On the magnetic field spectrum and its fluctuations in a primeval cold plasma in thermal equilibrium

Francisco Caruso*
Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, Brazil

Vitor Oguri and Felipe Silveira
Universidade do Estado do Rio de Janeiro, Rio de Janeiro, Brazil
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Magnetic fluctuations in a non-magnetized gaseous plasma is revisited and calculated without any approximation, based on the fluctuation-dissipation theorem. Also the simultaneous dependence of this intensity on the plasma and on the collisional frequencies is discussed. Finally, the total emitted plasma energy is compared to the Stefan-Boltzmann law of a pure black-body.

I. INTRODUCTION

Our universe is filled with magnetic fields present in almost all galaxies and clusters of galaxies, which are essential for many physical processes, such as synchrotron radiation generated by astronomical objects like pulsars and quasars. One possible explanation for these fields is built by considering an initially weak field, which is amplified by a dynamo mechanism [1–4]. However, this process needs a so-called *seed field*. These seed fields are presumed to be generated shortly after the Big Bang. Several theoretical explanations of how they were created can be found in the literature, such as the Biermann mechanism [5], supernova explosion [6–8], electromagnetic fluctuations in plasma [9] and others. Considering this last topic, Tajima, *et al.*, in 1992, [10] noted a lack of a concrete expression of the low-frequency spectrum of fluctuations of magnetic fields in thermal plasma and argued that this low-frequency spectrum can be the origin of magnetic fields in the Universe.

Fluctuations of physical quantities near zero frequency have been investigated by several authors since the papers of Johnson [11] and Nyquist [12]. A general theory on the fluctuation-dissipation theorem, which will be the starting point of this paper, was developed by [13]. To the best of our knowledge, an approximated expression for the low-frequency spectrum of magnetic fields fluctuations in a thermal plasma was obtained for the first time by [10]. They found a peak around $\omega = 0$ magnetic fluctuation which was interpreted as the evanescent energy component of electromagnetic fluctuations "screened" in plasma, below the plasma frequency. The impact of such a result into the cosmic microwave background was then investigated by [14]. Although in these two references the authors claim that the fluctuations were rigorously computed, several approximations were indeed made and they were not able to get a unique formula covering both the low- and high-frequency spectrum. Some criticism concerning Tajima's results can be found in Refs. [15, 16], where a new model was developed including thermal effects as well as collisional effects.

The aim of this paper is very specific. We reevaluate the derivation of the spectrum of magnetic fluctuations, in the case of electron-positron plasma, avoiding any approximation in the low-frequency region and also in the transition between the low- and the high-frequency spectrum. Several different behaviors between ours and previous results [10, 16], mainly in the low-frequency part of the spectrum, are found and discussed. We are also able to make new quantitative predictions, such as how the energy density of the magnetic fields deviates from the Stefan-Boltzmann law of an ideal black-body.

II. THE FIRST PREDICTIONS

The fluctuation-dissipation theorem developed by [13] can deal with the thermal fluctuations inside a plasma in or near thermal equilibrium. The expression for the magnetic field fluctuation in a homogeneous isotropic non-magnetized equilibrium plasma was obtained by [10] looking at waves in such a plasma. In an electron-positron plasma, for example, the magnetic fluctuations in wavenumber and frequency space are given as a function of the plasma temperature T by

* francisco.caruso@gmail.com

$$\frac{\langle B^2\rangle_{\vec{k},\omega}}{8\pi} = \frac{2\hbar\omega}{e^{\hbar\omega/k_B T}-1}\,\eta\omega_p^2\,\times$$
$$\times\,\frac{k^2c^2}{(\omega^2+\eta^2)k^4c^4+2\omega^2(\omega_p^2-\omega^2-\eta^2)k^2c^2+[(\omega^2-\omega_p^2)^2+\eta^2\omega^2]\omega^2} \tag{1}$$

where k_B is the Boltzmann constant, and ω_p and η are, respectively, the plasma and collisional frequencies. In an electron-positron plasma, the plasma frequency ω_p is given by the relation $\omega_p^2 = \omega_{pe^+}^2 + \omega_{pe^-}^2$; since $\omega_{pe^+} = \omega_{pe^-}$ we have

$$\omega_p^2 = 2\omega_{pe}^2; \qquad \omega_{pe}^2 = \frac{n_e 4\pi e^2}{\gamma m_e}; \qquad \text{and} \qquad \gamma = 1 + \frac{k_B T}{m_e c^2}$$

with e and m_e being, respectively, the charge and the mass of the constituents (electrons and positrons) of the plasma, n_e being the electron (positron) density. In addition, the collisional frequency is

$$\eta_{e^-} = \eta_{e^+} = \eta = \eta_e = 2.91\times 10^{-6}\, n_e\, T^{-1.5}\,\ln(\Lambda) \tag{2}$$

with

$$\ln(\Lambda) = \ln\left(4\pi n_e \lambda_D^3\right)$$

where λ_D is the Debye's length

$$\lambda_D = \sqrt{\frac{k_B T}{4\pi n_e e^2}} \tag{3}$$

Integrating the former equation in $\mathrm{d}\vec{k} = 4\pi k^2 \mathrm{d}k$ we get (the Fourier transform)

$$S(\omega) \equiv \frac{\langle B^2\rangle_\omega}{8\pi} = \int \frac{\mathrm{d}\vec{k}}{(2\pi)^3}\,\frac{\langle B^2\rangle_{\vec{k},\omega}}{8\pi} \equiv \int_0^\infty S(\omega,k)\mathrm{d}k \tag{4}$$

Thus we have to solve the following integral:

$$S(\omega) \equiv \frac{2\hbar\omega}{e^{\hbar\omega/k_B T}-1}\,\frac{\eta}{(2\pi)^3}\,\frac{\omega_p^2}{c^2}\,\times$$
$$\times\int_0^\infty \frac{(4\pi)k^4\mathrm{d}k}{(\omega^2+\eta^2)k^4+\dfrac{2\omega^2}{c^2}(\omega_p^2-\omega^2-\eta^2)k^2+\left[\dfrac{(\omega^2-\omega_p^2)^2+\eta^2\omega^2}{c^4}\right]\omega^2} \tag{5}$$

The integral over wavenumbers to be solved in Eq. (5) clearly shows a high wavenumber linear divergence. According to [10], this is expected since the derivation is based on classical fluid equations of motion and the constant collision frequency η is considered to be independent of k. However, they prefer to carry on their analyzes in the simpler phenomenological approach. To overcome the large k dependence, they first take the limit $\eta \to 0$ and then they integrate over k to infinity, which corresponds to the vanishing cross section of collisions as $k \to \infty$. This is a very delicate point and we will turn back to this point in Section III. For both the high frequency and high wavenumber limits the authors emphasized that the expression of Eq. (1) has a substantial value only where $\omega^2 - c^2k^2 - \omega_p^2 \simeq 0$. The combined high-frequency and high wavenumber limits were got by letting $\eta \to 0$. The expression for the low-frequency spectrum was obtained by breaking up the k integral into two intervals, by introducing what the authors called "a cutoff value" k_{cut}, with $x_{\text{cut}} \equiv k_{\text{cut}}c/\omega_{pe}$. Technically, this k_{cut} is not really a cutoff. It would be better to be called a "convergence point" which was arbitrarily chosen by Tajima, *et al.* [10, 14] to obtain a smooth connection at the joining point of the low and high spectrum. Although these authors sustain that their results do not critically depend on this upper limit, it was shown in Ref. [16] that this is not true. In Ref. [10, 14], the integration from 0 to k_{cut}, η was kept finite while in the integral from k_{cut} to ∞ the approximation $\eta \to 0$ was considered. The expressions obtained for the high and low parts of the spectrum were, respectively:

$$\frac{\langle B^2\rangle_\omega}{8\pi} = \frac{T}{2\pi}\delta(\omega)\int \frac{\omega_p^2}{\omega_p^2 + c^2k^2}\,k^2\mathrm{d}k + \frac{1}{2\pi c^3}\,\frac{\hbar}{e^{\hbar\omega/k_BT}-1}\,(\omega^2-\omega_p^2)^{3/2} \tag{6}$$

and

$$S(\omega') \equiv \frac{\langle B^2\rangle_{\omega'}}{8\pi} = \frac{1}{\pi^2}\,\frac{\hbar\omega'}{e^{(\hbar\omega'_{pe}/k_BT)\omega'}-1}\,2\eta'\left(\frac{\omega_{pe}}{c}\right)^3 \times \int \frac{x^4}{(\omega'^2+\eta'^2)x^4+\cdots}\,\mathrm{d}x + $$
$$+ \frac{\hbar(\omega'^2-\omega_p'^2)^{3/2}}{2\pi e^{(\hbar\omega_{pe}/k_BT)\omega'}-1}\left(\frac{\omega_{pe}}{c}\right)^3 \times \Theta(\omega - \sqrt{c^2k_{\mathrm{cut}}^2+\omega_p^2}) \tag{7}$$

where Θ is the Heaviside step function; $\eta' \equiv \eta/\omega_{pe}$, $\omega' \equiv \omega/\omega_{pe}$, and $\omega_p' \equiv \omega_p/\omega_{pe}$.

Defining, as in Refs. [10, 16], the normalization factor,

$$S_0 = \frac{\omega_{pe}^2 k_B T}{c^3} \tag{8}$$

we can numerically reproduce the previous, Eq. (7), results of those references as shown in Fig. 1.

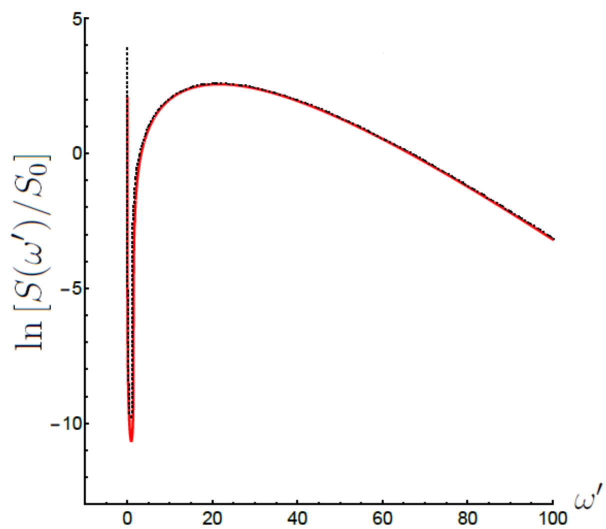

FIG. 1. Plot of the normalized magnetic field spectrum of Eq. (7) made by us (full line), compared to the plot given in Fig. 1.b of Ref. [16], both ploted for $T = 7\times10^9$ K, $n_e = 4.6\times10^{30}$ cm^{-3} ($\gamma = 1$).

Finally, the zero frequency limit of the magnetic fluctuations is give by

$$\lim_{\omega\to 0}\frac{\langle B^2\rangle_\omega}{8\pi} = \frac{\hbar\omega'}{\pi^2(e^{\hbar\omega_{pe}\omega'/k_BT}-1)}\,2\left(\frac{\omega_{pe}}{c}\right)^3\frac{1}{\eta'}\int_0^{x_{\mathrm{cut}}}\mathrm{d}x \tag{9}$$

At this point the frequency spectral intensity was plotted for a temperature $T = 10^{10}$ K, by requiring that the value of k_{cut} (or x_{cut}) provide a smooth behavior at the joint between the low-frequency spectrum and the black-body spectrum. The choice was $k_{\mathrm{cut}} \sim \omega_{pe}/c$ or ($x_{\mathrm{cut}} \sim 1$). The result for other temperature values were presented in another paper by [14]. The main claims by these authors was that the intensity of the spectrum does not vary sensitively with k_{cut} and that, near $\omega = 0$, the spectrum goes like ω^{-2}. Let us now show our general and exact results.

III. GENERAL RESULT

Our analytical solution for Eq. (5) was obtained by introducing a dimensionless variable $y = k/k_\circ$, where $k_\circ = \omega/c$, and reducing the integrand into partial fractions, namely

$$S(\omega) = Db\int_0^\infty \frac{y^4\mathrm{d}y}{y^4 - 2ay^2 + C} = Db\int_0^\infty \frac{F(y)}{f(y)\,\mathrm{d}y} \tag{10}$$

with $D \equiv \pi^{-2}\left(\frac{\hbar\omega^2}{e^{\hbar\omega/k_BT}-1}\right)\left(\frac{\omega_p^2}{c^3}\right)\eta$, $a \equiv \left(1 - \frac{\omega_p^2}{\omega^2+\eta^2}\right)$, $b = (\omega^2+\eta^2)^{-1}$, $C \equiv 1 + \frac{\omega_p^2}{\omega^2+\eta^2}\left(\frac{\omega_p^2}{\omega^2} - 2\right)$, $F(y) = -2ay^2 + C$ and $f(y) = y^4 - 2ay^2 + C$. The ratio between these two functions is expressed as

$$\frac{F(y)}{f(y)} = \frac{A_1}{y-y_1} + \frac{A_2}{y-y_2} + \frac{A_3}{y-y_3} + \frac{A_4}{y-y_4}$$

where y_i are the roots of $f(y)$ and $A_i = F(y_i)/f'(y_i)$, for $i = 1, 2, 3, 4$.

A straightforward calculation gives rise to our expression for $S(\omega)$, which will be expressed as a function of the variable $\omega' = \omega/\omega_{pe}$ to facilitate future comparisons, *i.e.*,

$$S(\omega') = \frac{1}{\pi^2\omega'^2}\left(\frac{\hbar\omega'^3}{e^{(\hbar\omega_{pe}/k_BT)\omega'} - 1}\right)\left(\frac{\omega_{pe}}{c}\right)^3 \times$$
$$\times\left\{2\lambda_c\frac{\eta'}{\omega'^2+\eta'^2} + f(\omega')\left[h(\omega')\sqrt{g(\omega')+h(\omega')} - 2\eta'\sqrt{g(\omega')-h(\omega')}\right]\right\} \tag{11}$$

where $\lambda_c = k_{cut}c/\omega_{pe}$, and f, g and h are functions defined by:

$$f(\omega') = \frac{\pi}{2\sqrt{2}}\frac{\omega'^{1/2}}{(\omega'^2+\eta'^2)^{3/2}}$$

$$g(\omega') = (\omega'^2+\eta'^2)^{1/2}\sqrt{\omega'^2(\omega'^2+\eta'^2) + 4(1-\omega'^2)}$$

$$h(\omega') = \omega'(\omega'^2+\eta'^2-2)$$

The constant λ_c represents, in our scheme, the cutoff to avoid the linear divergence in the wavenumber variable and, thus, must assume a high value. For us, λ_c has the same purpose of the $x_{\max} = k_{\max}c/\omega_{pe}$ used in the Tajima *et al* works, where $k_{\max}$ is introduced to avoid in a Coulomb collision that, for small distances, the Coulomb energy exceeds the kinetic energy. This occurs approximately for the closest approximation distance of a test particle and an electron in the plasma (See Ref. [16]). Therefore, λ_c cannot be of the order of 1. For the sake of future comparisons, we will fix the following plasma parameters: $T \simeq 10^{10}$ K, $n_e \simeq 4.8\times10^{30}$ cm^{-3} and $\lambda_c \simeq 2\,444.4$. In any case, we can show that we have a small dependency of Eq. (11) with the λ_c value as can be inferred from Fig. 2.

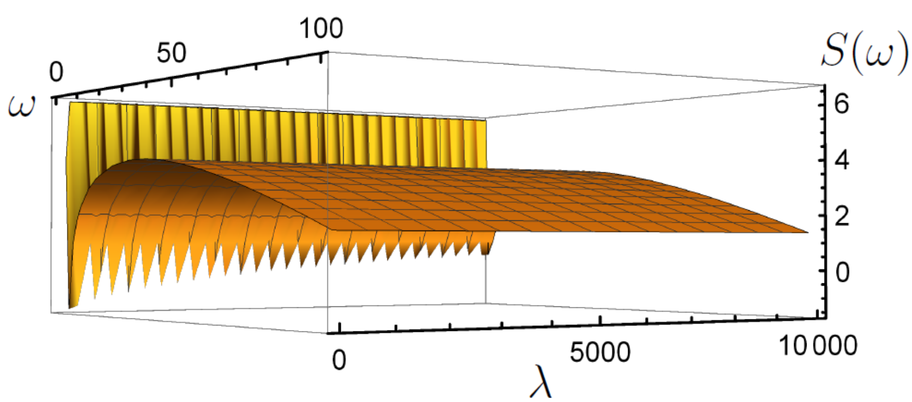

FIG. 2. Plot of $\ln[S(\omega')/S_0]$, where $S(\omega')$ is given by Eq. (11). For the following parameters: $0 < \omega' \leq 10$, $\gamma = 2.18724$, $T = 7\times10^9$ K, $n_e = 4.6\times10^{30}$ cm^{-3} and a huge range of λ_c values.

Our exact results, based on Eq. (11), are plotted in Figs. 3 and 4, considering, respectively, two different ranges for ω' ($0 < \omega' < 10$, and $0 < \omega' < 100$), with S_0 defined in Eq. (8):

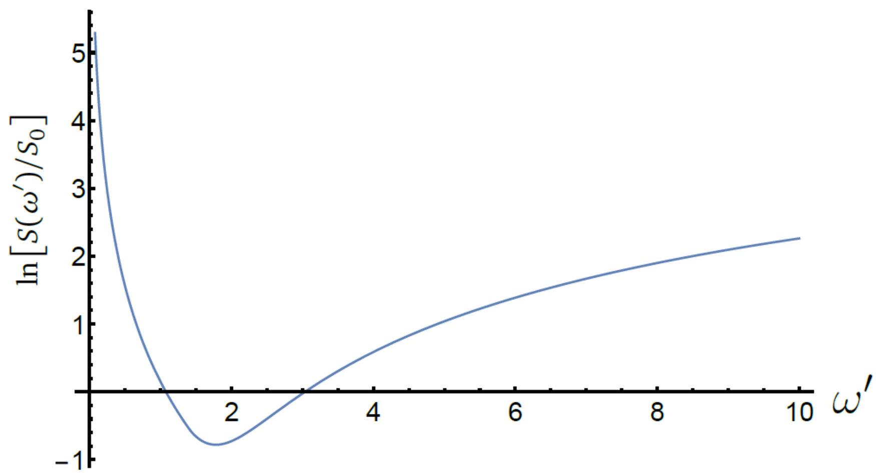

FIG. 3. Plot of $\ln[S(\omega')/S_0]$, where $S(\omega')$ is given by Eq. (11), for the following parameters: $0 < \omega' \leq 10$, $\gamma = 2.18724$, $T = 7 \times 10^9$ K, $n_e = 4.6 \times 10^{30}$ cm^{-3} and $\lambda = 2\,444.4$.

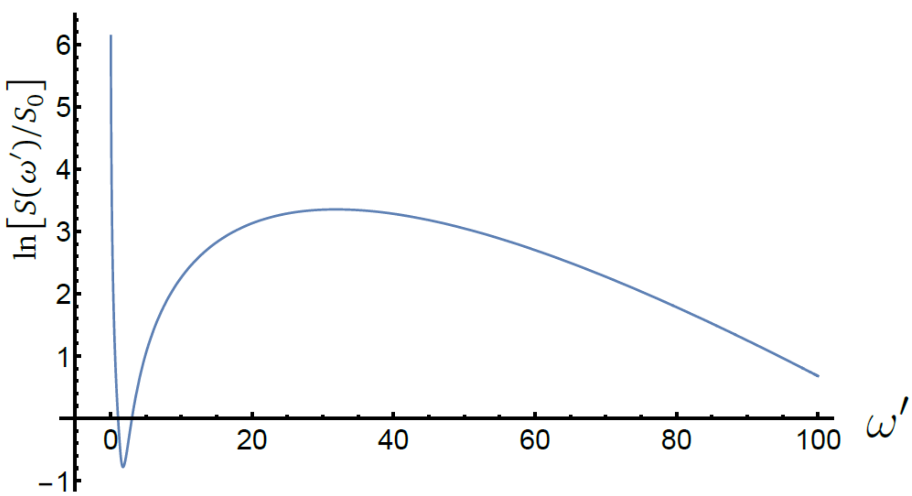

FIG. 4. Plot of $\ln[S(\omega')/S_0]$, where $S(\omega')$ is given by Eq. (11), for the following parameters: $0 < \omega' \leq 100$, $\gamma = 2.18724$, $T = 7 \times 10^9$ K, $n_e = 4.6 \times 10^{30}$ cm^{-3} and $\lambda = 2\,444.4$.

They are both in good agreement with the results of Ref. [16].

Lastly, in order to study the behavior of S by varying both the plasma's frequency and temperature, we have to come back to the variables ω and T, since $\omega' = \omega'(T)$.

We still need to know how the plasma density n_e varies with the temperature T.

Following the book of Paul M. Bellan [17], inside the plasma, *i.e.*, for $|x| \gg \lambda_D$, it is assumed that the electron distribution function is Maxwellian with temperature T. Since the distribution function depends only on constants of the motion, the one-dimensional electron velocity distribution function must depend only on the electron energy $mv^2/2 + qe\langle\varphi(x)\rangle$, a constant of the motion, and so must have the following dependence

$$f_e(v,x) = \frac{n_\circ}{2\pi k_B T/m_e} \exp\left[-\left(mv^2/2 + qe\langle\varphi(x)\rangle/k_B T\right)\right]$$

and the electron density is [Eq. (2.109), p. 55 of Ref. [17]]

$$n_e(x) = \int_{-\infty}^{\infty} \mathrm{d}v f_e(v,0) = n_\circ \exp(-qe\langle\varphi(x)\rangle/k_B T)$$

Let us take form the mean plasma density the expression

$$n_e = n_\circ e^{-\lambda/k_B T} \tag{12}$$

The parameters $n_\circ$ and λ are fixed by using two different inputs [16]: $n_e = 4.8 \times 10^{30}$ for $T = 1 \times 10^{10}$ K, and $n_e = 4.6 \times 10^{30}$ for $T = 7 \times 10^9$ K. We have to solve the system

$$4.8 \times 10^{30} = n_\circ \, e^{-\lambda/861730}$$
$$4.6 \times 10^{30} = n_\circ \, e^{-\lambda/603211}$$

which has the following solutions:

$$n_\circ = 5.30 \times 10^{30} \text{ cm}^{-3}; \quad \text{and} \quad \lambda = 8.56 \times 10^4 \text{ eV} \tag{13}$$

Thus, in our future calculations, we will adopt the following expression for $a(T)$:

$$a = \frac{2e\hbar}{k_B T} \left(\frac{2\pi n_\circ}{\gamma m_e} \right)^{1/2} e^{-k_B/(2k_B T)} \tag{14}$$

So, using Eqs. (12), (13) and (14), we get the result shown in Figure 5 for $S(\omega, T)$.

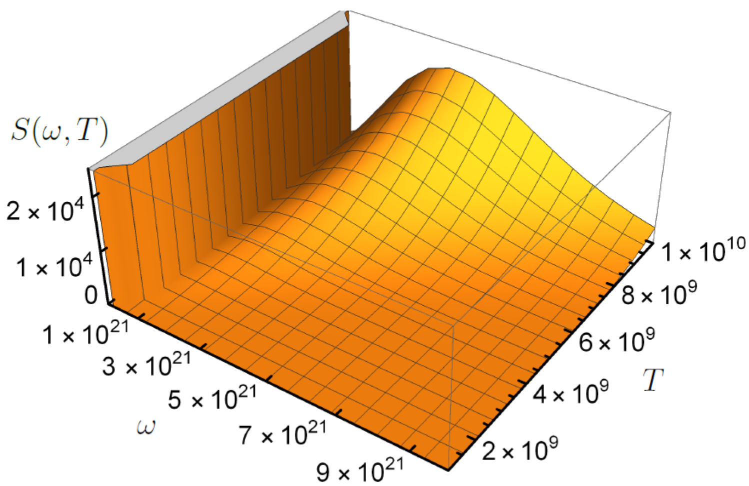

FIG. 5. Plot of $S(\omega, T)$ considering the same parameters as the Fig. 4 but with variable temperature.

In Fig. (6) it is shown how our prediction depends on the choice of the γ factor.

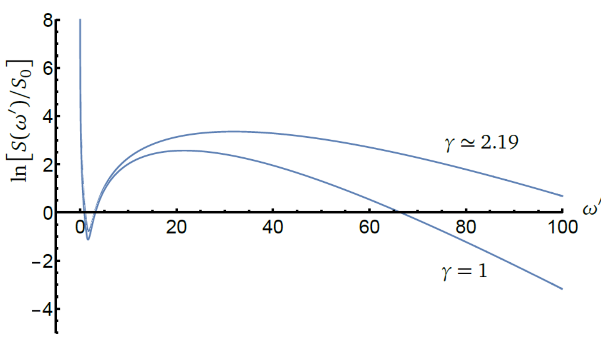

FIG. 6. Our prediction for $\ln[S(\omega')/S_0]$ for $\gamma = 1$ and $\gamma = 2.18724$.

Finally, the prediction of our model compared to that of Ref. [16] is shown in Figure 7. One should remember that the prediction of Ref. [16] is based on a model that extends that of Refs. [10] and [14] by including both thermal and collisional effects in the plasma description. Notice, however, that when this prediction is compared to ours we get a quite good agrement, which means that the previous discrepancy between the previously cited papers is mainly due to the approximations introduced in Refs. [10] and [14] which were not necessary in our approach.

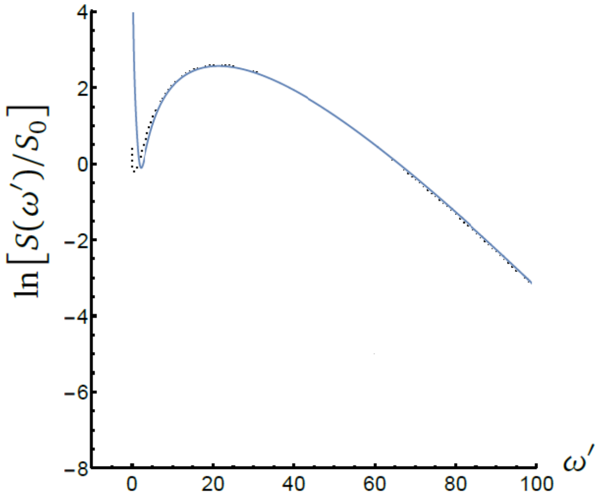

FIG. 7. Plot of the normalized magnetic field spectrum of Eq. (11) made by us (full line), compared to the plot given in Fig. 4.b of Ref. [16], both for $T = 7 \times 10^9$ K, $n_e = 4.6 \times 10^{30}$ cm^{-3} and $\lambda = 2\,444.4$ ($\gamma = 1$).

IV. SOME USEFUL LIMITS

A. The limit $\omega' \to 0$

Let us calculate now the limit $\omega' \to 0$, given by Eq. (11).

$$S_\circ\Big|_{\omega'\simeq 0} = \frac{1}{\pi^2}\left(\frac{\omega_{pe}}{c}\right)^3 \frac{\hbar\omega'}{1 + \dfrac{\hbar\omega_{pe}}{k_B T}\omega' - 1} \simeq \frac{1}{\pi^2}\frac{\omega_{pe}^2}{c^3} k_B T$$

Thus,

$$S(\omega')\Big|_{\omega'\simeq 0} = S_\circ\Big|_{\omega'\simeq 0} \times \left\{ \frac{2\lambda_c}{\eta'} + \underbrace{f(\omega')\Big|_{\omega'\simeq 0}}_{0} \times [\cdots] \right\}$$

or, finally, our prediction is

$$\boxed{S(\omega')\Big|_{\omega'\simeq 0} = \frac{2}{\pi^2}\frac{\omega_{pe}^2 k_B}{c^3\eta'}\lambda_c T}$$

This is exactly what Tajima has found ($k_B = 1$) [18]

$$\lim_{\omega\to 0} \frac{< B^2 >_\omega}{8\pi} = \frac{2}{\pi^2}\frac{\omega_{pe}^2}{c^3\eta'} x_{\rm cut}\, T$$

with

$$x_{\rm cut} = \frac{k_{\rm cut} c}{\omega_{pe}} \qquad \Rightarrow \qquad x_{\rm cut} \simeq 1$$

while for us

$$\frac{ck_{\max}}{\omega} = \frac{\lambda_c \omega_{pe}}{\omega} \qquad \Rightarrow \qquad \lambda_c = \frac{ck_{\max}}{\omega_{pe}}$$

which is the same factor.

B. The $\eta' \to 0$ limit of $S(\omega')$

Let us now determine the $\eta' \to 0$ limit of $S(\omega')$, given by Eq. (11). It is given by

$$S(\omega') = \frac{S_\circ}{\omega'^2} f(\omega')h(\omega')\sqrt{g(\omega') + h(\omega')}\Big|_{\eta'=0}$$

where,

$$\frac{S_\circ}{\omega'^2} = \frac{1}{\pi^2}\left(\frac{\hbar\omega'}{e^{(\hbar\omega_{pe}/k_B T)\omega'} - 1}\right)\left(\frac{\omega_{pe}}{c}\right)^3$$

$$f(\omega') \to \frac{\pi}{\sqrt{2}}\frac{\sqrt{\omega'}}{\omega'^3}$$

$$h(\omega') \to \omega'(\omega'^2 - 2)$$

$$g(\omega') \to \omega'\sqrt{\omega'^4 + 4 - 4\omega'^2} = \omega'(\omega'^2 - 2)$$

Thus,

$$S(\omega') \to \frac{1}{\pi^2}\left(\frac{\hbar\omega'}{e^{(\hbar\omega_{pe}/k_B T)\omega'} - 1}\right)\left(\frac{\omega_{pe}}{c}\right)^3 \frac{\pi}{\sqrt{2}}\frac{\sqrt{\omega'}}{\omega'^3}\,\omega'(\omega'^2 - 2)\sqrt{2\omega'(\omega'^2 - 2)}$$

or

$$\boxed{S(\omega') = \frac{1}{\pi}\left(\frac{\hbar(\omega'^2 - 2)^{3/2}}{e^{(\hbar\omega_{pe}/kT)\omega'} - 1}\right)\left(\frac{\omega_{pe}}{c}\right)^3} \tag{15}$$

This is exactly 2 times the second term of the principal formula of Tajima *et al*, which appears multiplied by the Heaviside function $\theta(\omega - \sqrt{c^2 k_{\rm cut} + \omega_p^2})$, which, for us, is just $\theta(\omega'^2 - 2)$ [19].

If $\omega' \gg \sqrt{2}$ (or in the limit $\omega_{pe} \to 0$), we get the well known Planck distribution [20]

$$S_{\rm Planck}(\omega') = \frac{1}{\pi}\left(\frac{\hbar\omega'^3}{e^{(\hbar\omega_{pe}/kT)\omega'} - 1}\right)\left(\frac{\omega_{pe}}{c}\right)^3$$

Thus, asymptotically, this result gives rise to the Stefan-Boltzmann law, $E_T \propto T^4$, if we integrate $S_{\rm Planck}(\omega')$ over ω'. However, fot the plasma, we have to integrate Eq. (15). It is clear that the contribution to this integral from the range $\sqrt{2} \le \omega' \le 10$, should yield a small deviation from this law. Let us now demonstrate it and determine its value.

V. DEVIATION FROM THE STEFAN-BOLTZMANN LAW

We have to solve the following integral to calculate the energy density of the cold plasma, E_T, with $S(\omega')$ given by Eq. (15):

$$E_T = \int_{\sqrt{2}}^{\infty} \frac{S(\omega)}{2\pi}\,\mathrm{d}\omega$$

Rewriting $S(\omega')$ as a function of ω, $S(\omega)$,

$$S(\omega) = \frac{1}{\pi}\left(\frac{\hbar(\omega^2 - 2\omega_{pe})^{3/2}}{e^{(\hbar\omega/kT)} - 1}\right)\left(\frac{1}{c}\right)^3$$

So,

$$E_T = \int_{\sqrt{2}}^{\infty} \frac{S(\omega)}{2\pi}\,\mathrm{d}\omega = \frac{\hbar}{2\pi^2 c^3} \int_{\sqrt{2}\omega_{pe}}^{\infty} \frac{(\omega^2 - 2\omega_{pe}^2)^{3/2}}{e^{\hbar\omega/k_B T}}\,\mathrm{d}\omega \tag{16}$$

Let us make

$$\omega = \frac{k_B T}{\hbar}\, z \quad \Rightarrow \quad \mathrm{d}\omega = \frac{k_B T}{\hbar}\mathrm{d}z$$

In terms of this new variable z,

$$E_T = \frac{1}{2\pi^2\hbar^3 c^3}\,(k_B T)^4 \underbrace{\int_a^{\infty} \frac{z^3(1 - a^2/z^2)^{3/2}}{e^z - 1}\,\mathrm{d}z}_{J} \tag{17}$$

where $a(T)$ is given by Eq. (14).

The integral of Eq. (17) can be numerically solved and we find $J = 6.42733$. For the values of T we are considering in the range $10^9 - 10^{10}$ K, $a^2 \simeq 0.0003$. So, we can made the approximation below, which can be numerically verified to be a good approximation. Indeed,

$$J \simeq \int_a^{\infty} \frac{(z^3 - 3a^2 z/2)}{e^z - 1}\,\mathrm{d}z = 6.42576$$

So, we have to compute two integrals:

$$J = \int_a^{\infty} \frac{z^3}{e^z - 1}\,\mathrm{d}z - \frac{3a^2}{2}\int_a^{\infty} \frac{z}{e^z - 1}\,\mathrm{d}z$$

It is convenient to have the integral from 0 to ∞ and, then, let us define $y = z - a$. With this change,

$$J = e^{-a}\left\{\int_0^{\infty} \frac{(y+a)^3}{e^y - e^{-a}}\,\mathrm{d}y - \frac{3a^2}{2}\int_0^{\infty} \frac{(y+a)}{e^y - e^{-a}}\,\mathrm{d}y\right\}$$

All those integrals are particular cases of the integral ([21], p. 354, Eq. (22)):

$$\int_0^{\infty} \frac{x^{p-1}}{e^{rx} - q}\,\mathrm{d}x = \frac{1}{qr^p}\Gamma(p)\sum_{k=1}^{\infty} \frac{q^k}{k^p} = \Gamma(p) r^{-p}\Phi(q,p,1)$$

if $[p > 0, r > 0, -1, q, 1]$, where Φ is the Lerch function ([21], p. 1039), and Γ is the usual gamma function. In our case, $r = 1$ and $q = e^{-a}$. Knowing this general result, we have to compute:

$$J = \underbrace{\int_0^{\infty} \frac{y^3 + 3ay^2 + 3a^2 y + a^3}{e^y - e^{-a}}\,\mathrm{d}y}_{J_1} + \underbrace{\int_0^{\infty} \frac{y + a}{e^y - e^{-a}}\,\mathrm{d}y}_{J_2}$$

$$J_2 = \Gamma(2)\Phi(e^{-a},2,1) + a\Phi(e^{-a},1,1)$$

and

$$J_1 = \Gamma(4)\Phi(e^{-a},4,1) + 3a\Gamma(3)\Phi(e^{-a},3,1) + 3a^2\Gamma(2)\Phi(e^{-a},2,1) + a^3\Phi(e^{-a},1,1)$$

But we know also that, in general,

$$\Phi(e^{-a},n,1) = \frac{\mathrm{Li}_n(e^{-a})}{e^{-a}}$$

where $Li_n(x)$ is the polylogarithm function.

Therefore, in terms of this function, $J = J_1 + J_2$ can be written as

$$J = 6\,\mathrm{Li}_4(e^{-a}) + 6a\,\mathrm{Li}_3(e^{-a}) + \frac{3}{2}a^2\,\mathrm{Li}_2(e^{-a}) - \frac{1}{2}a^3\,\mathrm{Li}_1(e^{-a}) \tag{18}$$

Knowing how a depends on T, Eq. (14), the above equation is the general expression for the T-dependence of our result given by Eq. (17), in the interval $2 \leq \omega' \leq 10$. This dependence was not discussed by Tajima. Note that the above equation gives the same numerical result previously found, *i.e.*, $J = 6.42576$. Therefore,

$$E_T = \frac{1}{2\pi^2\hbar^3c^3}\,(k_BT)^4 \; = 6\,\mathrm{Li}_4(e^{-a}) + 6a\,\mathrm{Li}_3(e^{-a}) + \frac{3}{2}a^2\,\mathrm{Li}_2(e^{-a}) - \frac{1}{2}a^3\,\mathrm{Li}_1(e^{-a})$$

or, in a more convenient formula,

$$E_T = \left(\frac{2\sigma}{c}\right)T^4 \times \frac{15}{\pi^4}\left[6\,\mathrm{Li}_4(e^{-a}) + 6a\,\mathrm{Li}_3(e^{-a}) + \frac{3}{2}a^2\,\mathrm{Li}_2(e^{-a}) - \frac{1}{2}a^3\,\mathrm{Li}_1(e^{-a})\right] \tag{19}$$

where we have introduced the usual Stefan-Boltzmann constant σ:

$$\sigma = \left(\frac{\pi^2k_B^4}{60\hbar^3c^2}\right) = 5.670 \times 10^{-5}\,\mathrm{erg}\cdot\mathrm{cm}^{-2}\cdot\mathrm{s}^{-1}\cdot\mathrm{K}^{-4}$$

To plot Eq. (19) we have used the expression for $a(T)$ given by Eq. (14).

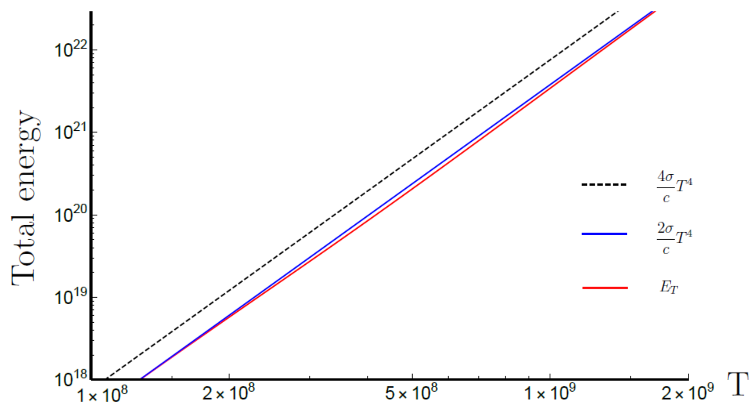

FIG. 8. Deviation from the Stefan-Boltzmann law.

Notice that the expression for plasma radiated energy, E_T (in red in the Fig. 8), is below the curve for the magnetic component of the black-body radiation (in blue), $(2\sigma/c)T^4$, for an intermediate region of temperature.

VI. DISCUSSIONS

In this paper, we have computed the spectrum of magnetic fluctuations of a homogeneous cosmic plasma avoiding any approximations. Several different behaviors between our results and the previous one obtained by [14], mainly in the low-frequency part of the spectrum, are found and discussed. It is important to stress that the exact result indicates that the peak of the zero-frequency spectrum is not so sensitively to the cut-off value λ_c, as shown in Fig. 2.

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[1] E. N. Parker, *Cosmical magnetic fields: Their origin and their activity*, 1st ed. (Oxford University Press, Oxford, 1979).

[2] M. J. Rees, The origin and cosmogonic implications of seed magnetic fields, Q. Jl R. astr. Soc. **28**, 197 (1987).
[3] G. Rüdiger and R. Hollerbach, *The Magnetic Universe. Geophysical and Astrophysical Dynamo Theory*, 1st ed. (Wiley VCH, Weinheim, 2004).
[4] R. M. Kulsrud and E. G. Zweibel, On the origin of cosmic magnetic fields, Reports on Progress in Physics **71**, 046901 (2008).
[5] L. Biermann, On the origin of magnetic fields on stars and in interstellar space, Z. Naturforsch. A **5**, 65 (1950).
[6] O. D. Miranda and R. Opher, The creation of large-scale voids by explosions of primordial supernovae, MNRAS **283**, 912 (1996).
[7] O. D. Miranda and R. Opher, Comparing single-cycle and multicycle explosive scenarios for the formation of voids, The Astrophysical Journal **482**, 573 (1997).
[8] O. D. Miranda, M. Opher, and R. Opher, Seed magnetic fields generated by primordial supernova explosions, Monthly Notices of the Royal Astronomical Society **301**, 547 (1998).
[9] N. Rostoker, R. Aamodt, and O. Eldridge, Kinetic theory of plasma and electromagnetic fields, Annals of Physics **31**, 243 (1965).
[10] T. Tajima, K. Cable, and R. M. Kulrsrud, On zero-frequency magnetic fluctuations in plasmas, Phys. Fluids B **4**, 2338 (1992).
[11] J. B. Johnson, Thermal agitation of electricity in conductors, Phys. Rev. **32**, 97 (1928).
[12] H. Nyquist, Thermal agitation of electric charge in conductors, Phys. Rev. **32**, 110 (1928).
[13] R. Kubo, Statistical–mechanical theory of irreversible processes. I. General theory and simple applications to magnetic and conduction problems, J. Phys. Soc. Jpn. **12**, 570 (1957).
[14] T. Tajima, S. Cable, K. Shibata, and R. M. Kulrsrud, On the origin of cosmological magnetic fields, ApJ **390**, 309 (1992).
[15] M. Opher and R. Opher, Was the electromagnetic spectrum a blackbody spectrum in the early universe?, Phys. Rev. Lett. **79**, 2628 (1997).
[16] M. Opher and R. Opher, Magnetic field spectrum in a plasma in thermal equilibrium in the epoch of primordial nucleosynthesis, Phys. Rev. D **56**, 3296 (1997).
[17] P. M. Bellan, *Fundamentals of Plasma Physics*, 1st ed. (Cambridge University Press, Cambridge, 2006).
[18] Eq. (19) of Ref. [10].
[19] Otherwise, the term $(\omega'^2 - 2)^{3/2}$ could be imaginary.
[20] We can see graphically that this is the case for $\omega' \geq 10$.
[21] I. M. Gradshteyn, I. S. & Ryzhik, *Table of Integrals, Series and Products*, seventh edition ed. (Elsevier, Amsterdam, 2007).