}$$

For the 3-particle case, an $n = 3$ solution of the (anti)commutators (132) is given by

$$\overline{\psi}_1 = \tfrac{1}{\sqrt{2}} \cdot Y \otimes I \otimes X \otimes X, \quad \overline{\psi}_2 = \tfrac{1}{\sqrt{2}} \cdot Y \otimes I \otimes X \otimes Y, \quad \overline{\psi}_3 = \tfrac{1}{\sqrt{2}} \cdot Y \otimes I \otimes Y \otimes I,$$
$$\overline{\xi}_1 = \tfrac{1}{\sqrt{2}} \cdot A \otimes A \otimes A \otimes I, \quad \overline{\xi}_2 = \tfrac{1}{\sqrt{2}} \cdot Y \otimes Y \otimes Y \otimes A, \quad \overline{\xi}_3 = \tfrac{1}{\sqrt{2}} \cdot Y \otimes A \otimes I \otimes I,$$
$$\mu = I \otimes Y \otimes A \otimes I. \tag{136}$$

We are now in the position to present the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded hermitian quantum Hamiltonians for $n = 2, 3$ (both interacting and non-interacting) $(1, 2, 1)_{[00]}$ multiplets. The general construction for $n \geqslant 4$ follows the scheme here outlined.

6.3 $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded quantum Hamiltonians of two interacting particles

It is convenient to indicate here as x, y the coordinates associated with the propagating bosons that, in the constant kinetic basis, were previously denoted as u, v.

The hermitian, constant matrices derived from the (134) solutions can be expressed as

$$\psi_1^A = \tfrac{1}{\sqrt{2}} \cdot Y \otimes X \otimes X, \quad \psi_2^A = \tfrac{1}{\sqrt{2}} \cdot Y \otimes X \otimes Y,$$
$$\xi_1^A = \tfrac{i}{\sqrt{2}} \cdot A \otimes Y \otimes I, \quad \xi_2^A = \tfrac{i}{\sqrt{2}} \cdot Y \otimes Y \otimes A, \qquad \mu^A = -i \cdot I \otimes A \otimes I. \tag{137}$$

The hermitian, constant matrices derived from the (135) solutions can be expressed as

$$\psi_1^B = \tfrac{1}{\sqrt{2}} \cdot Y \otimes I \otimes X, \quad \psi_2^B = \tfrac{1}{\sqrt{2}} \cdot Y \otimes I \otimes Y,$$
$$\xi_1^B = \tfrac{i}{\sqrt{2}} \cdot Y \otimes A \otimes I, \quad \xi_2^B = \tfrac{i}{\sqrt{2}} \cdot A \otimes A \otimes A, \qquad \mu^B = i \cdot X \otimes A \otimes I. \tag{138}$$

The two-particle hermitian Noether supercharges derived from (137) and (138) and corresponding to the quantization of formulas (124) are

$$\begin{aligned}
Q^A_{2P;10} &= -i(\psi_1^A \partial_x + \psi_2^A \partial_y + W_1 \mu^A \xi_1^A + W_2 \mu^A \xi_2^A), \\
Q^A_{2P;01} &= -i(\xi_1^A \partial_x + \xi_2^A \partial_y + W_1 \mu^A \psi_1^A + W_2 \mu^A \psi_2^A), \\
Q^B_{2P;10} &= -i(\psi_1^B \partial_x + \psi_2^B \partial_y + W_1 \mu^B \xi_1^B + W_2 \mu^B \xi_2^B), \\
Q^B_{2P;01} &= -i(\xi_1^B \partial_x + \xi_2^B \partial_y + W_1 \mu^B \psi_1^A + W_2 \mu^B \psi_2^B).
\end{aligned} \tag{139}$$

They depend on the real functions $W_1(x, y)$ and $W_2(x, y)$. In the non-interacting case we have $\partial_y W_1 = \partial_x W_2 = 0$, so that $W_1 \equiv W_1(x)$, $W_2 \equiv W_2(y)$. The selection of the normalized matrices μ^A, μ^B in (139) and not of their alternative choices respectively presented in (134) and (135) is made to ensure that the non-interacting Hamiltonian is a diagonal operator.

In the interacting case the weaker condition

$$\partial_y W_1(x,y) \;=\; \partial_x W_2(x,y), \tag{140}$$

which is solved by the positions

$$W_1 = \partial_x f(x,y) = f_x, \qquad W_2 = \partial_y f(x,y) = f_y \tag{141}$$

in terms of the unconstrained function $f(x,y)$, has to be enforced.

The condition (140) is derived, at the quantum level, by the requirement that the operators $Q^A_{2P;10}, Q^B_{2P;10}, Q^A_{2P;01}, Q^B_{2P;01}$ are square roots of the same Hamiltonian H_{2P}. Since, e.g., in the matrix representation (134) we have

$$\psi_1^A \xi_2^A \;\neq\; \psi_2^A \xi_1^A, \tag{142}$$

the condition

$$(Q^A_{2P;10})^2 - (Q^A_{2P;01})^2 \;=\; 0 \tag{143}$$

implies that the first order derivative contributions, appearing on the left hand side,

$$i\mu^A((\partial_x W_2 - \partial_y W_1)\psi_1^A\xi_2^A - (\partial_x W_2 - \partial_y W_1)\psi_2^A\xi_1^A) \;=\; 0, \tag{144}$$

should separately vanish for $\psi_1^A\xi_2^A$ and $\psi_2^A\xi_1^A$, thus leading to (140). The condition (142) is representation-dependent and not necessarily implied by the classical derivation.

It should be stressed that, for a given $f(x,y)$, the consistency of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded quantum theory does not require to solve the inverse equations which are induced, see (126), by the classical theory.

When (141) is enforced the four operators given in (139) are square-roots of the two-particle Hamiltonian H_{2P}. They satisfy

$$\{Q^A_{2P;10}, Q^A_{2P;10}\} = \{Q^A_{2P;01}, Q^A_{2P;01}\} = \{Q^B_{2P;10}, Q^B_{2P;10}\} = \{Q^B_{2P;01}, Q^B_{2P;01}\} = 2H_{2P}, \tag{145}$$

where

$$H_{2P} \;=\; \begin{pmatrix} H_0+V_{++} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & H_0+V_{--} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H_0+V_{++} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_0+V_{--} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & H_0+V_{-+} & -f_{xy} & 0 & 0 \\ 0 & 0 & 0 & 0 & -f_{xy} & H_0+V_{+-} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & H_0+V_{-+} & -f_{xy} \\ 0 & 0 & 0 & 0 & 0 & 0 & -f_{xy} & H_0+V_{+-} \end{pmatrix}$$

with $\qquad H_0 = \dfrac{1}{2}(-\partial_x^2 - \partial_y^2 + f_x^2 + f_y^2) \qquad$ and $\qquad V_{\epsilon\delta} = \dfrac{1}{2}(\epsilon f_{xx} + \delta f_{yy}), \quad$ for $\quad \epsilon,\delta = \pm 1.$ (146)

By construction, the 2-particle Hamiltonian (146) is hermitian. In the

$$f_{xy} \equiv \frac{\partial^2}{\partial_x\partial_y} f(x,y) \;=\; 0 \tag{147}$$

non-interacting case the Hamiltonian H_{2P} is a diagonal operator.

We can repeat for the $H_{2P} \in \mathcal{G}_{00}$ Hamiltonian the analysis of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetries given in Section **5** for the single-particle Hamiltonian.

Two copies of the Beckers-Debergh algebra (A.8) are obtained from the two sets of three operators

$$\mathfrak{S}_{2P,A} = \{Q^A_{2P;10}, Q^A_{2P;01}, H_{2P}\} \quad \text{and} \quad \mathfrak{S}_{2P,B} = \{Q^B_{2P;10}, Q^B_{2P;01}, H_{2P}\}. \tag{148}$$

We have indeed vanishing commutators

$$[Q^A_{2P;10}, Q^A_{2P;01}] = 0 \qquad \text{and} \qquad [Q^B_{2P;10}, Q^B_{2P;01}] = 0. \tag{149}$$

The two sets of four hermitian operators

$$\mathfrak{S}_{2P,1} = \{Q^B_{2P;10}, Q^A_{2P;01}, H_{2P}, Z_{2P}\} \quad \text{and} \quad \mathfrak{S}_{2P,2} = \{Q^A_{2P;10}, Q^B_{2P;01}, H_{2P}, \overline{Z}_{2P}\} \tag{150}$$

produce two copies of the one-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supertranslation algebra (A.7), with nonvanishing (anti)commutators

$$\{Q^B_{2P;10}, Q^B_{2P;10}\} = \{Q^A_{2P;01}, Q^A_{2P;01}\} = 2H_{2P}, \qquad [Q^B_{2P;10}, Q^A_{2P;01}] = iZ_{2P} \tag{151}$$

and, respectively,

$$\{Q^A_{2P;10}, Q^A_{2P;10}\} = \{Q^B_{2P;01}, Q^B_{2P;01}\} = 2H_{2P}, \qquad [Q^A_{2P;10}, Q^B_{2P;01}] = i\overline{Z}_{2P}. \tag{152}$$

The explicit expression of the $Z_{2P}, \overline{Z}_{2P} \in \mathcal{G}_{11}$ operators can be read from the commutators in formulas (151) and (152). We present, for completeness, Z_{2P}. It is given in terms of the 8×8 matrices $E_{i,j}$ with entry 1 at the intersection of the i-th row and j-th column and 0 otherwise. We have

$$\begin{aligned} Z_{2P} \;=\; & -(\partial_x^2 + \partial_y^2 - f_x^2 - f_y^2)(-E_{1,3} + E_{2,4} - E_{3,1} + E_{4,2}) + \\ & (\partial_x^2 - \partial_y^2 - f_x^2 + f_y^2)(-E_{5,7} + E_{6,8} - E_{7,5} + E_{8,6}) + \\ & -2(\partial_x\partial_y - f_x f_y)(E_{5,8} + E_{6,7} + E_{7,6} + E_{8,5}) + \\ & 2(f_x\partial_y - f_y\partial_x)(E_{5,8} - E_{6,7} + E_{7,6} - E_{8,5}) + \\ & -(f_{xx} + f_{yy})(E_{1,3} + E_{2,4} + E_{3,1} + E_{4,2} + E_{5,7} + E_{6,8} + E_{7,5} + E_{8,6}). \end{aligned} \tag{153}$$

6.4 $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded quantum Hamiltonians of three interacting particles

The $n = 3$ solutions (136) of the (anti)commutator relations (132) allow to define the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded, three-particle, interacting quantum Hamiltonian H_{3P}.

The three coordinates are now labeled, for simplicity, as x, y, z. The hermitian, constant matrices expressed in terms of the $\overline{\psi}_1, \overline{\psi}_2, \overline{\psi}_3, \overline{\xi}_1, \overline{\xi}_2, \overline{\xi}_3, \mu$ matrices defined in (136) are

$$\psi_1 = \overline{\psi}_1, \quad \psi_2 = \overline{\psi}_2, \quad \psi_3 = \overline{\psi}_3, \quad \xi_1 = i\overline{\xi}_1, \quad \xi_2 = i\overline{\xi}_2, \quad \xi_3 = i\overline{\xi}_3, \quad \overline{\mu} = i\mu. \tag{154}$$

The hermitian supercharges, respectively belonging to the $\mathcal{G}_{10}$ and $\mathcal{G}_{01}$ sectors, are given by

$$\begin{aligned} Q_{3P;10} &= -i(\psi_1\partial_x + \psi_2\partial_y + \psi_3\partial_z + f_x\overline{\mu}\xi_1 + f_y\overline{\mu}\xi_2 + f_z\overline{\mu}\xi_3), \\ Q_{3P;01} &= -i(\xi_1\partial_x + \xi_2\partial_y + \xi_3\partial_z + f_x\overline{\mu}\psi_1 + f_y\overline{\mu}\psi_2 + f_z\overline{\mu}\psi_3), \end{aligned} \tag{155}$$

where $f(x, y, z)$ is an arbitrary function of the three coordinates.

They are the square roots of the hermitian three-particle Hamiltonian H_{3P}, given by

$$\{Q_{3P;10}, Q_{3P;10}\} = \{Q_{3P;01}, Q_{3P;01}\} = 2H_{3P}. \tag{156}$$

The Hamiltonian H_{3P} is a 16×16 matrix differential operator. It is the sum of a diagonal part H_{diag} and of the off-diagonal terms H_{off}. In its turn the diagonal part is the sum of two terms, H_0 and V. We can set

$$H_{3P} \;=\; H_{diag} + H_{off}, \qquad H_{diag} = H_0 + V, \tag{157}$$

where

$$H_0 \;=\; \frac{1}{2}(-\partial_{xx}^2 - \partial_{yy}^2 - \partial_{zz}^2 + f_x^2 + f_y^2 + f_z^2) \cdot \mathbb{I}_{16}, \tag{158}$$

while V and H_{off} are conveniently expressed in terms of the 16×16 matrices $E_{i,j}$ with entry 1 at the intersection of the i-th row and j-th column and 0 otherwise. We get

$$\begin{aligned} V \;=\; & V_{--+}(E_{1,1} + E_{7,7}) + V_{+++}(E_{2,2} + E_{8,8}) + V_{+--}(E_{3,3} + E_{5,5}) + \\ & V_{-+-}(E_{4,4} + E_{6,6}) + V_{+-+}(E_{9,9} + E_{15,15}) + V_{-++}(E_{10,10} + E_{16,16}) + \\ & V_{---}(E_{11,11} + E_{13,13}) + V_{++-}(E_{12,12} + E_{14,14}), \end{aligned} \tag{159}$$

where

$$V_{\epsilon\delta\rho} \;=\; \frac{1}{2}(\epsilon f_{xx} + \delta f_{yy} + \rho f_{zz}), \qquad \text{for} \quad \epsilon, \delta, \rho = \pm 1. \tag{160}$$

The off-diagonal terms are

$$\begin{aligned} H_{off} \;=\; & f_{xy}(E_{3,4} + E_{4,3} + E_{5,6} + E_{6,5} + E_{9,10} + E_{10,9} + E_{15,16} + E_{16,15}) + \\ & f_{xz}(-E_{1,3} - E_{3,1} + E_{5,7} + E_{7,5} + E_{10,12} + E_{12,10} - E_{14,16} - E_{16,14}) + \\ & f_{yz}(-E_{1,4} - E_{4,1} + E_{6,7} + E_{7,6} - E_{9,12} - E_{12,9} + E_{14,15} + E_{15,14}). \end{aligned} \tag{161}$$

The off-diagonal part of the Hamiltonian vanishes ($H_{off} = 0$) if the three particles are not interacting, namely for

$$f_{xy} = f_{xz} = f_{yz} = 0. \tag{162}$$

The set of three operators $Q_{3P;10}$, $Q_{3P;01}$, H_{3P} close the Beckers-Debergh algebra (A.8) since the commutator between $Q_{3P;10}$, $Q_{3P;01}$ is vanishing:

$$[Q_{3P;10}, Q_{3P;01}] \;=\; 0. \tag{163}$$

7 Conclusions

In this paper we estalished the Hamiltonian formalism for classical $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded invariant mechanical theories and performed their canonical quantization. We had to carefully specify how the necessary ingredients (Poisson and Dirac brackets, canonical variables, etc.) apply to graded fields. We worked out the cases of $(1,2,1)_{[00]}$ and $(1,2,1)_{[11]}$ $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded multiplets of component fields. Both these multiplets contain, see [33], two types of fermionic fields, one ordinary boson and one exotic boson (in the $(1,2,1)_{[00]}$ multiplet the ordinary boson is propagating while the exotic boson is an auxiliary field, the situation being reversed for the $(1,2,1)_{[11]}$

multiplet). The theories derived by these multiplets are the simplest ones. For the moment we left aside the quantization of the theories based on the other type of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded multiplet (denoted as "$(2, 2, 0)$" since it possesses two propagating bosons and two propagating fermions) introduced in [33]. Based on the quantization of its counterpart in ordinary supersymmetric quantum mechanics, see [37], we are expecting the $(2, 2, 0)$ multiplet to produce more complicated quantum Hamiltonians than the ones here derived. They should present, in particular, a spin-orbit interaction. This class of theories will be left for future investigations.

Concerning the two types of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded $(1, 2, 1)$ multiplets we proved that the quantization of the single $(1, 2, 1)_{[11]}$ multiplet produces the same 4×4 matrix differential Hamiltonian induced by the quantization of $(1, 2, 1)_{[00]}$. We point out that we do not have a general argument that this should be necessarily the case for several interacting multiplets. The identification of the two Hamiltonians for the two single-multiplet cases can be the byproduct of their simplicity. Loosely speaking the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry, in this simple setting, does not leave room for an alternative, $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded, invariant Hamiltonian.

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded quantum Hamiltonian introduced in [23] is recovered from the quantization of the classical invariant actions for a single multiplet (either $(1, 2, 1)_{[00]}$ or $(1, 2, 1)_{[11]}$) which were presented in [33]. It turns out that this Hamiltonian is the $n = 1$ representative of the class of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded quantum Hamiltonians obtained by quantizing n interacting multiplets, see formula (146) for $n = 2$ and formulas (157-161) for $n = 3$.

The construction of multiparticle quantum Hamiltonians is particularly relevant because it helps answering the long-standing puzzle of the physical significance of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry. The single-particle quantum Hamiltonian (87) possesses different types of graded symmetries as discussed in Section **5**. Besides admitting $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded invariance, it is also an example of an ordinary $\mathcal{N} = 4$ supersymmetric quantum mechanics, see (108). Its $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry is emergent, but the construction of the Hilbert space and the computation of the energy eigenvalues of the model do not require it. A radical new feature appears when considering multiparticle quantum Hamiltonians based on $n > 1$ multiplets. In that case the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry directly affects the statistics of the particles and the construction of the, properly (anti)symmetrized, multiparticle wave functions. It implies measurable physical consequences (about the energy eigenvalues and their degeneracies, the partition function and the chemical potentials, etc.). In the next forthcoming paper, currently under writing, of this series devoted to $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded mechanics, we will present the distinct signatures of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry in multiparticle quantum Hamiltonians. In so doing we will be able to determine the experimentally testable consequences of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry within this class of theories. This work and the results here presented are the necessary steps in this line of research.

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Appendix: reminder of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras

We summarize for completeness, following [33], the basic properties of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras and conventions used in the text.

A $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathcal{G}$ is decomposed as

$$\mathcal{G} \;=\; \mathcal{G}_{00} \oplus \mathcal{G}_{10} \oplus \mathcal{G}_{01} \oplus \mathcal{G}_{11}. \tag{A.1}$$

It is endowed with the operation $[\cdot,\cdot\} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ satisfying, for any $g_a \in \mathcal{G}_{\vec{\alpha}}$, the properties

$$[g_a, g_b\} = g_a g_b - (-1)^{\vec{\alpha}\cdot\vec{\beta}} g_b g_a,$$
$$(-1)^{\vec{\gamma}\cdot\vec{\alpha}}[g_a,[g_b,g_c\}\} + (-1)^{\vec{\alpha}\cdot\vec{\beta}}[g_b,[g_c,g_a\}\} + (-1)^{\vec{\beta}\cdot\vec{\gamma}}[g_c,[g_a,g_b\}\} = 0. \tag{A.2}$$

The first equation gives the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded (anti)commutators; the second equation gives the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Jacobi identity. The generators g_a, g_b, g_c respectively belong to the sectors $\mathcal{G}_{\vec{\alpha}}, \mathcal{G}_{\vec{\beta}}, \mathcal{G}_{\vec{\gamma}}$, where $\vec{\alpha} = (\alpha_1, \alpha_2)$ for $\alpha_{1,2} = 0, 1$ and $\mathcal{G}_{\vec{\alpha}} \equiv \mathcal{G}_{\alpha_1\alpha_2}$ (and similarly for $\vec{\beta}$, $\vec{\gamma}$).

The scalar product $\vec{\alpha}\cdot\vec{\beta}$ is defined as

$$\vec{\alpha}\cdot\vec{\beta} \;=\; \alpha_1\beta_1 + \alpha_2\beta_2. \tag{A.3}$$

For the (anti)commutator one has $[g_a, g_b\} \in \mathcal{G}_{\vec{\alpha}+\vec{\beta}}$, with the vector sum defined mod 2.

According to the definitions, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded (anti)commutators $[A, B\}$ between two graded generators A, B are read from the table

$A\backslash B$	00	10	01	11
00	$[\cdot,\cdot]$	$[\cdot,\cdot]$	$[\cdot,\cdot]$	$[\cdot,\cdot]$
10	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$
01	$[\cdot,\cdot]$	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$	$\{\cdot,\cdot\}$
11	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$	$\{\cdot,\cdot\}$	$[\cdot,\cdot]$

. (A.4)

Let V be a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded vector space such that

$$V \;=\; V_{00} \oplus V_{10} \oplus V_{01} \oplus V_{11}, \tag{A.5}$$

with $v, v' \in V$ of respective ij, $i'j'$ gradings. If kl is the grading of the operator $M : V \to V$, where $v' = Mv$, then we have, *mod* 2,

$$i' = i + k, \qquad j' = j + l. \tag{A.6}$$

In the paper we consider the "one-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supertranslation algebra" with generators $H \in \mathcal{G}_{00}$, $Z \in \mathcal{G}_{11}$, $Q_{10} \in \mathcal{G}_{10}$, $Q_{01} \in \mathcal{G}_{01}$ and nonvanishing (anti)commutators

$$\{Q_{10}, Q_{10}\} = \{Q_{01}, Q_{01}\} = 2H, \qquad [Q_{10}, Q_{01}] = -2Z. \tag{A.7}$$

The algebra

$$\{Q_{10}, Q_{10}\} = \{Q_{01}, Q_{01}\} = 2H, \qquad [Q_{10}, Q_{01}] = [H, Q_{10}] = [H, Q_{10}] = 0, \tag{A.8}$$

with generators H, Q_{10}, Q_{01} and vanishing commutator between Q_{10}, Q_{01} is referred to (following [23, 38]) as "the Beckers-Debergh algebra".

In a 4×4 matrix representation of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra the nonvanishing entries (which can be either numbers or differential operators) of the $\mathcal{G}_{ij}$ graded sectors are accommodated according to

$$\mathcal{G}_{00} = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}, \qquad \mathcal{G}_{11} = \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix},$$
$$\mathcal{G}_{10} = \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix}, \qquad \mathcal{G}_{01} = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \\ 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}, \tag{A.9}$$

In the text the needed matrices are conveniently expressed as tensor products of the 4 real, 2×2 split-quaternion matrices I, X, Y, A given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad \text{(A.10)}$$

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