The octonionically-induced $\mathcal{N}=7$ exceptional $G(3)$ superconformal quantum mechanics

Francesco Toppan*

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* *CBPF, Rua Dr. Xavier Sigaud 150, Urca, cep 22290-180, Rio de Janeiro (RJ), Brazil.*

Abstract

Both the $\mathcal{N}=7$ superconformal quantum mechanics possessing the exceptional $G(3)$ Lie superalgebra as dynamical symmetry and its associated deformed oscillator with $G(3)$ as spectrum-generating superalgebra are presented.

This superconformal quantum mechanics, uniquely defined up to similarity transformations, is obtained via the octonionically-induced "quasi-nonassociative" method employed to derive the exceptional $\mathcal{N}=8$ $F(4)$ model.

To construct the $G(3)$ theories, the covariant embedding of the 7-dimensional representation of the Lie algebra g_2 within the 8×8 matrices spanning the $Cl(0,7)$ Clifford algebra is derived.

The Hilbert space of the $G(3)$ deformed oscillator is given by a 16-ple of square-integrable functions of a real space coordinate. The spectrum of the theory is computed.

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*E-mail: *toppan@cbpf.br*

1 Introduction

This work presents the construction of both the $\mathcal{N}=7$ superconformal quantum mechanics possessing the exceptional $G(3)$ Lie superalgebra as dynamical symmetry and of its associated (via the de Alfaro-Fubini-Furlan construction [1]) deformed oscillator with $G(3)$ as spectrum-generating superalgebra. This superconformal quantum mechanics, uniquely defined up to similarity transformations, is obtained via the octonionically-induced "quasi-nonassociative" method employed [2] to derive the exceptional $\mathcal{N}=8$ $F(4)$ superconformal quantum mechanics. Unlike the $F(4)$ theory, the $G(3)$ superconformal mechanics is inherently a quantum theory since the $\mathcal{N}=7$ worldline $(1,7,7,1)$ supermultiplet [3] (which carries a $G(3)$ representation [4]) does not admit a classical Lagrangian defining a world-line sigma-model. The Hilbert space of the $G(3)$ deformed oscillator is given by a 16-ple of square-integrable functions of the real space coordinate x. The spectrum of the deformed oscillator (discrete and bounded from below) is derived.

A consequence of the present construction is that both the exceptional Lie superalgebras $G(3)$ and $F(4)$ define their respective, unique, superconformal quantum mechanics. It should be stressed that, as a byproduct, a new differential matrix representation of $G(3)$ is obtained.

It is worth recalling that the 5 exceptional Lie algebras g_2, f_4, e_6, e_7, e_8, as well as the 2 exceptional Lie superalgebras $G(3)$, $F(4)$ in Kac's classification [5], are related to the octonions: g_2 is the Lie algebra of the group of automorphisms of the octonions, while f_4, e_6, e_7 and e_8 are recovered from the octonionic cases in the Freudenthal-Tits magic square construction; the octonionic construction of $G(3)$ and $F(4)$ was presented in [6].

The "quasi-nonassociativity" is based on the double role of the octonionic structure constants C_{ijk} which also enter seven 8×8 gamma matrices γ_i $(i=1,\ldots,7)$ defining the Euclidean $Cl(0,7)$ Clifford algebra. These matrices are induced by the left action of each one of the seven imaginary octonions over a real octonion, see [2] for details. Since the quasi-nonassociative derivation was presented in [2], it is sufficient here to pinpoint the differences between the $F(4)$ and the $G(3)$ constructions. In both cases 16×16 matrices with differential entries are required. The R-symmetry subalgebra of G_3 is g_2, since the $G(3)$ even subsector is decomposed as $sl(2)\oplus g_2$. The 14-generator exceptional Lie algebra g_2 admits a 7-dimensional representation. The key ingredient consists in expressing this representation in terms of the antisymmetric, covariant rank-2 tensors defined by the γ_i matrices. Once this is done (the construction is given in Section **2**), the 16×16 matrix differential representation of $G(3)$ is carried out by taking into account that the seven supercharges Q_i and their seven superconformal partners $\widetilde{Q}_i$ are labeled by the octonionic vector index i. This construction is presented in Section **3**. The deformed oscillator with $G(3)$ spectrum-generating superalgebra, the derivation of its Hilbert space from the $G(3)$ lowest weight reprentations and the computation of its spectrum are all given in Section **4**. A more detailed discussion of the results, together with future outlines, is presented in the Conclusions.

2 Octonionic covariance and the g_2 representation

The first part of this Section follows [2].

The octonionic multiplication is defined, for the seven imaginary octonions e_i ($i = 1, 2, \ldots, 7$), as

$$e_i e_j \;=\; -\delta_{ij} + C_{ijk} e_k \tag{1}$$

(here and in the following the sum over repeated indices is understood). The rank-3 totally antisymmetric tensor C_{ijk} defines the octonionic structure constants. Two more totally antisymmetric constant tensors are compatible with the octonionic multiplication; they are expressed as C_{ijkl} and $\epsilon_{ijklmnp}$ (their rank is 4 and 7, respectively). The following normalizations are assumed

$$\begin{gathered} C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1, \\ C_{4567} = C_{2356} = C_{2437} = C_{1357} = C_{1346} = C_{1276} = C_{1245} = 1, \\ \epsilon_{1234567} = 1. \end{gathered} \tag{2}$$

Due to the relation

$$6C_{ijkl} \;=\; \epsilon_{ijklmnp} C_{mnp}, \tag{3}$$

only two of the three constant, totally antisymmetric tensors are independent.

The left action of any given imaginary octonion e_i over a real octonion $x = x_0 + x_j e_j$ ($x_0, x_j \in \mathbb{R}$) induces a 8×8 matrix γ_i defined by the linear transformation

$$\vec{x}'_{(i)} \;=\; \gamma_i \vec{x} \quad \text{for} \quad x \mapsto e_i x = x'_{(i)} = -\delta_{ij} x_j + (x_0 \delta_{ik} + C_{ijk} x_j) e_k. \tag{4}$$

The 8 real numbers entering the real octonions x, $x'_{(i)}$ are arranged as 8-component vectors $\vec{x}$, $\vec{x}'_{(i)}$ so that, e.g., $\vec{x} = (x_0, x_j)^T$.

The seven matrices γ_i satisfy the $Cl(0,7)$ Euclidean Clifford algebra relations

$$\gamma_i \gamma_j + \gamma_j \gamma_i \;=\; -2\delta_{ij} \mathbb{I}_8, \qquad i, j = 1, 2, \ldots, 7 \tag{5}$$

(here and in the following $\mathbb{I}_n$ denotes the $n \times n$ identity matrix).

The γ_i entries are expressed in terms of the octonionic structure constants C_{ijk} through

$$(\gamma_i)_{LM} \;=\; \left(\begin{array}{c|c} 0 & \delta_{im} \\ \hline -\delta_{il} & C_{ilm} \end{array} \right), \tag{6}$$

where L, M take values $L = 0, l$ and $M = 0, m$, with $l, m = 1, 2, \ldots, 7$.
Up to an overall sign, the (6) matrices are obtained from the (4) map.

The totally antisymmetric constants C_{ijk} play a double role. They define the non-associative octonionic multiplication (1) (where, in particular, $(e_1 e_2) e_4 = e_3 e_4 = -e_5 \neq e_1 (e_2 e_4) = e_1 e_6 = e_5$) and they enter the matrix representation of the associative $Cl(0,7)$ Clifford algebra.

The $Cl(0,7)$ gamma matrices γ_i provide a basis to span the 64-dimensional vector space of 8×8 real matrices. The rank $r=0,1,2,3$ products of r different γ_i matrices are expressed as

$$\gamma^{(0)}\equiv \mathbb{I}_8, \quad \gamma^{(1)}\equiv \gamma_i, \quad \gamma^{(2)}\equiv \gamma_i\gamma_j \quad (i<j), \quad \gamma^{(3)}\equiv \gamma_i\gamma_j\gamma_k \quad (i<j<k). \tag{7}$$

Due to Hodge duality, the product of $7-r$ different matrices γ_i is equivalent to the product of matrices of rank r. One has that $\gamma^{(0)}$ and $\gamma^{(3)}$ provide the basis for the $1+35=36$ symmetric 8×8 matrices, while $\gamma^{(1)}$ and $\gamma^{(2)}$ provide the basis for the $7+21=28$ antisymmetric 8×8 matrices.

Up to now, this is the construction presented in [2]. The extra ingredients required for the construction of the $G(3)$ superconformal quantum mechanics are the following.

Scalars (rank-0 tensors), vectors (rank-1 tensors), rank-2 and rank-3 tensors are obtained by (partially) saturating the indices labeling the three constant totally antisymmetric tensors (2) with the γ_i matrices. These tensors provide a basis for the "octonionic covariant" matrices of given order. It is convenient to illustrate them by presenting the following table

$0A:$	$C_{ijk}\gamma_i\gamma_j\gamma_k$	1	S
$0B:$	$C_{ijkl}\gamma_i\gamma_j\gamma_k\gamma_l$	1	S
$0C:$	$\epsilon_{ijklmnp}\gamma_i\gamma_j\gamma_k\gamma_l\gamma_m\gamma_n\gamma_p$	1	S
$1A:$	$C_{ijk}\gamma_j\gamma_k$	7	A
$1B:$	$C_{ijkl}\gamma_j\gamma_k\gamma_l$	7	S
$1C:$	$\epsilon_{ijklmnp}\gamma_j\gamma_k\gamma_l\gamma_m\gamma_n\gamma_p$	7	A
$2A:$	$C_{ijk}\gamma_k$	7	A
$2B:$	$C_{ijkl}\gamma_k\gamma_l$	21	A
$2C:$	$\epsilon_{ijklmnp}\gamma_k\gamma_l\gamma_m\gamma_n\gamma_p$	21	A
$3A:$	$C_{ijk}\mathbb{I}_8$	1	S
$3B:$	$C_{ijkl}\gamma_l$	7	A
$3C:$	$\epsilon_{ijklmnp}\gamma_l\gamma_m\gamma_n\gamma_p$	35	S

(8)

The third column reports the number of spanning matrices of given type, while their symmetry (S) or antisymmetry (A) under matrix transposition is specified in the fourth column.

Different types of octonionic-covariant matrices do not necessarily determine different matrices. As an example, the $\mathbb{I}_8$ identity can be expressed either as the $0C$ scalar ($\mathbb{I}_8=\frac{1}{7!}\epsilon_{ijklmnp}\gamma_i\gamma_j\gamma_k\gamma_l\gamma_m\gamma_n\gamma_p$) or as the rank-3 tensor $3A$ ($C_{ijk}\mathbb{I}_8$). A relevant identification, due to Hodge duality, is

$$1C\equiv 2A \qquad (\epsilon_{ijklmnp}\gamma_j\gamma_k\gamma_l\gamma_m\gamma_n\gamma_p\sim C_{ijk}\gamma_k). \tag{9}$$

There are only two (up to normalization) octonionic-covariant scalar matrices. Besides the identity expressed by "$0C$", both $0A$ and $0B$ determine the diagonal matrix

$$0A\equiv 0B \quad \sim \quad diag(7,-1,-1,-1,-1,-1,-1,-1). \tag{10}$$

One should further note that the 7 antisymmetric matrices individuated by $1A$ differ from the 7 antisymmetric gamma matrices γ_i, covariantly expressed as $1C$, $2A$ or $3B$.

There are three types of octonionically-induced rank-1 (vector) matrices, the 7 symmetric matrices from $1B$, which will be denoted as "b_i", the 7 antisymmetric matrices γ_i and the 7 antisymmetric matrices from $1A$, which will be denoted as "m_i". One can set

$$\begin{aligned} b_i &= \frac{1}{24}C_{ijkl}\gamma_j\gamma_k\gamma_l, \qquad \text{so that} \quad (b_i)_{LM} = \begin{pmatrix} 0 & \delta_{im} \\ \delta_{il} & 0 \end{pmatrix}, \\ m_i &= \frac{1}{2}C_{ijk}\gamma_j\gamma_k. \end{aligned} \tag{11}$$

A more convenient basis to express the generic rank-1 antisymmetric matrices a_i is to present them as linear combinations of γ_i, n_i ($a_i = w_1\gamma_i + w_2 n_i$, with $w_1, w_2 \in \mathbb{R}$), where n_i is introduced as

$$n_i = \frac{1}{4}(m_i + 3\gamma_i), \qquad \text{so that} \quad (n_i)_{LM} = \begin{pmatrix} 0 & 0 \\ 0 & C_{ilm} \end{pmatrix}. \tag{12}$$

The matrices n_i are nonvanishing in the 7×7 lower-right block only.

The most general octonionically-induced rank-2 antisymmetric matrices a_{ij} are given by a linear combination of $2A$, $2B$ and $2C$. One can set

$$a_{ij} = z_1 C_{ijk}\gamma_k + z_2 C_{ijkl}\gamma_k\gamma_l + z_3\epsilon_{ijklmnp}\gamma_k\gamma_l\gamma_m\gamma_n\gamma_p, \qquad \text{with} \quad z_1, z_2, z_3 \in \mathbb{R}. \tag{13}$$

One restricts the coefficients z_1, z_2, z_3 by imposing two conditions. The first one is the request that the a_{ij} matrices are nonvanishing only in the 7×7 lower-right block. This condition is fulfilled if z_3 is constrained to satisfy

$$z_3 = -\frac{1}{120}z_1 - \frac{1}{30}z_2. \tag{14}$$

This condition leaves at most 21 linearly independent a_{ij} matrices.

The second condition comes from satisfying the covariant constraint

$$C_{ijk}a_{jk} = 0. \tag{15}$$

This condition is satisfied if z_1 is set to vanish:

$$z_1 = 0. \tag{16}$$

The (15) constraint implies 7 relations, leaving $21 - 7 = 14$ linearly independent matrices a_{ij}. These matrices realize, in their 7×7 lower-right block, the fundamental 7-dimensional representation of the 14 generators of the exceptional g_2 Lie algebra. By suitably setting the overall normalization, one can express the 14 linearly independent matrices as r_{ij}, defined as

$$r_{ij} = \frac{1}{2}C_{ijkl}\gamma_k\gamma_l - \frac{1}{60}\epsilon_{ijklmnp}\gamma_k\gamma_l\gamma_m\gamma_n\gamma_p. \tag{17}$$

The 7 rank-1 matrices n_i introduced in (12) can be expressed in the basis of the a_{qj} matrices satisfying the (14) condition. One gets

$$n_i = \frac{1}{6}C_{iqj}\cdot(\frac{3}{4}C_{qjk}\gamma_k - \frac{1}{8}C_{qjkl}\gamma_k\gamma_l - \frac{1}{480}\epsilon_{qjklmnp}\gamma_k\gamma_l\gamma_m\gamma_n\gamma_p). \tag{18}$$

The set of r_{ij} and n_i matrices produces the 21 linearly independent antisymmetric matrices with nonvanishing 7×7 lower-right block (and vanishing otherwise). Schematically, their commutation relations satisfy

$$[r,r]\sim r, \qquad [r,n]\sim n, \qquad [n,n]\sim r+n. \tag{19}$$

The matrices n_i belong to the 7-dimensional representation of the g_2 algebra. The whole set of matrices r_{ij}, n_i realize the 7-dimensional matrix representation of the $so(7)$ Lie algebra (while the commutators $[\gamma_i,\gamma_j]$ realize the 8-dimensional matrix representation of $so(7)$ since their first columns/rows are nonvanishing).

The following remark is worth to be stressed: the octonionically-induced covariant decomposition of matrices produces a nice embedding of the 7-dimensional matrix representations of both g_2 and $so(7)$ inside the 8-dimensional representation of the $Cl(0,7)$ Clifford algebra.

The explicit commutators among the r_{ij}, n_i matrices, schematically presented in (19), are covariantly written as

$$\begin{aligned}
[r_{ij},r_{kl}] &= a(\delta_{ik}r_{jl}-\delta_{il}r_{jk}-\delta_{jk}r_{il}+\delta_{jl}r_{ik})+\\
&\quad b(\delta_{ik}C_{jlmn}-\delta_{il}C_{jkmn}-\delta_{jk}C_{ilmn}+\delta_{jl}C_{ikmn})r_{mn}+\\
&\quad (3-a-\frac{b}{2})(C_{ijkm}r_{ml}-C_{ijlm}r_{mk}-C_{klim}r_{mj}+C_{kljm}r_{mi})+\\
&\quad (4-a-2b)C_{ijm}C_{kln}r_{mn},\\
[r_{ij},n_k] &= 4\delta_{ik}n_j-4\delta_{jk}n_i+2C_{ijkl}n_l,\\
[n_i,n_j] &= \frac{1}{2}r_{ij}+C_{ijk}n_k.
\end{aligned}\tag{20}$$

As a consequence of the $C_{ijk}r_{jk}=0$ constraint, on the right hand side of the first equation two real parameters a,b can be arbitrarily chosen since any pair of their selected values define the same generator. The first equation gives the octonionic-covariant expression of the structure constants of the g_2 Lie algebra. The whole set of three equations gives the structure constants of $so(7)$.

The second equation in (20) shows that the n_i's belong to a representation of g_2. The covariant rank-1 matrices γ_i, n_i, b_i satisfy the same 7-dimensional representation of g_2. Indeed, in all these cases one gets

$$[r_{ij},v_k] = 4\delta_{ik}v_j-4\delta_{jk}v_i+2C_{ijkl}v_l, \tag{21}$$

where the v_i's can be respectively replaced by γ_i, n_i or b_i.

3 The $G(3)$ superconformal quantum mechanics

The exceptional, finite, $G(3)$ Lie superalgebra can be interpreted, see [4], as a one-dimensional superconformal algebra with $\mathcal{N}=7$ extension. $G(3)$ admits a 5-grading decomposition, given by

$$G(3) = \mathcal{G}_{-1}\oplus\mathcal{G}_{-\frac{1}{2}}\oplus\mathcal{G}_0\oplus\mathcal{G}_{\frac{1}{2}}\oplus\mathcal{G}_1. \tag{22}$$

The half-integer sectors $\mathcal{G}_{\pm\frac{1}{2}}$ are odd; the integer sectors $\mathcal{G}_0$, $\mathcal{G}_{\pm 1}$ are even.
The (anti)commutators (collectively denoted as "$[\cdot,\cdot\}$" when needed) respect the above decomposition, so that

$$[\mathcal{G}_{s_1},\mathcal{G}_{s_2}\} \quad\subset\quad \mathcal{G}_{s_1+s_2}, \qquad \text{for} \quad s_1,s_2=0,\pm\frac{1}{2},\pm 1. \tag{23}$$

Each $\mathcal{G}_{\pm 1}$ sector is spanned by a single generator, respectively denoted as "H,K", with $H\in\mathcal{G}_1$ and $K\in\mathcal{G}_{-1}$. The $\mathcal{G}_0$ sector is a subalgebra, given by the direct sum

$$\mathcal{G}_0 \quad=\quad u(1)\oplus g_2. \tag{24}$$

In application to superconformal mechanics, the exceptional Lie algebra g_2 is known as the "R-symmetry". The $u(1)$ generator is denoted as "D". It corresponds to the dilatation operator. The set of D,H,K generators close an $sl(2)$ subalgebra, with D as the Cartan element and H (K) as the positive (negative) root.

The 5-grading decomposition (22) is related to the scaling dimension of the generators, defined by their commutators with D. Indeed, one gets

$$[D,Z_s]=isZ_s, \qquad \forall Z_s\in\mathcal{G}_s. \tag{25}$$

The odd sector $\mathcal{G}_{\frac{1}{2}}$ is spanned by seven supercharges, denoted as "Q_i" $(i=1,2,\ldots,7)$, while the $\mathcal{G}_{-\frac{1}{2}}$ sector is spanned by their 7 conformal superpartners, denoted as "$\widetilde{Q}_i$".

The positive sector $\mathcal{G}_{>0}=\mathcal{G}_{\frac{1}{2}}\oplus\mathcal{G}_1$ is isomorphic to the $\mathcal{N}=7$-extended worldline superalgebra (see [7]), defined by the (anti)commutators

$$\{Q_i,Q_j\}=2\delta_{ij}H, \qquad [H,Q_i]=0. \tag{26}$$

The nonvanishing (anti)commutators of $G(3)$ are presented for completeness; due to the octonionic covariance the structure constants are expressed, as in formula (20), in terms of the constant octonionic tensors. In this presentation the g_2 subalgebra generators are given in terms of the R_{ij} antisymmetric tensor satisfying the constraints

$$C_{ijk}R_{jk} \quad=\quad 0. \tag{27}$$

The nonvanishing (anti)commutators are

$$\begin{aligned}
&[D,H]=iH, \qquad [D,K]=-iK, \qquad [D,Q_i]=\frac{i}{2}Q_i, \qquad [D,\widetilde{Q}_i]=-\frac{i}{2}\widetilde{Q}_i,\\
&[H,K]=-2iD, \qquad [H,\widetilde{Q}_i]=-iQ_i, \qquad [K,Q_i]=i\widetilde{Q}_i,\\
&[R_{ij},R_{kl}]=\frac{3i}{4}(C_{ijkm}r_{ml}-C_{ijlm}R_{mk}-C_{klim}R_{mj}+C_{kljm}R_{mi})+iC_{ijm}C_{kln}R_{mn},\\
&[R_{ij},Q_k]=i(\delta_{ik}Q_j-\delta_{jk}Q_i)+\frac{i}{2}C_{ijkl}Q_l, \qquad [R_{ij},\widetilde{Q}_k]=i(\delta_{ik}\widetilde{Q}_j-\delta_{jk}\widetilde{Q}_i)+\frac{i}{2}C_{ijkl}\widetilde{Q}_l,\\
&\{Q_i,Q_j\}=2\delta_{ij}H, \qquad \{\widetilde{Q}_i,\widetilde{Q}_j\}=2\delta_{ij}K, \qquad \{Q_i,\widetilde{Q}_j\}=2D\delta_{ij}+R_{ij}.
\end{aligned} \tag{28}$$

The structure constants of the g_2 subalgebra are here presented by setting equal to zero the parameters a,b entering the right hand side of the first equation in (20): $a=b=0$. Please note that the g_2 generators are now given in capitalized form (R_{ij} instead of r_{ij}).

3.1 The differential representation

The construction of the superconformal quantum mechanics with $G(3)$ as dynamical symmetry requires the following steps. One introduces the space coordinate x and its associated derivative ∂_x with assigned scaling dimensions

$$[x] = -\frac{1}{2}, \qquad [\partial_x] = \frac{1}{2}. \tag{29}$$

The generators entering (28) are realized by Hermitian differential operators (since no confusion arises, the same symbol will be used to denote a $G(3)$ generator and its differential representative). As customary, the fermionic operators Q_i, $\widetilde{Q}_i$ are block-antidiagonal. For this reason the size of the matrices introduced in Section **2** has to be doubled. Therefore, the differential representation is given by 16×16 matrices with differential entries. The superconformal Hamiltonian is given by $H \in \mathcal{G}_1$. Its $s = 1$ scaling property implies that it is a diagonal operator with an inverse square potential. With a suitable normalization of its Laplacian term, H is of the form

$$H = -\frac{1}{2}{\partial_x}^2 \cdot \mathbb{I}_{16} + \frac{1}{x^2}V, \qquad V = diag(V_0, V_1, \ldots, V_{15}), \tag{30}$$

where V is a diagonal matrix whose V_I $(I = 0, \ldots, 15)$ real entries have to be determined.

The conformal partner of the Hamitonian is the operator $K \in \mathcal{G}_{-1}$. It is expressed by the quadratic term

$$K = \frac{1}{2}x^2 \cdot \mathbb{I}_{16}. \tag{31}$$

The dilatation operator D, obtained from the $[H, K]$ commutator, is

$$D = -\frac{i}{2}(x\partial_x + \frac{1}{2}) \cdot \mathbb{I}_{16}. \tag{32}$$

The remaining 16×16 matrix differential operators are constructed in terms of the 8×8 matrices γ_i, b_i, n_i, r_{ij} introduced in (6), (11), (12) and (17), respectively. The following 2×2 matrices are further introduced to produce 16×16 matrices as tensor products. It is convenient to set

$$\mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{33}$$

In octonionic-covariant notation, the most general 16-dimensional representations of g_2 can be expressed, in convenient normalization, either as the Hermitian matrices

$$R_{ij} = \frac{i}{4}\mathbb{I}_2 \otimes r_{ij}, \tag{34}$$

or as the Hermitian matrices

$$R^{\pm}_{ij} = \frac{i}{8}(\mathbb{I}_2 \pm X) \otimes r_{ij} \tag{35}$$

(in the latter case $R_{ij}^{+} \leftrightarrow R_{ij}^{-}$ are mutually recovered via a similarity transformation).
The differential representation of $G(3)$ forces the g_2 subalgebra to be represented by (34), while the (35) solutions have to be discarded because they do not allow the Q_i, $\widetilde{Q}_i$ operators introduced below to be in a representation of g_2.

The most general block-antidiagonal, Hermitian operators of scaling dimension $s = \frac{1}{2}$ ($s = -\frac{1}{2}$) and carrying a vector index i are expressed as Q_i ($\widetilde{Q}_i$); they are given by

$$Q_i = -\frac{i}{\sqrt{2}}\left(A_i\partial_x + \frac{B_i}{x}\right), \qquad \widetilde{Q}_i = \frac{x}{\sqrt{2}}C_i, \tag{36}$$

where the matrices A_i, B_i are respectively given by the linear combinations

$$\begin{array}{rcl} A_i &=& k_1 A\otimes\gamma_i + k_2 A\otimes n_i + k_3 Y\otimes b_i + N_F\cdot(k_4 Y\otimes\gamma_i + k_5 Y\otimes n_i + k_6 A\otimes b_i),\\ B_i &=& \widetilde{k}_1 Y\otimes\gamma_i + \widetilde{k}_2 Y\otimes n_i + \widetilde{k}_3 A\otimes b_i + N_F\cdot(\widetilde{k}_4 A\otimes\gamma_i + \widetilde{k}_5 A\otimes n_i + \widetilde{k}_6 Y\otimes b_i). \end{array} \tag{37}$$

The matrix N_F is the fermion parity operator. It is defined as

$$N_F = X\otimes\mathbb{I}_8. \tag{38}$$

The real coefficients $k_1,\ldots,k_6$ and $\widetilde{k}_1,\ldots,\widetilde{k}_6$ have to be determined by requiring the closure of the (28) (anti)-commutators.

The matrices C_i entering the second equation of (36) are expressed by the same linear combinations as the matrices A_i. Without loss of generality the requirement from (28) that, at given i, the anticommutator $\{Q_i, \widetilde{Q}_i\}$ would be proportional to the dilatation operator D, implies that one can set

$$C_i = A_i. \tag{39}$$

The matrices A_i, B_i must fulfill the following conditions

$$\begin{array}{rcl} \{A_i, A_j\} &=& 2\delta_{ij}\mathbb{I}_{16},\\ \{A_i, B_j\} + \{A_j, B_i\} &=& 0,\\ \{B_i, B_j\} - A_iB_j - A_jB_i &=& 0, \end{array} \tag{40}$$

resulting from the closure of the worldline superalgebra (26) with the identification of the Hamiltonian H given in (30). This identification further implies that

$$V = -\frac{1}{2}(B_i^2 - A_iB_i) \qquad \text{for any given} \quad i = 1, 2, \ldots, 7. \tag{41}$$

The anticommutators $\{Q_i, \widetilde{Q}_j\}$, for $i \neq j$, should produce antisymmetric matrices proportional to the g_2 R-symmetry generators R_{ij} from (34), so that

$$A_iA_j + B_iA_j + A_jB_i \;\propto\; R_{ij}. \tag{42}$$

Solving the constraints (40,41,42) guarantees the closure of the $G(3)$ superconformal algebra. The solutions are obtained with the following steps.

The first equation in (40) implies that k_4, k_5, k_6 have to be set to

$$k_4 = \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2 - 2k_1), \quad k_5 = \tfrac{1}{2}(-\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 - 2k_2), \quad k_6 = \tfrac{1}{2}(-2k_3 + \varepsilon_1 + \varepsilon_2), \tag{43}$$

where ε_a are three independent signs ($\varepsilon_a = \pm 1$ for $a = 1,2,3$), while k_1, k_2, k_3 remain arbitrary real numbers.

The second equation in (40) is automatically satisfied, while relations for $\widetilde{k}_i$'s are obtained from the third equation in (40) and from (41,42). The linearity of the (42) constraint makes more convenient to solve it first. Then, the two other conditions unambiguously fix all remaining $\widetilde{k}_i$'s.

This analysis has to be repeated for each one of the 8 different cases corresponding to the 3 sign assignments of the ε_a's. It is easily checked that in all these cases the same diagonal matrix V, defining the potential term of the (30) Hamiltonian, is recovered. This means that V does not depend on the arbitrary choices of the real parameters k_1, k_2, k_3 and of the signs $\varepsilon_1, \varepsilon_2, \varepsilon_3$. Therefore, the superconformal Hamiltonian H is uniquely determined. It is given by

$$H = -\frac{1}{2}{\partial_x}^2 \cdot \mathbb{I}_{16} + \frac{1}{x^2}V, \qquad V = diag(V_0, V_1, \ldots, V_{15}), \quad \text{with}$$
$$V_0 = V_8 = 1, \quad V_i = V_{8+i} = 0 \quad \text{for} \quad i = 1, \ldots, 7. \tag{44}$$

The most suitable presentation of the $G(3)$ differential matrix representation is obtained by setting

$$k_1 = 1, \qquad k_2 = k_3 = k_4 = k_5 = k_6 = 0 \qquad (\text{therefore} \quad \varepsilon_1 = -\varepsilon_2 = \varepsilon_3 = 1). \tag{45}$$

With this choice of parameters one gets

$$Q_i = -\frac{i}{\sqrt{2}}\left(A \otimes \gamma_i \partial_x - \frac{A \otimes b_i}{x}\right) \tag{46}$$

and

$$\widetilde{Q}_i = \frac{x}{\sqrt{2}} A \otimes \gamma_i. \tag{47}$$

The operators $H, K, D, R_{ij}, Q_i, \widetilde{Q}_i$, respectively introduced in (44,31,32,34,46,47), close the $G(3)$ superalgebra (anti)commutators (28).

The coupling constants of the square inverse potential are expressed by the entries of the diagonal matrix V presented in (44). In the octonionic-covariant formalism, V is given by the scalar combination

$$V = \frac{1}{8}\mathbb{I}_{16} - \frac{1}{48}C_{ijk}\Gamma_i\Gamma_j\Gamma_k\Gamma_8\Gamma_9, \quad \text{with} \quad \Gamma_i = A \otimes \gamma_i, \quad \Gamma_8 = Y \otimes \mathbb{I}_8, \quad \Gamma_9 = X \otimes \mathbb{I}_8. \tag{48}$$

The supertrace of V is vanishing:

$$str(V) = 0. \tag{49}$$

4 The $G(3)$ deformed oscillator

Following the de Alfaro-Fubini-Furlan [1] construction, the deformed matrix oscillator with $G(3)$ as spectrum-generating superalgebra is given by the Hamiltonian H_{osc},

$$H_{osc} = H + K, \tag{50}$$

with H, K respectively introduced in (44), (31). Therefore

$$H_{osc} = -\tfrac{1}{2}{\partial_x}^2 \cdot \mathbb{I}_{16} + \tfrac{1}{x^2}V + \tfrac{1}{2}x^2 \cdot \mathbb{I}_{16},$$
$$\text{with} \quad V = diag(V_0, V_1, \ldots, V_{15}), \quad V_0 = V_8 = 1, \quad V_i = V_{8+i} = 0 \quad \text{for} \quad i = 1, \ldots, 7. \tag{51}$$

This matrix Hamiltonian corresponds to the direct sum of 14 undeformed oscillators plus 2 oscillators which are deformed by the presence of the extra $\frac{1}{x^2}$ potential term.

7 pairs of deformed creation (a_i^+) and annihilation (a_i^-) operators are introduced through the positions

$$a_i^\pm = \widetilde{Q}_i \mp iQ_i = \frac{1}{\sqrt{2}}\left(A \otimes \gamma_i(\mp\partial_x + x) \pm \frac{A \otimes b_i}{x}\right). \tag{52}$$

They satisfy the commutation relations

$$[H_{osc}, a_i^\pm] = \pm a_i^\pm. \tag{53}$$

For any given i the H_{osc} Hamiltonian is expressed by the anticommutators (no summation over the repeated indices)

$$H_{osc} = \frac{1}{2}\{a_i^+, a_i^-\}. \tag{54}$$

For any given i the commutator of the creation/annihilation operators produces a deformed (due to the presence on the right hand side of a Klein operator) Heisenberg algebra:

$$[a_i^+, a_i^-] = \mathbb{I}_{16} + 2S_i, \tag{55}$$

where S_i is a diagonal matrix. For any $i = 1, \ldots, 7$, the matrix S_i is a Klein operator since it satisfies

$${S_i}^2 = \mathbb{I}_{16}, \qquad \{S_i, a_i^+\} = \{S_i, a_i^-\} = 0. \tag{56}$$

Explicitly, one has

$$S_i = diag(s_0, \ldots, s_{15}), \quad \text{with} \quad s_0 = s_8 = -1, \quad s_j = s_{8+j} = \delta_{ij} \quad \text{for} \quad j = 1, \ldots, 7. \tag{57}$$

Each one of the seven annihilation operators a_i^- defines 16 lowest weight vectors Ψ_{lwv} as solutions of the equation

$$a_i^- \Psi_{lwv} = 0. \tag{58}$$

The sixteen solutions of (58) are denoted as $\Psi_i^{(J)}$ $(J = 0, 1, \ldots, 15)$; only the J-th component of the $\Psi_i^{(J)}$ vector is nonvanishing. The lowest weight vectors are given by

$$\Psi_i^{(J)} \;=\; (\Psi_{i,0}^{(J)}, \Psi_{i,1}^{(J)}, \ldots, \Psi_{i,15}^{(J)})^T, \quad \text{with} \quad \Psi_{i,K}^{(J)} = \delta_{JK}\psi_{i,K} \tag{59}$$

and, up to a proportionality factor,

$$\psi_{i,0} = \psi_{i,8} = \tfrac{1}{x}e^{-\frac{1}{2}x^2}, \quad \psi_{i,i} = \psi_{i,8+i} = xe^{-\frac{1}{2}x^2}, \quad \psi_{i,K} = e^{-\frac{1}{2}x^2} \quad (K \neq 0, 8, i, 8+i). \tag{60}$$

It follows in particular that, irrespective of the value of $i = 1, \ldots, 7$, the vectors $\Psi_i^{(0)}$ denote the same wave function (similarly, independently of i, the vectors $\Psi_i^{(8)}$ denote another unique wave function).

Each lowest weight vector $\Psi_i^{(J)}$ defines a lowest weight representation spanned by the vectors $(a_1^+)^{n_1}(a_2^+)^{n_2}\ldots(a_7^+)^{n_7}\Psi_i^{(J)}$, with $n_1, \ldots, n_7$ arbitrary non-negative integers. The Hilbert space of the model is given by a direct sum of those lowest weight representations which allow normalized wave functions. The selection of the admissible lowest representations proceeds as follows.

At first one has to notice that $\psi_{i,0}$, $\psi_{i,8}$ given in (60) do not produce normalized square-integrable functions due to the singular integration $\sim \int dx \frac{1}{x^2}$ at the origin. Therefore, the lowest weight representations induced by the lowest weight vectors $\Psi_i^{(0)}$ and $\Psi_i^{(8)}$ are not admissible. A further analysis similarly proves that all lowest weight representations induced by the gaussian lowest weight vectors $\psi_{i,K}$ with $K \neq 0, 8, i, 8+i$, are also not admissible. Indeed, each such a gaussian wave function produces $\Psi_i^{(0)}$, $\Psi_i^{(8)}$ as descendant states via the application of some creation operator a_j^+. Let's take, as an example, $\Psi_2^{(1)}$, leading to the gaussian function $\psi_{2,1}$ for $i = 2$, $K = J = 1$. A straightforward check shows that

$$a_1^+\Psi_2^{(1)} \;\propto\; \Psi_i^{(8)} \quad \forall i = 1, \ldots, 7. \tag{61}$$

A similar construction applies to all gaussian functions entering (60). Their lowest weight representations under the action of the 7 creation operators a_j^+ produce non normalizable wave functions and, therefore, cannot be used to construct a Hilbert space.

It turns out that the Hilbert space can be defined to be the direct sum of the 14 lowest weight vectors $\Psi_i^{(i)}$, $\Psi_i^{(8+i)}$ whose component wave functions are the odd-parity (under $x \leftrightarrow -x$ transformation) functions $xe^{-\frac{1}{2}x^2}$ entering (60).

These lowest weight vectors are energy eigenstates with common (degenerate) energy eigenvalue $E = \frac{3}{2}$.

The diagonal fermion parity operator N_F introduced in (38) defines bosonic (fermionic) states as its $+1$ (-1) eigenspaces. Since N_F commutes with the Hamiltonian H_{osc},

$$[N_F, H_{osc}] \;=\; 0, \tag{62}$$

it can be used to introduce a superselected Hilbert space $\mathcal{H}$. The normalizable vectors $\Psi \in \mathcal{H}$ satisfy the superselection condition

$$P\Psi \;=\; \Psi, \tag{63}$$

where P is the projection operator ($P^2 = P$) given by

$$P = N_F \cdot e^{\pi i (H_{osc} - \frac{3}{2})}. \tag{64}$$

The superselected Hilbert space admits a bosonic, 7 times degenerate, vacuum defined by the states $\Psi_i^{(i)}$ with $i = 1, \ldots, 7$. It is convenient, for simplicity, to denote the vacuum states as $|\Psi_i\rangle = \Psi_i^{(i)}$.

The energy spectrum of the theory is given by the eigenvalues

$$E_n = \frac{3}{2} + n, \qquad n \in \mathbb{N}_0, \tag{65}$$

The excited states ($n > 0$) are obtained by applying the creation operators a_j^+ which, by construction, anticommute with the fermion parity operator,

$$\{N_F, a_j^+\} = 0. \tag{66}$$

Accordingly, the eigenstates of energy level E_n are bosonic (fermionic) if n is even (odd).

The 7 creation operators a_j^+ satisfy the "soft supersymmetry version", see [8], of the $\mathcal{N} = 7$ worldline superalgebra (26):

$$\{a_i^+, a_j^+\} = 2\delta_{ij} Z, \qquad [Z, a_i^+] = 0, \qquad i, j = 1, \ldots, 7, \tag{67}$$

with Z given by

$$Z = -H + K - 2iD, \tag{68}$$

for H, K, D respectively introduced in (44,31,32). The notion of soft supersymmetry refers to the fact that the operator Z is not a Hamiltonian, but a raising operator.

The determination of the degeneracy of the spectrum of the excited states proceeds similarly as for the $F(4)$ [2] oscillator. At the first ($n = 1$) excited level, $7 \times 7 = 49$ excited states $a_i^+|\Psi_j\rangle$ can be written. The 7 cases corresponding to $i = j$ produce the same eigenstate. The 42 remaining cases with $i \neq j$ determine a total number of 7 inequivalent eigenstates (each one obtained from 6 different combinations; as an example, up to normalization, the same eigenstate is expressed as $a_1^+|\Psi_2\rangle$, $a_2^+|\Psi_1\rangle$, $a_4^+|\Psi_5\rangle$, $a_5^+|\Psi_4\rangle$, $a_6^+|\Psi_7\rangle$ or $a_7^+|\Psi_6\rangle$). This leaves a total number of 8 fermionic eigenstates of energy $E = \frac{5}{2}$.

The construction gets repeated at each excited level producing 8 degenerate states at any given integer $n > 0$. The semi-infinite $(7; 8; 8; 8; \ldots)$ tower of energy eigenstates is a consequence of the $\mathcal{N} = 7$ $(7, 8, 1)$ worldline supermultiplet [3] applied to the soft superalgebra (67).

Let's summarize these results: the vacuum energy E_{vac}, the excited energy eigenstates E_n and their respective degeneracies $d(E_{vac})$, $d(E_n)$ are

$$E_{vac} = \tfrac{3}{2}, \quad d(E_{vac}) = 7, \quad E_n = E_{vac} + n, \quad d(E_n) = 8, \quad n = 1, 2, \ldots. \tag{69}$$

The 7 degenerate, bosonic, normalized ground energy wave functions $|\Psi_i\rangle$ are

$$|\Psi_i\rangle = (\overline{\psi}_{i,0}, \overline{\psi}_{i,1}, \ldots, \overline{\psi}_{i,15})^T, \tag{70}$$

where

$$\overline{\psi}_{i,K} \;=\; \delta_{iK}\frac{2^{\frac{1}{2}}}{\pi^{\frac{1}{4}}}xe^{-\frac{1}{2}x^2}, \quad \text{for} \quad i=1,\ldots,7, \quad K=0,1,\ldots,15. \tag{71}$$

A convenient (unnormalized) presentation for the 8 distinct fermionic wave functions $\Psi_I^{[1]}$, where $I=0,1,\ldots,7$, of the first ($n=1$, $E=\frac{5}{2}$) excited level is given by

$$\begin{array}{l}\Psi_0^{[1]}=a_1^+|\Psi_1\rangle,\;\Psi_1^{[1]}=a_2^+|\Psi_3\rangle,\;\Psi_2^{[1]}=a_3^+|\Psi_1\rangle,\;\Psi_3^{[1]}=a_1^+|\Psi_2\rangle,\\ \Psi_4^{[1]}=a_7^+|\Psi_1\rangle,\;\Psi_5^{[1]}=a_1^+|\Psi_6\rangle,\;\Psi_6^{[1]}=a_5^+|\Psi_1\rangle,\;\Psi_7^{[1]}=a_1^+|\Psi_4\rangle.\end{array} \tag{72}$$

The unique nonvanishing component wave function of $\Psi_0^{[1]}$ is at the $K=8$ position and proportional to $x^2e^{-\frac{1}{2}x^2}$, while the unique nonvanishing component wavefunction of $\Psi_i^{[1]}$ is at the $K=8+i$ position and proportional to $(x^2-1)e^{-\frac{1}{2}x^2}$.

The (unnormalized) 8 distinct wave-functions $\Psi_I^{[n+1]}$ of the $(n+1)$-th excited level, for $n\geq 1$, can be expressed via the recursive formula

$$\Psi_0^{[n+1]}=a_1^+\Psi_1^{[n]}, \qquad \Psi_i^{[n+1]}=a_i^+\Psi_0^{[n]} \quad \text{for} \quad i=1,2,\ldots,7. \tag{73}$$

It is worth to compare the $G(3)$ energy spectrum with the one obtained from the exceptional $\mathcal{N}=8$ $F(4)$ deformed oscillator. The $\mathcal{N}=7$ $G(3)$ case corresponds to a shifted version with the same degeneracy of the vacuum energy and of the excited states, but with different vacuum energy. The vacuum energy of the $F(4)$ model is $E_{vac}^{F(4)}=\frac{2}{3}$.

Another difference with respect to the $F(4)$ case concerns the component wave functions. Some of the component wave functions of the $F(4)$ oscillator are necessarily singular (but square normalizable) at the origin. Therefore, they require the [9, 10] framework of square integrable functions on the real line (which extends the [11] and [1] quantization prescribing wavefunctions defined on the $x\geq 0$ half-line and satisfying the Dirichlet boundary condition at the origin). All component wave functions of the $G(3)$ oscillator are regular on the real line, including the origin.

5 Conclusions

Unlike its $F(4)$ counterpart, the $G(3)$ superconformal quantum mechanics does not arise as a quantization of a classical action. The reason is the following. The $F(4)$ model [2] is a quantization of a world-line sigma model formulated in the classical Lagrangian setting. Indeed, the scale-invariant restriction [12] of the global $\mathcal{N}=8$-invariant sigma model [3] based on the $(1,8,7)$ supermultiplet gives a classical theory which possesses $F(4)$ as dynamical symmetry. On the other hand, the analysis in [4] proves that $G(3)$ is classically realized as a D-module representation on a long $(1,7,7,1)$ supermultiplet; this supermultiplet presents fields of four different scaling dimensions and, in particular, a fermionic auxiliary field. A simple argument shows that the presence of this fermionic auxiliary field prevents the construction of a non-trivial invariant classical action with no higher-derivative terms.

On a technical note, the construction of the $G(3)$ superconformal quantum mechanics (and its associated deformed oscillator) is only made possible by the preliminary embedding of the 7-dimensional representation of the g_2 subalgebra within the 8×8 matrices spanning the $Cl(0,7)$ Clifford algebra. A detailed presentation of this embedding, within the octonionic-covariant framework, was given in Section **2**.

The method of the octonionically-induced representations, discussed here and in [2], implies the so-called "quasi-nonassociativity", that is the restriction obtained on the moduli space of the coupling constants as a consequence of the non-associative structure. It gets reflected, e.g., in the determination of V in (48) from the octonionic structure constants. As a consequence, at these critical values, an emergent exceptional dynamical symmetry appears.

This work concludes the construction of superconformal quantum mechanical models with exceptional finite Lie superalgebras as dynamical symmetry. What is next? There is no reason to limit the application of the method of octonionically-induced representations to finite algebras. In [13] the so-called "Non-associative" $\mathcal{N}=8$ Superconformal algebra as an $\mathcal{N}=8$ extension of the Virasoro algebra was introduced. The term "non-associative" here refers to the property that the graded Jacobi identities are not satisfied. This feature allows to overcome the constraints on the existence of non-trivial central extensions for Virasoro algebras, which are only allowed up to $\mathcal{N} \leq 4$ (see the classifications given in [14] and [15]) if the graded Jacobi identities are assumed. The [13] $\mathcal{N}=8$ Non-associative Superconformal Algebra is recovered [16] via a Sugawara construction of an octonionic $\mathcal{N}=8$ superaffine algebra of Mal'cev type. It is therefore a natural candidate to explore the consequences of the octonionically-induced representations in an infinite dimensional setting.

Probably, the most promising application of octonionically-induced representations is in connection with the octonionic M-algebra introduced in [17]. Based on the [18] octonionic realizations of Clifford gamma matrices, it gives surprising features like a 5-brane sector which is no longer independent from the particle and 2-brane sectors as in ordinary M-algebra. Its bosonic subalgebra consists of 4×4 octonionic-valued Hermitian matrices. This poses problems for the consistency of its quantization because this structure does not satisfy the Jordan algebra's axioms. It is worth recalling that the only genuine non-associative system which is quantized within the Jordan's framework is the algebra of 3×3 Hermitian octonionic matrices introduced in [19]. A detailed analysis of the quantization of this model was given in [20]. Interestingly, at the end of decade of 1960s, P. Jordan himself investigated the possibility of maintaining a consistent quantization even when Jordan's axioms are relaxed. In this context, the example of the 4×4 Hermitian octonionic matrices was investigated as a toy model (see [21] for an historical account of this attempt). The method of octonionically-induced representations can offer an alternative approach to derive a consistent quantization of the 4×4 Hermitian octonionic matrices and of the octonionic M-algebra. It is a completely uncharted territory which deserves unravelling.

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