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Target duality in $\mathcal{N}=8$ superconformal mechanics and the coupling of dual pairs

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Abstract

We couple dual pairs of $\mathcal{N}=8$ superconformal mechanics with conical targets of dimension d and $8-d$. The superconformal coupling generates an oscillator-type potential on each of the two target factors, with a frequency depending on the respective dual coordinates. In the case of the inhomogeneous (3,8,5) model, which entails a monopole background, it is necessary to add an extra supermultiplet of constants for half of the supersymmetry. The $\mathcal{N}=4$ analog, joining an inhomogeneous (1,4,3) with a (3,4,1) multiplet, is also analyzed in detail.

1 Introduction and summary

$\mathcal{N}$-extended superconformal mechanics (for a review, see [1]) is defined on off-shell supermultiplets containing propagating bosons, fermions and auxiliary fields and, following the conventions of [2], being denoted by $(d,\mathcal{N},\mathcal{N}-d)$. Their associated invariant actions define one-dimensional sigma models with a d-dimensional conical target manifold. The case of $\mathcal{N}$=8 has been studied less extensively than those with $\mathcal{N}$=2 or $\mathcal{N}$=4. However, in the literature one finds invariant actions for the supermultiplets (1,8,7) [3, 4], (3,8,5) [5, 6] and (5,8,3) [5, 7]. The (2,8,6) model is free.

In this paper, we make use of a $d \leftrightarrow \mathcal{N}-d$ duality observed in [8] to couple for the first time two dually related superconformal mechanics. Depending on the target dimension d, for $\mathcal{N}$=8 the coupled systems are invariant under one of the four one-dimensional finite superconformal algebras $A(3,1)$, $D(4,1)$, $D(2,2)$ or $F(4)$. Their target manifold is a product of two asymptotically flat cones of dimension d and $8-d$ over the spheres S^{d-1} and S^{7-d}, respectively.

The possibility of consistently coupling dually related supermultiplets was first observed, for homogeneous supersymmetry transformations, in [9]. This produces $\mathcal{N}$=8 superconformal systems with targets of dimension d = 1+7, 2+6 or 3+5 (the 4-dimensional system is degenerate, and the dual of the 8-dimensional system is empty). However, for the particular cases of ($\mathcal{N}$=4, d=1) and ($\mathcal{N}$=8, d=3), an inhomogeneous deformation of the supersymmetry is admissible (see, e.g., [10] and [8], respectively). The presence of an inhomogeneity parameter is responsible for the appearance of a Calogero potential in the $\mathcal{N}$=4, $A(1,1)$-invariant, (1,4,3) model and of a Dirac monopole in the $\mathcal{N}$=8, $D(2,1)$-invariant, (3,8,5) model, as will be reviewed below. In these instances, a consistent superconformal coupling of the inhomogeneous supermultiplet with its (homogeneous) dual is non-trivial, as will be shown here. It requires the introduction of an extra supermultiplet of constants for half of the supersymmetries and leads to new superconformal interactions in the presence of a Calogero potential or a monopole. In particular, in all cases (homogeneous or not), an oscillator potential on each of the two cones is generated, with a frequency depending on the mutually dual coordinate.

The description of the models is given in a Lagrangian framework. By setting all fermionic fields to zero and eliminating the auxiliary fields, we are led to the dynamics of two interacting bosonic sigma models whose parameters are fixed by superconformal invariance. Passing to conical radial variables then reveals the geometry and the physical content of the coupled model. In this fashion, our results provide an extension of the class of known superconformal models.

Some interesting questions are left for future investigations. In particular, it seems quite plausible that the bosonic sector of the dually coupled models, whose parameters are fixed by superconformal invariance, turn out to be integrable, as a remnant of the off-shell invariant transformations.

The paper is structured as follows. After reviewing general features of $(d,8,8-d)$ supermultiplets in Section 2, we present in Section 3 the superconformal pairing of dually related multiplets and work out the coupled Lagrangian in the case of homogeneous supersymmetry, ending up with the general bosonic potential on the cone product in the presence of Fayet-Iliopoulos terms. Sections 4 and 5 deal with the inhomogeneous (3,8,5) supermultiplet, its Dirac monopole background and the corresponding gauge transformations. In Section 6, the dual (5,8,3) supermultiplet is displayed, before Section 7 couples it to the inhomogeneous (3,8,5) model. Here one finds the central results of the paper. In Section 8 we reduce the coupled system back to the (5,8,3) supermultiplet. Complete actions and the $\mathcal{N}$=4 coupling of the inhomogeneous (1,4,3) supermultiplet with its dual (3,4,1) partner are presented in detail in three Appendices.

2 Generalities for (d,8,8−d) supermultiplets

$\mathcal{N}$=8 superconformal mechanical systems realize the one-dimensional global supersymmetry algebra

$$\{Q_i, Q_j\} \;=\; 2\delta_{ij}H \qquad \text{with} \qquad i,j=1,\ldots,8 \qquad \text{and} \qquad H \;=\; \partial_t \;, \tag{2.1}$$

where t parametrizes the particle worldline. The corresponding supermultiplets are denoted by (d,8,8−d), indicating d propagating bosonic, 8 propagating fermionic and 8−d auxiliary bosonic coordinate functions for the superparticle, which thus moves on some d-dimensional target space parametrized by $x=\{x^a\,|\,a=1,\ldots,d\}$.

In the construction of $\mathcal{N}$=8 superconformal actions we can make manifest at most four of the eight supersymmetries. Picking by convention Q_1, Q_2, Q_3 and Q_8, an $\mathcal{N}$=4 invariant action reads

$$S_d \;=\; \int \mathrm{d}t\; \mathcal{L}_d \;=\; \int \mathrm{d}t\; Q_8Q_1Q_2Q_3F(x)\;, \tag{2.2}$$

where $F(x)$ is a yet unconstrained function of all coordinates. The restriction to the manifest $\mathcal{N}$=4 superalgebra splits the $\mathcal{N}$=8 supermultiplet,

$$(d,8,8{-}d) \;\longrightarrow\; (d_1,4,4{-}d_1)\;\oplus\;(d_2,4,4{-}d_2) \qquad \text{with} \quad d_1,d_2\le 4 \quad \text{and} \quad d_1+d_2=d \tag{2.3}$$

and opposite chiralities.[1] It turns out that the action depends only on two combinations of second derivatives of F, namely [2]

$$\Phi_1 \;=\; -\Delta_{d_1}F \;\equiv\; -F_{1\,1}-\ldots-F_{d_1d_1} \qquad \text{and} \qquad \Phi_2 \;=\; \Delta_{d_2}F \;\equiv\; F_{d_1+1\,d_1+1}+\ldots+F_{d\,d}\;, \tag{2.4}$$

where we grouped the coordinates according to the decomposition above.

To enhance to $\mathcal{N}$=8 invariance, we must impose

$$Q_\ell\mathcal{L}_d=\partial_t W_\ell \qquad \text{for} \qquad \ell=4,5,6,7\;. \tag{2.5}$$

This produces a harmonicity condition on F,

$$\Delta_d F \;\equiv\; \delta^{ab}F_{ab} \;=\; 0\;. \tag{2.6}$$

As a consequence, we have

$$\Phi_1 \;=\; \Phi_2 \;=:\; \Phi \qquad \text{with} \qquad \Delta_d\Phi=0\;. \tag{2.7}$$

Clearly, for $d\le 5$, we may take $d_2=1$ so that $\Phi=F_{dd}$, singling out the last coordinate. Hence, taking F to be harmonic, we obtain an $\mathcal{N}$=8 sigma model, with a conformally flat target space for the propagating bosonic coordinates,

$$\mathrm{d}s^2 \;=\; \Phi(x)\,\delta_{ab}\mathrm{d}x^a\mathrm{d}x^b\;. \tag{2.8}$$

The remaining generators of the conformal $sl(2)$ algebra are realized as

$$K \;=\; -t^2\partial_t-2t\,\lambda_\varphi \qquad \text{and} \qquad D \;=\; -t\,\partial_t-\lambda_\varphi \tag{2.9}$$

[1] The construction fails if $d_1=0$ or $d_2=0$. There exists, however, a different method which works in all cases [11].

[2] Except for d=3, where an inhomogeneous deformation yields a background gauge potential, see below.

on functions φ with engineering dimension $[\varphi] = \lambda_\varphi$. They give rise to 8 superconformal generators [10, 8]

$$\widetilde{Q}_i \;=\; [K, Q_i] \; . \tag{2.10}$$

Superconformal symmetry is imposed by also demanding that [3]

$$D\,\mathcal{L}_d \;=\; \partial_t M_D \qquad \text{and} \qquad K\,\mathcal{L}_d \;=\; \partial_t M_K \; , \tag{2.11}$$

which yields two conditions on Φ, namely

$$[\Phi] = -1 - 2\lambda_x \qquad \text{and} \qquad \Phi = \Phi(r) \quad \text{with} \quad r^2 \;=\; \delta_{ab}x^a x^b \; . \tag{2.12}$$

The closure of the D-module representation for the $\mathcal{N}$=8 superconformal algebra determines a critical value for the engineering dimension of x,

$$\lambda_x \;=\; \tfrac{1}{d-4} \qquad \Rightarrow \qquad [\Phi] \;=\; -1 - \tfrac{2}{d-4} \;=\; \tfrac{d-2}{4-d} \;=\; (2-d)\lambda_x \; . \tag{2.13}$$

As a consequence, the conformal factor is indeed fixed to the proper harmonic expression,

$$\Phi(r) \;=\; r^{2-d} \; . \tag{2.14}$$

Introducing the spherical line element $\mathrm{d}\Omega_{d-1}$ on S^{d-1} and changing the radial coordinate via

$$\rho \;=\; \tfrac{2}{|4-d|} r^{(4-d)/2} \; , \tag{2.15}$$

the metric reads

$$\mathrm{d}s^2 \;=\; r^{2-d}\big(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{d-1}\big) \;=\; \mathrm{d}\rho^2 + \tfrac14(4-d)^2\rho^2\mathrm{d}\Omega^2_{d-1} \; . \tag{2.16}$$

It reveals the target space to be a specific cone over S^{d-1}, asymptotically flat with a linear relative deficit of $|4-d|/2$. Its scalar curvature comes out as

$$R \;=\; \tfrac14(d-1)(d-2)^2(d-6)r^{d-4} \;=\; (d-1)(d-2)^2(d-6)(d-4)^{-2}\rho^{-2} \; , \tag{2.17}$$

which is negative for $d = 3, 5$ and positive for $d = 7, 8$. At $d = 2, 6$ we encounter flat space.

In any dimension d up to 8, the manifest $\mathcal{N}$=4 superconformal algebra must be a particular member of the $D(2, 1; \alpha)$ family. It turns out that the value of α is determined (up to an S_3 automorphism) by the relation

$$\alpha \;=\; -\tfrac12|4-d| \;=\; -\tfrac{1}{2|\lambda_x|} \; . \tag{2.18}$$

In fact, only for the special values

$$\alpha \in \big\{-3, -2, -\tfrac32, -1, -\tfrac23, -\tfrac12, -\tfrac13,\ 0,\ \tfrac12,\ 1,\ 2,\ \infty\big\} \tag{2.19}$$

attained via (2.18) (and its S_3 orbit) is $D(2, 1; \alpha)$ extendable to an $\mathcal{N}$=8 superconformal algebra.

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[3] The 'D condition' actually follows from the 'K condition'.

3 Duality and coupling in the homogeneous case

From the results of the previous section, an obvious duality relates

$$d \;\leftrightarrow\; 8{-}d \qquad \Leftrightarrow \qquad \lambda_x \;\leftrightarrow\; -\lambda_x \qquad \Leftrightarrow \qquad \big\{ r \leftrightarrow \tfrac{1}{r} \quad \text{and} \quad S^{d-1} \leftrightarrow S^{7-d} \big\}\ . \tag{3.1}$$

The self-dual point at d=4, however, represents a degenerate case, and the case of d=0 is empty. We summarize the values for all dimensions in the following table, which displays also the manifest $\mathcal{N}$=4 superalgebra $\mathcal{G}_4$ and the full $\mathcal{N}$=8 superalgebra $\mathcal{G}_8$ for each case.

d	0	1	2	3	4	5	6	7	8
Φ	r^2	r	1	r^{-1}	r^{-2}	r^{-3}	r^{-4}	r^{-5}	r^{-6}
λ_x	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	∞	$+1$	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{1}{4}$
α	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2
$\mathcal{G}_4$	$D(2,1)$	$D(2,1;\frac{1}{2})$	$A(1,1)$	$D(2,1)$	$A(1,1)$	$D(2,1)$	$A(1,1)$	$D(2,1;\frac{1}{2})$	$D(2,1)$
$\mathcal{G}_8$	$D(4,1)$	$F(4)$	$A(3,1)$	$D(2,2)$	–	$D(2,2)$	$A(3,1)$	$F(4)$	$D(4,1)$

The duality map indicated in (3.1) is easily performed by interchanging propagating and auxiliary bosons and flipping the direction of the supersymmetry transformations. If we summarily denote the propagating bosons, fermions and auxiliary bosons by x^a, ψ^i and f^α, respectively, and indicate the components of the dual multiplet by overtildes and lowered indices, the structure schematically takes the following form,

$$\begin{array}{ccccccccc}
 & & x^a & \xrightarrow{Q} & \psi^i & \xrightarrow{Q} & (f^\alpha, \dot{x}^a) & \xrightarrow{Q} & \dot{\psi}^i \\
 & & \updownarrow & & \updownarrow & & \updownarrow & & \\
\dot{\widetilde{\psi}}_i & \xleftarrow{Q} & (\dot{\widetilde{x}}_a, \widetilde{f}_a) & \xleftarrow{Q} & \widetilde{\psi}_i & \xleftarrow{Q} & \widetilde{x}_\alpha & &
\end{array}$$

where the horizontal arrows encode the various supersymmetry transformations and the vertical arrows depict the duality relations.

We have essentially three different cases of such a duality for $\mathcal{N}$=8 superconformal theories:

$$(1,8,7) \;\leftrightarrow\; (7,8,1) \quad , \qquad (2,8,6) \;\leftrightarrow\; (6,8,2) \quad , \qquad (3,8,5) \;\leftrightarrow\; (5,8,3)\ . \tag{3.2}$$

The two members of each pair have different target dimensions but share the same superconformal algebra. For this reason, they can be coupled together in a Lagrangian

$$\mathcal{L}_{d+(8-d)} \;=\; \mathcal{L}_d + \mathcal{L}_{8-d} + \gamma\, \mathcal{L}_{d,(8-d)}\ , \tag{3.3}$$

with a coupling of dimensionless strength γ provided by a canonical pairing,

$$\mathcal{L}_{d,(8-d)} \;=\; x^a \widetilde{f}_a + \psi^i \widetilde{\psi}_i - f^\alpha \widetilde{x}_\alpha\ . \tag{3.4}$$

Note that the dimensions in each pairing add up to one, and the duality guarantees the $\mathcal{N}$=8 superconformal invariance of the coupling term, as long as the transformations remain homogeneous. This is the case for d=1 and d=2. In three dimensions, there exists an inhomogeneous deformation of the (3,8,5) multiplet. When this is turned on, the coupling to the dual (5,8,3) becomes less obvious. We will dwell on this point later on.

Let us take a look at the bosonic part of the Lagrangian in the homogeneous case. It takes the form

$$\mathcal{L}_{d+(8-d)}\big| \;=\; \Phi(r)\left(\dot{x}^a\dot{x}^a + f^\alpha f^\alpha\right) + \widetilde{\Phi}(\widetilde{r})\left(\dot{\widetilde{x}}_\alpha\dot{\widetilde{x}}_\alpha + \widetilde{f}_a\widetilde{f}_a\right) + \gamma\left(x^a\widetilde{f}_a - f^\alpha\widetilde{x}_\alpha\right) . \tag{3.5}$$

We may add Fayet-Iliopoulos terms with dimensionful parameters μ_α and $\widetilde{\mu}^a$ to get

$$\mathcal{L}'_{d+(8-d)}\big| \;=\; \mathcal{L}_{d+(8-d)}\big| + \mu_\alpha f^\alpha - \widetilde{\mu}^a\widetilde{f}_a . \tag{3.6}$$

Eliminating the auxiliary components by their equations of motion,

$$f_\alpha \;=\; \tfrac12\Phi^{-1}\big(\gamma\widetilde{x}_\alpha - \mu_\alpha\big) \qquad\text{and}\qquad \widetilde{f}^a \;=\; -\tfrac12\widetilde{\Phi}^{-1}\big(\gamma x^a - \widetilde{\mu}^a\big) , \tag{3.7}$$

we arrive at

$$\mathcal{L}''_{d+(8-d)}\big| \;=\; \Phi\,\dot{x}^a\dot{x}^a + \widetilde{\Phi}\,\dot{\widetilde{x}}_\alpha\dot{\widetilde{x}}_\alpha - \tfrac14\Phi^{-1}\big(\gamma\widetilde{x}_\alpha{-}\mu_\alpha\big)\big(\gamma\widetilde{x}_\alpha{-}\mu_\alpha\big) - \tfrac14\widetilde{\Phi}^{-1}\big(\gamma x^a{-}\widetilde{\mu}^a\big)\big(\gamma x^a{-}\widetilde{\mu}^a\big) , \tag{3.8}$$

which features a very specific potential in the joint target space of both multiplets.

For a physical interpretation, it is useful to fix $\Phi(r) = r^{2-d}$ and $\widetilde{\Phi} = \widetilde{r}^{d-6}$ and pass to standard radial coordinates (up to a factor of $\frac12$),

$$\rho(r) \;=\; \tfrac{2}{|4-d|}\,r^{(4-d)/2} \qquad\text{and}\qquad \widetilde{\rho}(\widetilde{r}) \;=\; \tfrac{2}{|4-d|}\,\widetilde{r}^{(d-4)/2} . \tag{3.9}$$

Introducing total angular momenta ℓ and $\widetilde{\ell}$ for the d- and $(8{-}d)$-dimensional targets and unit vectors via $x^a = re^a$ and $\widetilde{x}_\alpha = \widetilde{r}\widetilde{e}_\alpha$, one arrives at

$$\mathcal{L}^{\text{cone}}_{d+(8-d)}\big| \;=\; \dot{\rho}^2 + \tfrac{4\ell^2}{|d-4|^2}\rho^{-2} + \dot{\widetilde{\rho}}^2 + \tfrac{4\widetilde{\ell}^2}{|d-4|^2}\widetilde{\rho}^{-2} - \tfrac14\Phi^{-1}\big(\gamma\widetilde{r}\vec{\widetilde{e}} - \vec{\mu}\big)^2 - \tfrac14\widetilde{\Phi}^{-1}\big(\gamma r\vec{e} - \vec{\widetilde{\mu}}\big)^2 , \tag{3.10}$$

where $r = r(\rho)$ and $\widetilde{r} = \widetilde{r}(\widetilde{\rho})$ is understood. Apart from the standard angular momentum 'barriers', the potential for the coordinates r and $\widetilde{r}$ is of oscillator type, centered around $\vec{r} = \vec{\widetilde{\mu}}/\gamma$ and $\vec{\widetilde{r}} = \vec{\mu}/\gamma$ and with (position-dependent) frequencies $\omega = \frac{\gamma}{2}\widetilde{\Phi}^{-1/2}$ and $\widetilde{\omega} = \frac{\gamma}{2}\Phi^{-1/2}$, respectively.

4 D-module representation of the (3,8,5) supermultiplet

Let us adopt a convenient notation for the components of the (3,8,5) multiplet:

$$\begin{cases} \text{bosons } x^a\text{:} & x, y, z \quad\text{or}\quad x_1, x_2, x_3 \\ \text{fermions } \psi^i\text{:} & \psi_0,\psi_1,\psi_2,\psi_3,\xi_0,\xi_1,\xi_2,\xi_3 \\ \text{auxiliaries } f^\alpha\text{:} & f_1, f_2, g, g_1, g_2 \end{cases} . \tag{4.1}$$

For simplicity, we lower all indices. According to the relations of Section 2, we have $\lambda_x = -1$ and $\alpha = -\frac12$, so the $\mathcal{N}$=4 algebra $D(2,1;-\frac12) \simeq D(2,1;1) \simeq osp(4|2)$ should get enlarged to an $D(2|2) \simeq osp(4|4)$ algebra. For the conformal factor we expect $\Phi = \frac{1}{r}$.

A unique feature specific to d=3 is the option to deform the homogeneous superconformal transformations by a constant shift in some transformations of fermions to auxiliaries. Without loss of generality, we choose a frame in which only the auxiliary coordinate g appears shifted, and only in the action of Q_2, Q_3, Q_6 and Q_7. Hence, half of the deformation is taken to be contained in manifestly realized $\mathcal{N}$=4 supersymmetry.

The $\mathcal{N}$=8 transformations are captured in the following array:

	Q_8	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
x_1	ψ_0	ψ_1	ψ_2	ψ_3	ξ_0	ξ_1	ξ_2	ξ_3
x_2	ψ_1	$-\psi_0$	ψ_3	$-\psi_2$	ξ_1	$-\xi_0$	$-\xi_3$	ξ_2
x_3	ξ_0	$-\xi_1$	$-\xi_2$	$-\xi_3$	$-\psi_0$	ψ_1	ψ_2	ψ_3
ψ_0	$\dot{x}_1$	$-\dot{x}_2$	$-f_1$	$-f_2$	$-\dot{x}_3$	$-g$	$-g_1$	$-g_2$
ψ_1	$\dot{x}_2$	$\dot{x}_1$	$-f_2$	f_1	$-g$	$\dot{x}_3$	g_2	$-g_1$
ψ_2	f_1	f_2	$\dot{x}_1$	$-\dot{x}_2$	$-g_1$	$-g_2$	$\dot{x}_3$	$g+c$
ψ_3	f_2	$-f_1$	$\dot{x}_2$	$\dot{x}_1$	$-g_2$	g_1	$-g-c$	$\dot{x}_3$
ξ_0	$\dot{x}_3$	g	g_1	g_2	$\dot{x}_1$	$-\dot{x}_2$	$-f_1$	$-f_2$
ξ_1	g	$-\dot{x}_3$	g_2	$-g_1$	$\dot{x}_2$	$\dot{x}_1$	f_2	$-f_1$
ξ_2	g_1	$-g_2$	$-\dot{x}_3$	$g+c$	f_1	$-f_2$	$\dot{x}_1$	$\dot{x}_2$
ξ_3	g_2	g_1	$-g-c$	$-\dot{x}_3$	f_2	f_1	$-\dot{x}_2$	$\dot{x}_1$
f_1	$\dot{\psi}_2$	$-\dot{\psi}_3$	$-\dot{\psi}_0$	$\dot{\psi}_1$	$\dot{\xi}_2$	$\dot{\xi}_3$	$-\dot{\xi}_0$	$-\dot{\xi}_1$
f_2	$\dot{\psi}_3$	$\dot{\psi}_2$	$-\dot{\psi}_1$	$-\dot{\psi}_0$	$\dot{\xi}_3$	$-\dot{\xi}_2$	$\dot{\xi}_1$	$-\dot{\xi}_0$
g	$\dot{\xi}_1$	$\dot{\xi}_0$	$-\dot{\xi}_3$	$\dot{\xi}_2$	$-\dot{\psi}_1$	$-\dot{\psi}_0$	$-\dot{\psi}_3$	$\dot{\psi}_2$
g_1	$\dot{\xi}_2$	$\dot{\xi}_3$	$\dot{\xi}_0$	$-\dot{\xi}_1$	$-\dot{\psi}_2$	$\dot{\psi}_3$	$-\dot{\psi}_0$	$-\dot{\psi}_1$
g_2	$\dot{\xi}_3$	$-\dot{\xi}_2$	$\dot{\xi}_1$	$\dot{\xi}_0$	$-\dot{\psi}_3$	$-\dot{\psi}_2$	$\dot{\psi}_1$	$-\dot{\psi}_0$

(4.2)

The action for the (3,8,5) multiplet reads

$$S_3 \;=\; \int \mathrm{d}t\ \mathcal{L}_3 \;=\; \int \mathrm{d}t\ Q_8 Q_1 Q_2 Q_3 F(x,y,z) \tag{4.3}$$

with

$$F_{xx} + F_{yy} + F_{zz} \;=\; 0\ , \tag{4.4}$$

and the conformal factor comes out as

$$F_{zz} \;=\; \Phi \;=\; \tfrac{1}{r} \qquad \text{with} \qquad r^2 = x^2 + y^2 + z^2\ . \tag{4.5}$$

Without loss of generality, the z coordinate is singled out because we had to make a choice in the supersymmetry transformations.

The dependence on the inhomogeneous shift parameter c is linear, so we write

$$\mathcal{L}_3 \;=\; \mathcal{L}_3^{(0)} + c\,\mathcal{L}_3^{(1)}\ . \tag{4.6}$$

After a lengthy but straightforward computation, we find

$$\mathcal{L}_3^{(0)} \;=\; \Phi\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + f_1^2 + f_2^2 + g^2 + g_1^2 + g_2^2) \;+\; \text{fermionic terms} \tag{4.7}$$

and

$$\begin{aligned}\mathcal{L}_3^{(1)} \;=\;& \Phi\, g + A_x \dot{x} + A_y \dot{y} + \\ & \Phi_x(\psi_0\xi_1 + \psi_1\xi_0) + \Phi_y(\psi_1\xi_1 - \psi_0\xi_0) - \Phi_z(\psi_1\psi_0 + \xi_1\xi_0)\ ,\end{aligned} \tag{4.8}$$

where we introduced

$$A_x \;=\; F_{zy} \qquad \text{and} \qquad A_y \;=\; -F_{zx}\ . \tag{4.9}$$

The complete expression of $\mathcal{L}_3^{(0)}$ is displayed in Appendix A. Setting all fermions to zero, we extract the bosonic part

$$\mathcal{L}_3\big| \;=\; \Phi\,(\dot{x}^2+\dot{y}^2+\dot{z}^2+f_1^2+f_2^2+g^2+g_1^2+g_2^2) \;+\; c\,(\Phi\,g + A_x\dot{x} + A_y\dot{y})\ . \tag{4.10}$$

We remark that only the z derivative of F appears, so it makes sense to define a prepotential

$$G \;:=\; F_z \qquad\Rightarrow\qquad G_x \;=\; -A_y\ ,\quad G_y \;=\; A_x\ ,\quad G_z \;=\; \Phi\ , \tag{4.11}$$

which inherits the harmonicity from F.

It is admissible to slightly deform our model by adding Fayet-Iliopoulos terms. This extends the bosonic Lagrangian to

$$\mathcal{L}_3'\big| \;=\; \mathcal{L}_3\big| + \mu_\alpha f_\alpha + \zeta g + \zeta_\alpha g_\alpha \qquad \text{with} \quad \alpha=1,2 \tag{4.12}$$

and five real parameters μ_α, ζ and ζ_α.

We solve the equations of motion for the auxiliary fields,

$$f_\alpha \;=\; -\frac{\mu_\alpha}{2\Phi}\ ,\qquad g \;=\; -\frac{\zeta+c\Phi}{2\Phi}\ ,\qquad g_\alpha \;=\; -\frac{\zeta_\alpha}{2\Phi}\ , \tag{4.13}$$

and eliminate them from the Lagrangian to arrive at

$$\mathcal{L}_3''\big| \;=\; \Phi\,(\dot{x}^2+\dot{y}^2+\dot{z}^2) \;-\; \tfrac14\Phi^{-1}(\mu_\alpha^2+\zeta^2+\zeta_\alpha^2) \;-\; \tfrac12 c\,\zeta \;-\; \tfrac14 c^2\Phi \;+\; c\,(A_x\dot{x}+A_y\dot{y})\ . \tag{4.14}$$

Apparently, there is not only a magnetic but also an electric field, together

$$B_x = c\,G_{xz}\ ,\quad B_y = c\,G_{yz}\ ,\quad B_z = -c(G_{xx}{+}G_{yy}) = c\,G_{zz} \qquad \text{and} \qquad E_a = -\tfrac14 c^2 G_{za}\ , \tag{4.15}$$

both being simply proportional to the gradient of $G_z=\Phi$. With $\Phi=\frac{1}{r}$, we identify a magnetic monopole, while for the interpretation of the electric field we better pass to the conical coordinates,

$$r=\tfrac14\rho^2 \qquad\Rightarrow\qquad \mathrm{d}s^2 \;=\; \mathrm{d}\rho^2+\tfrac14\rho^2\mathrm{d}\Omega_2^2 \qquad \text{and} \qquad A_0 \;=\; c^2\rho^{-2}\ . \tag{4.16}$$

The bosonic dynamics of this theory has been analyzed for general values of α in [12].

5 Gauge freedom

In order to explicitly write down the Lagrangian, we must 'integrate' Φ to find the prepotential G, from which the gauge potential A is obtained. The answer is not unique, due to abelian gauge invariance,

$$\delta A_a \;=\; \partial_a u \qquad \text{and} \qquad \delta G \;=\; v \tag{5.1}$$

with a priori arbitrary harmonic gauge functions u and v. However, the invariance of $\Phi{=}G_z$ enforces $v_z=0$, and the relation between G and A_a connects the two functions,

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x\ . \tag{5.2}$$

The (local) solution introduces another function $h(x,y)$ via

$$u = h_y(x,y) + \widetilde{u}(z) \qquad \text{and} \qquad v = h_x(x,y) \qquad \text{with} \qquad h_{xx} + h_{yy} = 0\ . \tag{5.3}$$

The harmonicity of u implies that $\widetilde{u}$ is at most linear in z. Alternatively, we may interpret the above relation as Cauchy-Riemann equations for the real and imaginary part of a holomorphic function of $w = x + \mathrm{i}y$,

$$v - \mathrm{i}(u{-}\widetilde{u}) \;=\; E(w) \;=:\; \partial_w H(w) \qquad \Rightarrow \qquad h \;=\; H(w) + \overline{H}(\overline{w})\ , \tag{5.4}$$

where H and h are determined up to a constant. Therefore, the gauge freedom for the prepotential is encoded in a single holomorphic function E.

For the magnetic monopole there does not exist a globally regular gauge potential; we must be content with configurations on a 'northern' (N) and on a 'southern' (S) patch, related by a regular gauge transformation in the equatorial overlap. The standard expressions obtained from $G_z = \frac{1}{r}$ read

$$G^N \;=\; +\ln(r{+}z) \qquad \Rightarrow \qquad A^N_x = G^N_y = \tfrac{y}{r(z+r)} \qquad \text{and} \qquad A^N_y = -G^N_x = -\tfrac{x}{r(z+r)}\ , \tag{5.5}$$

$$G^S \;=\; -\ln(r{-}z) \qquad \Rightarrow \qquad A^S_x = G^S_y = \tfrac{y}{r(z-r)} \qquad \text{and} \qquad A^S_y = -G^S_x = -\tfrac{x}{r(z-r)}\ , \tag{5.6}$$

so that indeed (for $a = x, y$)

$$G^N - G^S \;=\; \ln(x^2{+}y^2) \;=:\; h_x \qquad \Rightarrow \qquad A^N_a - A^S_a \;=\; -2\partial_a \arctan\tfrac{y}{x} \;=:\; \partial_a h_y\ , \tag{5.7}$$

and the holomorphic combination

$$E \;=\; \ln(x^2{+}y^2) + 2\mathrm{i}\arctan\tfrac{y}{x} \;=\; \ln w^2 \tag{5.8}$$

gives rise to the correct 'pre-gauge' function

$$H \;=\; 2w(\ln w{-}1) \qquad \Rightarrow \qquad h \;=\; x\ln(x^2{+}y^2) - 2x - 2y\arctan\tfrac{y}{x} \tag{5.9}$$

in the class described above and regular away from the poles. The singularity of the northern functions along the negative z-axis and likewise for the southern patch signify the would-be Dirac string in a global configuration.

6 The dual (5,8,3) supermultiplet

Applying the duality reflection to the (3,8,5) multiplet, we obtain a (5,8,3) multiplet. However, we must first put the inhomogeneous deformation parameter c to zero, since such a deformation does not exist for d=5. Section 2 tells us that $\lambda_x = +1$ and $\Phi = r^{-3}$, and we again realize an $D(2,2) \simeq osp(4|4)$ superalgebra. Naming the components as follows,

$$\begin{cases} \text{bosons } \tilde{x}_\alpha\text{:} & v_1, v_2, w, w_1, w_2 \\ \text{fermions } \tilde{\psi}_i\text{:} & \chi_0, \chi_1, \chi_2, \chi_3, \lambda_0, \lambda_1, \lambda_2, \lambda_3 \\ \text{auxiliaries } \tilde{f}_a\text{:} & h_1, h_2, h_3 \end{cases}\ , \tag{6.1}$$

the array (4.2) gets transformed into the $\mathcal{N}$=8 transformations for the (5,8,3) multiplet:

	Q_8	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
v_1	λ_2	$-\lambda_3$	$-\lambda_0$	λ_1	χ_2	χ_3	$-\chi_0$	$-\chi_1$
v_2	λ_3	λ_2	$-\lambda_1$	$-\lambda_0$	χ_3	$-\chi_2$	χ_1	$-\chi_0$
w	χ_1	χ_0	$-\chi_3$	χ_2	$-\lambda_1$	$-\lambda_0$	$-\lambda_3$	λ_2
w_1	χ_2	χ_3	χ_0	$-\chi_1$	$-\lambda_2$	λ_3	$-\lambda_0$	$-\lambda_1$
w_2	χ_3	$-\chi_2$	χ_1	χ_0	$-\lambda_3$	$-\lambda_2$	λ_1	$-\lambda_0$
χ_0	h_3	$\dot{w}$	$\dot{w}_1$	$\dot{w}_2$	h_1	$-h_2$	$-\dot{v}_1$	$-\dot{v}_2$
χ_1	$\dot{w}$	$-h_3$	$\dot{w}_2$	$-\dot{w}_1$	h_2	h_1	$\dot{v}_2$	$-\dot{v}_1$
χ_2	$\dot{w}_1$	$-\dot{w}_2$	$-h_3$	$\dot{w}$	$\dot{v}_1$	$-\dot{v}_2$	h_1	h_2
χ_3	$\dot{w}_2$	$\dot{w}_1$	$-\dot{w}$	$-h_3$	$\dot{v}_2$	$\dot{v}_1$	$-h_2$	h_1
λ_0	h_1	$-h_2$	$-\dot{v}_1$	$-\dot{v}_2$	$-h_3$	$-\dot{w}$	$-\dot{w}_1$	$-\dot{w}_2$
λ_1	h_2	h_1	$-\dot{v}_2$	$\dot{v}_1$	$-\dot{w}$	h_3	$\dot{w}_2$	$-\dot{w}_1$
λ_2	$\dot{v}_1$	$\dot{v}_2$	h_1	$-h_2$	$-\dot{w}_1$	$-\dot{w}_2$	h_3	$\dot{w}$
λ_3	$\dot{v}_2$	$-\dot{v}_1$	h_2	h_1	$-\dot{w}_2$	$\dot{w}_1$	$-\dot{w}$	h_3
h_1	$\dot{\lambda}_0$	$\dot{\lambda}_1$	$\dot{\lambda}_2$	$\dot{\lambda}_3$	$\dot{\chi}_0$	$\dot{\chi}_1$	$\dot{\chi}_2$	$\dot{\chi}_3$
h_2	$\dot{\lambda}_1$	$-\dot{\lambda}_0$	$\dot{\lambda}_3$	$-\dot{\lambda}_2$	$\dot{\chi}_1$	$-\dot{\chi}_0$	$-\dot{\chi}_3$	$\dot{\chi}_2$
h_3	$\dot{\chi}_0$	$-\dot{\chi}_1$	$-\dot{\chi}_2$	$-\dot{\chi}_3$	$-\dot{\lambda}_0$	$\dot{\lambda}_1$	$\dot{\lambda}_2$	$\dot{\lambda}_3$

(6.2)

The full Lagrangian $\mathcal{L}_5$ is found in Appendix B. Its bosonic part is obvious,

$$\mathcal{L}_5\Big| \;=\; \widetilde{\Phi}\,(\dot{v}_\alpha^2 + \dot{w}^2 + \dot{w}_\alpha^2 + h_a^2)\;, \tag{6.3}$$

where the prepotential function is

$$\widetilde{\Phi} \;=\; F_{v_1v_1} + F_{v_2v_2} \;=\; -(F_{ww} + F_{w_1w_1} + F_{w_2w_2})\;. \tag{6.4}$$

7 Coupling (3,8,5) to (5,8,3)

Since both (3,8,5) and (5,8,3) multiplets represent the same $D(2,2)$ superalgebra, it is natural to couple them. The duality provides a canonical interaction term $\mathcal{L}_{3,5}^{(0)}$ in the joint Lagrangian

$$\mathcal{L}_3^{(0)} \;+\; \mathcal{L}_5 \;+\; \gamma\mathcal{L}_{3,5}^{(0)} \tag{7.1}$$

of the form

$$\mathcal{L}_{3,5}^{(0)} \;=\; x_a h_a - f_\alpha v_\alpha - g\,w - g_\alpha w_\alpha + \psi_i\lambda_i + \xi_i\chi_i \qquad \text{with} \quad a=1,2,3\;, \quad \alpha=1,2\;, \quad i=0,1,2,3\;, \tag{7.2}$$

with some dimensionless coupling constant γ. It is easy to check that $\mathcal{L}_{3,5}^{(0)}$ is invariant (up to total time derivatives) under all eight supersymmetries and their conformal partners, because the dimensions of any two duality partners add up to one.

The superscript (0) reminds us that we turned off the inhomogeneous deformation in the (3,8,5) multiplet. So the question arises as to whether it is possible to extend this coupling to the deformed multiplet as well, and what this entails for the dual (5,8,3) multiplet. To answer this, we first observe that

$$\mathcal{L}_3 \; + \; \mathcal{L}_5 \; + \; \gamma \mathcal{L}^{(0)}_{3,5} \tag{7.3}$$

is indeed invariant (up to total time derivatives) under Q_8, Q_1, Q_4 and Q_5, but

$$Q_2 \mathcal{L}^{(0)}_{3,5} = -c\chi_3 \ , \quad Q_3 \mathcal{L}^{(0)}_{3,5} = c\chi_2 \ , \quad Q_6 \mathcal{L}^{(0)}_{3,5} = -c\lambda_3 \ , \quad Q_7 \mathcal{L}^{(0)}_{3,5} = c\lambda_2 \tag{7.4}$$

do not vanish. Yet, since c is a constant, these terms are linear and may be cancelled by adding other linear terms to the interaction. To achieve this feat, however, one must view the deformation parameter c as the highest component of an $\mathcal{N}$=4 multiplet of type (3,4,1) involving the supercharges Q_j for $j = 2, 3, 6, 7$. Denoting the components of dimension -1, $-\frac{1}{2}$ and 0 by e_a, ω_i and c, respectively, the transformation table takes the form

	Q_8	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
e_1	0	0	ω_2	ω_3	0	0	ω_0	ω_1
e_2	0	0	ω_3	$-\omega_2$	0	0	$-\omega_1$	ω_0
e_3	0	0	$-\omega_0$	$-\omega_1$	0	0	ω_2	ω_3
ω_0	0	0	0	$-c$	0	0	0	0
ω_1	0	0	c	0	0	0	0	0
ω_2	0	0	0	0	0	0	0	$-c$
ω_3	0	0	0	0	0	0	c	0
c	0	0	0	0	0	0	0	0

(7.5)

It is important to realize that all these components are constants, i.e. time independent, otherwise there could not be zeros in this table. For the same reason, it is admissible to have this multiplet annihilated by the other four supercharges, Q_k for $k = 8, 1, 4, 5$. If we add to our interaction Lagrangian two extra pieces,

$$\mathcal{L}^{(1)}_{3,5} \; = \; \omega_0\chi_2 + \omega_1\chi_3 + \omega_2\lambda_2 + \omega_3\lambda_3 \qquad \text{and} \qquad \mathcal{L}^{(2)}_{3,5} \; = \; e_1 h_1 + e_2 h_2 + e_3 h_3 \ , \tag{7.6}$$

it is not hard to check that all unwanted terms get cancelled, and only total time derivatives remain. In other words,

$$\mathcal{L}_{3+5} \; := \; \mathcal{L}_3 \; + \; \mathcal{L}_5 \; + \; \gamma \mathcal{L}_{3,5} \tag{7.7}$$

is fully $\mathcal{N}$=8 superconformally invariant for

$$\begin{aligned} \mathcal{L}_{3,5} \; &= \; (x_a{+}e_a)h_a - f_\alpha v_\alpha - g\,w - g_\alpha w_\alpha \\ &+ \; \xi_0\chi_0 + \xi_1\chi_1 + (\xi_2{+}\omega_0)\chi_2 + (\xi_3{+}\omega_1)\chi_3 + \psi_0\lambda_0 + \psi_1\lambda_1 + (\psi_2{+}\omega_2)\lambda_2 + (\psi_3{+}\omega_3)\lambda_3 \ , \end{aligned} \tag{7.8}$$

which adds to the pairings (7.2) a term linear in a (1,4,3) submultiplet $(w; \chi_2, \chi_3, \lambda_2, \lambda_3; h_a)$ inside our dual (5,8,3) multiplet. Another interpretation is that the (1,8,5) components with inhomogeneous transformation receive constant shifts which cancel the inhomogeneity produced in the canonical coupling term.

Interestingly, there is another way to cancel the non-invariant terms (7.4). Observing that

$$Q_j \mathcal{L}^{(0)}_{3,5} \; = \; c\,Q_j w \qquad \text{for} \qquad j = 2, 3, 6, 7 \tag{7.9}$$

suggests repairing the deficit by adding

$$\mathcal{L}_{3,5}^{(0')} \;=\; -c\,w \tag{7.10}$$

to the interaction. While $Q_j\mathcal{L}_{3,5}^{(0')}$ indeed just cancels the unwanted terms, now the other four supersymmetries are compromised, however, as

$$Q_8\mathcal{L}_{3,5}^{(0')} = -c\,\chi_1\;, \quad Q_1\mathcal{L}_{3,5}^{(0')} = -c\,\chi_0\;, \quad Q_4\mathcal{L}_{3,5}^{(0')} = c\,\lambda_1\;, \quad Q_5\mathcal{L}_{3,5}^{(0')} = c\,\lambda_0\;. \tag{7.11}$$

Comparing with (7.4), we see that the deficiency has simply been shifted from the Q_j to the Q_k with $k=8,1,4,5$, and the relevant fermionic components carry indices 0 and 1 instead of 2 and 3. Hence, adding a suitable constant (3,4,1) multiplet for those supersymmetries and the appropriate terms $\mathcal{L}_{3,5}^{(1')}$ and $\mathcal{L}_{3,5}^{(2')}$ to the interaction will accomplish the job just as well. The only difference for the bosonic Lagrangians is an additional term of $-\gamma c\,w$.

Sticking with the first resolution and adding Fayet-Iliopoulos terms for all auxiliary components, the bosonic part of the total action reads

$$\begin{aligned}\mathcal{L}'_{3+5}\big| \;&=\; \Phi\left(\dot{x}_a^2+f_\alpha^2+g^2+g_\alpha^2\right)+c\,\vec{A}\cdot\dot{\vec{x}}+\widetilde{\Phi}\left(\dot{v}_\alpha^2+\dot{w}^2+\dot{w}_\alpha^2+h_a^2\right)\\ &-\;\left(\gamma v_\alpha-\mu_\alpha\right)f_\alpha-\left(\gamma w-\zeta-c\Phi\right)g-\left(\gamma w_\alpha-\zeta_\alpha\right)g_\alpha+\left(\gamma(x_a+e_a)-\widetilde{\mu}_a\right)h_a\;,\end{aligned} \tag{7.12}$$

and elimination of the auxiliary components produces

$$\begin{aligned}\mathcal{L}''_{3+5}\big| \;&=\; \Phi\,\dot{x}_a^2+c\,\vec{A}\cdot\dot{\vec{x}}+\widetilde{\Phi}\left(\dot{v}_\alpha^2+\dot{w}^2+\dot{w}_\alpha^2\right)\\ &-\;\tfrac14\Phi^{-1}\big((\gamma v_\alpha-\mu_\alpha)^2+(\gamma w-\zeta-c\Phi)^2+(\gamma w_\alpha-\zeta_\alpha)^2\big)-\tfrac14\widetilde{\Phi}^{-1}\big(\gamma(x_a+e_a)-\widetilde{\mu}_a\big)^2\;.\end{aligned} \tag{7.13}$$

The constant Lagrange multipliers e_a serve to eliminate the zero modes of the h_a. For convenience, we relabel $w_\alpha=v_{2+\alpha}$ and $w=v_5$ and define $v^2=v_\alpha v_\alpha+w^2+w_\alpha w_\alpha$. In conical radial coordinates $\rho=2r^{1/2}$ and $\sigma=2v^{-1/2}$, the bosonic action then takes the form

$$\begin{aligned}\mathcal{L}^{\text{cone}}_{3+5}\big| \;&=\; \dot{\rho}^2+4\ell^2\rho^{-2}+\dot{\sigma}^2+4\widetilde{\ell}^2\sigma^{-2}+c\,\vec{A}\cdot\dot{\vec{x}}\\ &-\;\tfrac{1}{16}\rho^2\big(4\gamma\sigma^{-2}\vec{e}_\sigma-\vec{\mu}-4c\rho^{-2}\vec{e}_5\big)^2-\sigma^{-6}\big(\gamma(\rho^2\vec{e}_\rho+4\vec{e})-4\vec{\widetilde{\mu}}\big)^2\;,\end{aligned} \tag{7.14}$$

where we introduced the angular momenta ℓ and $\widetilde{\ell}$ in the three- and five-dimensional targets, and the vectors in the first and second brackets are five- and three-dimensional, respectively.

8 A deformed (5,8,3) supermultiplet

If in (7.7) we set to zero the complete (5,8,3) multiplet, we simply come back to the original deformed (3,8,5) theory. Let us then try the opposite and see whether we recover the (5,8,3) model. However, due to (7.4) it is not consistent to put the (3,8,5) components to zero completely, but we must keep the zero modes of x_a, ψ_2, ψ_3, ξ_2, ξ_3 and g, which we denote by an overbar. With this provision, the full Lagrangian (7.7) reduces to

$$\begin{aligned}\widehat{\mathcal{L}}_5 \;&=\; \mathcal{L}_5\;+\;\gamma\left((\overline{x}_a+e_a)h_a+(\overline{\xi}_2+\omega_0)\chi_2+(\overline{\xi}_3+\omega_1)\chi_3+(\overline{\psi}_2+\omega_2)\lambda_2+(\overline{\psi}_3+\omega_3)\lambda_3-(\overline{g}+c)w\right)\\ &=:\;\mathcal{L}_5\;+\;\gamma\left(e'_ah_a+\omega'_0\chi_2+\omega'_1\chi_3+\omega'_2\lambda_2+\omega'_3\lambda_3-c'w\right)\;,\end{aligned} \tag{8.1}$$

and to the transformations (7.5) of the constants we must add

	Q_8	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
$\overline{x}_1$	0	0	$\overline{\psi}_2$	$\overline{\psi}_3$	0	0	$\overline{\xi}_2$	$\overline{\xi}_3$
$\overline{x}_2$	0	0	$\overline{\psi}_3$	$-\overline{\psi}_2$	0	0	$-\overline{\xi}_3$	$\overline{\xi}_2$
$\overline{x}_3$	0	0	$-\overline{\xi}_2$	$-\overline{\xi}_3$	0	0	$\overline{\psi}_2$	$\overline{\psi}_3$
$\overline{\xi}_2$	0	0	0	$\overline{g}+c$	0	0	0	0
$\overline{\xi}_3$	0	0	$-\overline{g}-c$	0	0	0	0	0
$\overline{\psi}_2$	0	0	0	0	0	0	0	$\overline{g}+c$
$\overline{\psi}_3$	0	0	0	0	0	0	$-\overline{g}-c$	0
$\overline{g}$	0	0	0	0	0	0	0	0

(8.2)

which is what remains of (4.2). We see that only the four Q_j are effective. The upshot is a deformation of the original (5,8,3) Lagrangian by linear terms in a (1,4,3) submultiplet. The linear coefficients (e'_a, ω'_i, c') are just the sum of the (3,4,1) zero-mode submultiplet (8.2) of the original (3,8,5) multiplet and the constant auxiliary (3,4,1) multiplet (7.5). This combination transforms as follows,

	Q_8	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
e'_1	0	0	ω'_2	ω'_3	0	0	ω'_0	ω'_1
e'_2	0	0	ω'_3	$-\omega'_2$	0	0	$-\omega'_1$	ω'_0
e'_3	0	0	$-\omega'_0$	$-\omega'_1$	0	0	ω'_2	ω'_3
ω'_0	0	0	0	$-c'$	0	0	0	0
ω'_1	0	0	c'	0	0	0	0	0
ω'_2	0	0	0	0	0	0	0	$-c'$
ω'_3	0	0	0	0	0	0	c'	0
c'	0	0	0	0	0	0	0	0

(8.3)

Hence, the coupling of the (5,8,3) multiplet to a dual inhomogeneous (3,8,5) multiplet leads to a deformation of the former, which consists of the coupling of a (1,4,3) submultiplet to an auxiliary constant (3,4,1) dual multiplet. The deformation is parametrized by γ and contains the (3,8,5) inhomogeneity c as part of it. Of course, we may also add standard Fayet-Iliopoulos terms.

A Appendix: Action for the (3,8,5) supermultiplet

The complete Lagrangian for the (3,8,5) multiplet reads

$$
\begin{aligned}
\mathcal{L}_3^{(0)} = \;& \Phi\,(\dot{x}^2+\dot{y}^2+\dot{z}^2+f_1^2+f_2^2+g^2+g_1^2+g_2^2)+ \\
& \Phi\,(\dot{\psi}_0\psi_0+\dot{\psi}_1\psi_1+\dot{\psi}_2\psi_2+\dot{\psi}_3\psi_3+\dot{\xi}_0\xi_0+\dot{\xi}_1\xi_1+\dot{\xi}_2\xi_2+\dot{\xi}_3\xi_3)+ \\
& \Phi_x((\dot{y}\xi_0\xi_1+f_1\xi_0\xi_2+f_2\xi_0\xi_3)+(\dot{y}\psi_0\psi_1+f_1\psi_0\psi_2+f_2\psi_0\psi_3) \\
& +(g\psi_0\xi_1+g_1\psi_0\xi_2+g_2\psi_0\xi_3)+(g\psi_1\xi_0+g_1\psi_2\xi_0+g_2\psi_3\xi_0) \\
& -(\dot{y}\xi_2\xi_3+f_1\xi_3\xi_1+f_2\xi_1\xi_2)+(\dot{y}\psi_2\psi_3+f_1\psi_3\psi_1+f_2\psi_1\psi_2) \\
& +(g\xi_3\psi_2+g_1\xi_1\psi_3+g_2\xi_2\psi_1)-(g\xi_2\psi_3+g_1\xi_3\psi_1+g_2\xi_1\psi_2) \\
& +\dot{z}(\psi_0\xi_0+\xi_1\psi_1+\xi_2\psi_2+\xi_3\psi_3))+ \\
& \Phi_y((-\dot{x}\xi_0\xi_1-f_1\xi_0\xi_3+f_2\xi_0\xi_2)+(\dot{x}\psi_1\psi_0-f_1\psi_3\psi_0+f_2\psi_2\psi_0) \\
& -(\dot{z}\xi_1\psi_0-g_1\xi_3\psi_0+g_2\xi_2\psi_0)-(\dot{z}\xi_0\psi_1+g_1\xi_0\psi_3-g_2\xi_0\psi_2) \\
& +g(\xi_0\psi_0-\xi_1\psi_1+\xi_2\psi_2+\xi_3\psi_3) \\
& +\dot{x}(\xi_2\xi_3-\psi_2\psi_3)-\dot{z}(\xi_3\psi_2-\xi_2\psi_3) \\
& +g_1(\psi_1\xi_2-\xi_1\psi_2)+g_2(\psi_1\xi_3-\xi_1\psi_3))+ \\
& \Phi_z((g\xi_0\xi_1+g_1\xi_0\xi_2+g_2\xi_0\xi_3)+(g\psi_0\psi_1+g_1\psi_0\psi_2+g_2\psi_0\psi_3) \\
& -(\dot{y}\psi_0\xi_1+f_1\psi_0\xi_2+f_2\psi_0\xi_3)-(\dot{y}\psi_1\xi_0+f_1\psi_2\xi_0+f_2\psi_3\xi_0) \\
& +(g\xi_2\xi_3+g_1\xi_3\xi_1+g_2\xi_1\xi_2)+(g\psi_2\psi_3+g_1\psi_3\psi_1+g_2\psi_1\psi_2) \\
& +(\dot{y}\xi_3\psi_2+f_1\xi_1\psi_3+f_2\xi_2\psi_1)-(\dot{y}\xi_2\psi_3+f_1\xi_3\psi_1+f_2\xi_1\psi_2) \\
& +\dot{x}(\psi_0\xi_0+\xi_1\psi_1+\xi_2\psi_2+\xi_3\psi_3))+ \\
& \Phi_{xx}(\psi_3\psi_1\xi_2\xi_0+\psi_3\psi_0\xi_2\xi_1-\psi_2\psi_1\xi_3\xi_0-\psi_2\psi_0\xi_3\xi_1)+ \\
& \Phi_{yy}(\psi_2\psi_0\xi_2\xi_0+\psi_3\psi_0\xi_3\xi_0-\psi_2\psi_1\xi_2\xi_1-\psi_3\psi_1\xi_3\xi_1)+ \\
& \Phi_{zz}(-\xi_3\xi_2\xi_1\xi_0+\psi_3\psi_2\xi_1\xi_0-\psi_1\psi_0\xi_3\xi_2+\psi_3\psi_2\psi_1\psi_0)+ \\
& \Phi_{xy}(\psi_2\psi_0\xi_3\xi_0-\psi_2\psi_0\xi_2\xi_1+\psi_3\psi_1\xi_2\xi_1-\psi_3\psi_0\xi_2\xi_0 \\
& -\psi_2\psi_1\xi_3\xi_1-\psi_3\psi_0\xi_3\xi_1-\psi_2\psi_1\xi_2\xi_0-\psi_3\psi_1\xi_3\xi_0)- \\
& \Phi_{xz}(\psi_2\xi_3\xi_1\xi_0+\psi_2\psi_1\psi_0\xi_3-\psi_3\psi_1\psi_0\xi_2+\psi_3\psi_2\psi_1\xi_0 \\
& +\psi_3\xi_2\xi_1\xi_0-\psi_0\xi_3\xi_2\xi_1+\psi_3\psi_2\psi_0\xi_1-\psi_1\xi_3\xi_2\xi_0)- \\
& \Phi_{yz}(\psi_0\xi_3\xi_2\xi_0+\psi_2\xi_2\xi_1\xi_0+\psi_3\xi_3\xi_1\xi_0-\psi_1\xi_3\xi_2\xi_1 \\
& -\psi_3\psi_2\psi_0\xi_0+\psi_3\psi_1\psi_0\xi_3+\psi_2\psi_1\psi_0\xi_2+\psi_3\psi_2\psi_1\xi_1)
\end{aligned}
\tag{A.1}
$$

and

$$
\begin{aligned}
\mathcal{L}_3^{(1)} = \;& \Phi\, g+A_x\dot{x}+A_y\dot{y}+ \\
& \Phi_x(\psi_0\xi_1+\psi_1\xi_0)+\Phi_y(\psi_1\xi_1-\psi_0\xi_0)-\Phi_z(\psi_1\psi_0+\xi_1\xi_0)\;.
\end{aligned}
\tag{A.2}
$$

B Appendix: Action for the (5,8,3) supermultiplet

The complete Lagrangian for the (5,8,3) multiplet reads

$$
\begin{aligned}
\mathcal{L}_5 \;=\;& \widetilde{\Phi}(\dot{v}_1^2+\dot{v}_2^2+\dot{w}^2+\dot{w}_1^2+\dot{w}_2^2+h_1^2+h_2^2+h_3^2)+\\
&\widetilde{\Phi}(\dot{\lambda}_0\lambda_0+\dot{\lambda}_1\lambda_1+\dot{\lambda}_2\lambda_2+\dot{\lambda}_3\lambda_3+\dot{\chi}_0\chi_0+\dot{\chi}_1\chi_1+\dot{\chi}_2\chi_2+\dot{\chi}_3\chi_3)+\\
&\widetilde{\Phi}_{v_1}[\dot{v}_2(\lambda_0\lambda_1+\lambda_2\lambda_3+\chi_1\chi_0+\chi_2\chi_3)+h_1(\lambda_1\lambda_3+\lambda_2\lambda_0+\chi_3\chi_1+\chi_2\chi_0)\\
&+h_2(\lambda_2\lambda_1+\lambda_3\lambda_0+\chi_0\chi_3+\chi_2\chi_1)+h_3(\lambda_0\chi_2+\lambda_3\chi_1+\lambda_2\chi_0+\chi_3\lambda_1)]+\\
&\widetilde{\Phi}_{v_2}[\dot{v}_1(\lambda_1\lambda_0+\lambda_3\lambda_2+\chi_0\chi_1+\chi_3\chi_2)+h_1(\lambda_2\lambda_1+\lambda_3\lambda_0+\chi_1\chi_2+\chi_3\chi_0)\\
&+h_2(\lambda_0\lambda_2+\lambda_3\lambda_1+\chi_3\chi_1+\chi_2\chi_0)+h_3(\chi_1\lambda_2+\lambda_3\chi_0+\lambda_1\chi_2+\lambda_0\chi_3)]+\\
&\widetilde{\Phi}_{w}[\dot{w}_1(\lambda_3\lambda_0+\lambda_1\lambda_2+\chi_0\chi_3+\chi_1\chi_2)+\dot{w}_2(\lambda_0\lambda_2+\lambda_3\lambda_1+\chi_2\chi_0+\chi_1\chi_3)\\
&+h_1(\chi_2\lambda_3+\chi_1\lambda_0+\chi_0\lambda_1+\lambda_2\chi_3)+h_2(\chi_1\lambda_1+\lambda_2\chi_2+\lambda_0\chi_0+\lambda_3\chi_3)\\
&+h_3(\lambda_1\lambda_0+\lambda_2\lambda_3+\chi_1\chi_0+\chi_3\chi_2)]+\\
&\widetilde{\Phi}_{w_1}[\dot{w}(\lambda_0\lambda_3+\lambda_2\lambda_1+\chi_2\chi_1+\chi_3\chi_0)+\dot{w}_2(\lambda_2\lambda_3+\lambda_1\lambda_0+\chi_0\chi_1+\chi_2\chi_3)\\
&+h_1(\lambda_3\chi_1+\chi_3\lambda_1+\chi_2\lambda_0+\chi_0\lambda_2)+h_2(\chi_1\lambda_2+\chi_2\lambda_1+\lambda_0\chi_3+\chi_0\lambda_3)\\
&+h_3(\lambda_2\lambda_0+\lambda_3\lambda_1+\chi_1\chi_3+\chi_2\chi_0)]+\\
&\widetilde{\Phi}_{w_2}[\dot{w}(\lambda_3\lambda_1+\lambda_2\lambda_0+\chi_0\chi_2+\chi_3\chi_1)+\dot{w}_1(\lambda_0\lambda_1+\lambda_3\lambda_2+\chi_1\chi_0+\chi_3\chi_2)\\
&+h_1(\chi_0\lambda_3+\lambda_1\chi_2+\chi_3\lambda_0+\chi_1\lambda_2)+h_2(\chi_3\lambda_1+\lambda_2\chi_0+\chi_2\lambda_0+\chi_1\lambda_3)\\
&+h_3(\lambda_3\lambda_0+\lambda_1\lambda_2+\chi_3\chi_0+\chi_2\chi_1)]+\\
&\widetilde{\Phi}_{ww}(\lambda_0\chi_0\lambda_1\chi_1+\lambda_2\chi_2\lambda_3\chi_3+\chi_0\chi_1\chi_2\chi_3)+\\
&\widetilde{\Phi}_{v_1v_1}(\lambda_0\chi_0\chi_2\lambda_2+\lambda_1\chi_1\lambda_3\chi_3+\lambda_0\chi_1\chi_2\lambda_3-\chi_0\lambda_1\lambda_2\chi_3+\lambda_0\lambda_1\lambda_2\lambda_3)+\\
&\widetilde{\Phi}_{v_2v_2}(\lambda_0\chi_0\chi_3\lambda_3+\lambda_1\chi_1\lambda_2\chi_2+\lambda_0\chi_1\lambda_2\chi_3-\chi_0\lambda_1\chi_2\lambda_3+\lambda_0\lambda_1\lambda_2\lambda_3)+\\
&\widetilde{\Phi}_{w_1w_1}(\lambda_0\chi_0\chi_2\lambda_2+\lambda_1\chi_1\lambda_3\chi_3-\lambda_0\lambda_1\chi_2\chi_3+\lambda_0\chi_1\lambda_2\chi_3+\chi_0\chi_1\lambda_2\lambda_3+\\
&-\chi_0\lambda_1\chi_2\lambda_3+\chi_0\chi_1\chi_2\chi_3)+\\
&\widetilde{\Phi}_{w_2w_2}(\lambda_0\chi_0\chi_3\lambda_3+\lambda_1\chi_1\lambda_2\chi_2-\lambda_0\lambda_1\chi_2\chi_3+\lambda_0\chi_1\chi_2\lambda_3+\chi_0\chi_1\lambda_2\lambda_3+\\
&-\chi_0\lambda_1\lambda_2\chi_3+\chi_0\chi_1\chi_2\chi_3)+\\
&\widetilde{\Phi}_{v_1v_2}(\lambda_0\chi_0\chi_2\lambda_3-\lambda_0\chi_0\lambda_2\chi_3-\lambda_0\chi_1\chi_2\lambda_2+\lambda_0\chi_1\chi_3\lambda_3+\chi_0\lambda_1\lambda_2\chi_2\\
&+\chi_0\lambda_1\chi_3\lambda_3+\lambda_1\chi_1\chi_2\lambda_3-\lambda_1\chi_1\lambda_2\chi_3)+\\
&\widetilde{\Phi}_{v_1w}(-\lambda_0\chi_0\lambda_1\lambda_2-\lambda_0\chi_0\chi_1\chi_2+\lambda_0\lambda_3\lambda_1\chi_1+\lambda_0\lambda_3\chi_2\lambda_2+\chi_0\chi_3\chi_1\lambda_1\\
&+\chi_0\chi_3\lambda_2\chi_2+\lambda_1\lambda_2\chi_3\lambda_3+\chi_1\chi_2\chi_3\lambda_3)+\\
&\widetilde{\Phi}_{v_1w_1}(\lambda_0\lambda_1\chi_2\lambda_3-\lambda_0\chi_1\lambda_2\lambda_3-\chi_0\lambda_1\lambda_2\lambda_3-\lambda_0\lambda_1\lambda_2\chi_3-\chi_0\lambda_1\chi_2\chi_3\\
&+\chi_0\chi_1\lambda_2\chi_3-\chi_0\chi_1\chi_2\lambda_3-\lambda_0\chi_1\chi_2\chi_3)+\\
&\widetilde{\Phi}_{v_1w_2}(\lambda_0\chi_0\chi_2\chi_3+\lambda_0\chi_0\lambda_2\lambda_3+\chi_0\chi_1\chi_2\lambda_2+\chi_0\chi_1\lambda_3\chi_3+\lambda_0\lambda_1\lambda_2\chi_2+\\
&\lambda_0\lambda_1\chi_3\lambda_3+\lambda_1\chi_1\lambda_2\lambda_3+\lambda_1\chi_1\chi_2\chi_3)+\\
&\widetilde{\Phi}_{v_2w}(\lambda_0\chi_0\chi_3\chi_1+\lambda_0\chi_0\lambda_3\lambda_1+\chi_0\chi_2\lambda_1\chi_1+\chi_0\chi_2\chi_3\lambda_3+\lambda_0\lambda_2\chi_1\lambda_1\\
&+\lambda_0\lambda_2\lambda_3\chi_3+\lambda_2\chi_2\lambda_3\lambda_1+\chi_2\lambda_2\chi_1\chi_3)+\\
&\widetilde{\Phi}_{v_2w_1}(-\lambda_0\chi_0\chi_2\chi_3-\lambda_0\chi_0\lambda_2\lambda_3+\chi_0\chi_1\chi_2\lambda_2+\chi_0\chi_1\lambda_3\chi_3+\lambda_0\lambda_1\lambda_2\chi_2\\
&+\lambda_0\lambda_1\chi_3\lambda_3+\lambda_1\chi_1\lambda_3\lambda_2+\lambda_1\chi_1\chi_3\chi_2)+\\
&\widetilde{\Phi}_{v_2w_2}(-\chi_0\lambda_1\lambda_2\lambda_3+\lambda_0\lambda_1\lambda_2\chi_3-\lambda_0\lambda_1\chi_2\lambda_3-\lambda_0\chi_1\lambda_2\lambda_3-\chi_0\chi_1\lambda_2\chi_3\\
&-\lambda_0\chi_1\chi_2\chi_3+\chi_0\chi_1\chi_2\lambda_3-\chi_0\lambda_1\chi_2\chi_3)+\\
&\widetilde{\Phi}_{ww_1}(-\lambda_0\chi_0\lambda_1\chi_2+\lambda_0\chi_0\chi_1\lambda_2-\chi_0\lambda_3\lambda_1\chi_1+\chi_0\lambda_3\lambda_2\chi_2-\lambda_0\chi_3\lambda_1\chi_1\\
&+\lambda_0\chi_3\lambda_2\chi_2-\lambda_3\chi_3\lambda_1\chi_2+\lambda_3\chi_3\chi_1\lambda_2)+\\
&\widetilde{\Phi}_{ww_2}(\lambda_0\chi_0\chi_3\lambda_1-\lambda_0\chi_0\lambda_3\chi_1-\chi_0\lambda_2\chi_1\lambda_1+\chi_0\lambda_2\chi_3\lambda_3+\lambda_0\chi_2\lambda_1\chi_1\\
&+\lambda_0\chi_2\chi_3\lambda_3+\lambda_1\chi_3\chi_2\lambda_2-\chi_1\lambda_3\chi_2\lambda_2)+\\
&\widetilde{\Phi}_{w_1w_2}(\lambda_0\chi_0\chi_2\lambda_3-\lambda_0\chi_0\lambda_2\chi_3+\chi_0\lambda_1\chi_2\lambda_2+\chi_0\lambda_1\lambda_3\chi_3+\lambda_0\chi_1\chi_2\lambda_2\\
&+\lambda_0\chi_1\lambda_3\chi_3+\chi_2\lambda_3\lambda_1\chi_1+\lambda_2\chi_3\chi_1\lambda_1)\;. \qquad\qquad (B.1)
\end{aligned}
$$

C Appendix: $\mathcal{N}$=4 duality

It is instructive to display the simpler case of $\mathcal{N}$=4 duality. Since only the (1,4,3) multiplet allows for an inhomogeneous deformation, we concentrate on the d=1 / d=3 duality and the coupling of these two multiplets.

Like in the $\mathcal{N}$=8 cases, the $\mathcal{N}$=4 Lagrangians have the form

$$\mathcal{L}_d \;=\; \Phi\,\delta_{ab}\dot{x}^a\dot{x}^b \;+\; \ldots \;. \tag{C.1}$$

Scale (D) and special conformal (K) invariance require

$$\Phi \;=\; r^\beta\,Y(\text{angles}) \qquad \text{and} \qquad \dot{\Phi}\,r^2 = \tfrac{\mathrm{d}}{\mathrm{d}t}Z \qquad \text{for} \qquad r^2 = x^a x^a \;, \tag{C.2}$$

with some exponent β and functions Y and Z. It follows that $Z = \frac{c}{c+2}r^{\beta+2}Y$ and Y = constant.

Let us denote the components of the two multiplets by

$$\begin{cases} (1,4,3): & x;\psi_0,\psi_1,\psi_2,\psi_3;f_1,f_2,f_3 \\ (3,4,1): & v_1,v_2,v_3;\lambda_0,\lambda_1,\lambda_2,\lambda_3;h \end{cases} \tag{C.3}$$

and assign scaling dimensions ($i=0,1,2,3$ and $a=1,2,3$)

$$[x,\psi_i,f_a] = -1,-\tfrac{1}{2},0 \qquad \text{and} \qquad [v_a,\lambda_i,h] = 1,\tfrac{3}{2},2 \;, \tag{C.4}$$

so that the conformal factors for a dimensionless action become

$$\Phi \;=\; x^{-1} \qquad \text{and} \qquad \widetilde{\Phi} \;=\; v^{-3} \quad \text{with} \quad v^2 = v_a v_a \;. \tag{C.5}$$

The bosonic target space is therefore the product of a (half) line with a three-dimensional cone. The supersymmetry transformations are given by

	Q_1	Q_2	Q_3	Q_4
x	ψ_1	ψ_2	ψ_3	ψ_0
ψ_0	f_1	f_2	f_3	$\dot{x}$
ψ_1	$\dot{x}$	f_3+c	$-f_2$	$-f_1$
ψ_2	$-f_3-c$	$\dot{x}$	f_1	$-f_2$
ψ_3	f_2	$-f_1$	$\dot{x}$	$-f_3$
f_1	$\dot{\psi}_0$	$-\dot{\psi}_3$	$\dot{\psi}_2$	$-\dot{\psi}_1$
f_2	$\dot{\psi}_3$	$\dot{\psi}_0$	$-\dot{\psi}_1$	$-\dot{\psi}_2$
f_3	$-\dot{\psi}_2$	$\dot{\psi}_1$	$\dot{\psi}_0$	$-\dot{\psi}_3$

	Q_1	Q_2	Q_3	Q_4
v_1	λ_0	$-\lambda_3$	λ_2	$-\lambda_1$
v_2	λ_3	λ_0	$-\lambda_1$	$-\lambda_2$
v_3	$-\lambda_2$	λ_1	λ_0	$-\lambda_3$
λ_0	$\dot{v}_1$	$\dot{v}_2$	$\dot{v}_3$	h
λ_1	h	$\dot{v}_3$	$-\dot{v}_2$	$-\dot{v}_1$
λ_2	$-\dot{v}_3$	h	$\dot{v}_1$	$-\dot{v}_2$
λ_3	$\dot{v}_2$	$-\dot{v}_1$	h	$-\dot{v}_3$
h	$\dot{\lambda}_1$	$\dot{\lambda}_2$	$\dot{\lambda}_3$	$\dot{\lambda}_0$

(C.6)

with inhomogeneous parameter c. The transformations can be written in terms of the quaternionic structure constants δ_{ab} and ϵ_{abc} (with $\epsilon_{123}=1$). We note that the two multiplets must have the same chirality to be coupled. Therefore, the overall sign of ϵ_{123} in the second multiplet is fixed in order to allow the supersymmetric pairing of the multiplets.

The superconformally invariant action of the coupled system is given as a sum of three terms,

$$\mathcal{L}_{1+3} \;=\; \mathcal{L}_1 \;+\; \mathcal{L}_3 \;+\; \gamma\mathcal{L}_{1,3}\;, \tag{C.7}$$

with

$$\mathcal{L}_1 \;=\; Q_4Q_3Q_2Q_1F(x) \qquad \text{and} \qquad \mathcal{L}_3 \;=\; Q_4Q_3Q_2Q_1\widetilde{F}(\vec{v})\;. \tag{C.8}$$

The supersymmetric pairing term reads ($a = 1,2,3$ and $i = 0,1,2,3$)

$$\begin{aligned}
\mathcal{L}_{1,3} &= \mathcal{L}_{1,3}^{(0)} + \mathcal{L}_{1,3}^{(1)} + \mathcal{L}_{1,3}^{(2)}\;,\\
\mathcal{L}_{1,3}^{(0)} &= x\,h - f_a v_a + \psi_i\lambda_i\;,\\
\mathcal{L}_{1,3}^{(1)} &= \omega_1\lambda_1 + \omega_2\lambda_2\;,\\
\mathcal{L}_{1,3}^{(2)} &= e\,h\;,
\end{aligned} \tag{C.9}$$

where the extra constants ω_1, ω_2 and e have been added, with scaling dimensions $[\omega_1] = [\omega_2] = -\frac{1}{2}$ and $[e] = -1$. The supersymmetry transformations of the constant (1,2,1) multiplet are

	Q_1	Q_2	Q_3	Q_4
e	ω_1	ω_2	0	0
ω_1	0	$-c$	0	0
ω_2	c	0	0	0
c	0	0	0	0

(C.10)

An alternative coupling possibility is the following,

$$\begin{aligned}
\mathcal{L}_{1,3} &= \mathcal{L}_{1,3}^{(0)} + \mathcal{L}_{1,3}^{(0')} + \mathcal{L}_{1,3}^{(1')} + \mathcal{L}_{1,3}^{(2')}\;,\\
\mathcal{L}_{1,3}^{(0)} &= x\,h - f_a v_a + \psi_i\lambda_i\;,\\
\mathcal{L}_{1,3}^{(0')} &= -c\,v_3\;,\\
\mathcal{L}_{1,3}^{(1')} &= \omega_0\lambda_0 + \omega_3\lambda_3\;,\\
\mathcal{L}_{1,3}^{(2')} &= e'h\;,
\end{aligned} \tag{C.11}$$

where the extra constants ω_0, ω_3 and e' have been added, with scaling dimensions $[\omega_0] = [\omega_3] = -\frac{1}{2}$ and $[e'] = -1$. The supersymmetry transformations of this constant (1,2,1) multiplet are

	Q_1	Q_2	Q_3	Q_4
e'	0	0	ω_3	ω_0
ω_0	0	0	c	0
ω_3	0	0	0	$-c$
c	0	0	0	0

(C.12)

The Lagrangians of the one- and three-dimensional systems read

$$\begin{aligned}
\mathcal{L}_1 &= \Phi\left\{\dot{x}^2 + f_a^2 + \dot{\psi}_0\psi_0 + \dot{\psi}_a\psi_a\right\}\\
&+ \Phi_x\left\{\psi_0\psi_a f_a + \tfrac{1}{2}\epsilon_{abc}\psi_a\psi_b f_c\right\} \;+\; \Phi_{xx}\left\{\tfrac{1}{6}\epsilon_{abc}\psi_0\psi_a\psi_b\psi_c\right\}\\
&+ c\,\Phi\,f_3 \;+\; c\,\Phi_x\psi_0\psi_3
\end{aligned} \tag{C.13}$$

and

$$\begin{aligned}\mathcal{L}_3 \;&=\; \widetilde{\Phi}\,\big\{\dot{v}_a^2 + h^2 + \dot{\lambda}_0\lambda_0 + \dot{\lambda}_a\lambda_a\big\} \\ &+\; \widetilde{\Phi}_a\big\{\lambda_a\lambda_b\dot{v}_b + \epsilon_{abc}\big(\tfrac{1}{2}\lambda_b\lambda_c h - \lambda_0\lambda_b\dot{v}_c\big)\big\} + \tfrac{1}{6}\Delta\widetilde{\Phi}\,\epsilon_{abc}\lambda_0\lambda_a\lambda_b\lambda_c\ ,\end{aligned} \tag{C.14}$$

where

$$\Phi \;=\; F_{xx} \qquad \text{and} \qquad \widetilde{\Phi} \;=\; \Delta\widetilde{F} \;\equiv\; \widetilde{F}_{aa}\ , \tag{C.15}$$

respectively.

Finally we add Fayet-Iliopoulos terms which are superconformal (not just supersymmetric) invariants and introduce dimensionful constants μ_a and ν,

$$\mathcal{L}_{\rm FI} \;=\; \mu_a f_a - \nu\, h\ , \tag{C.16}$$

with $[\mu_a] = 1$ and $[\nu] = -1$. The supersymmetry transformations act trivially on μ_a and ν.

Setting all fermions to zero, the total bosonic Lagrangian based on (C.7) with (C.9) becomes

$$\mathcal{L}'_{1+3}\big| \;=\; \Phi\,(\dot{x}^2 + f_a^2) + \widetilde{\Phi}\,(\dot{v}_a^2 + h^2) - (\gamma v_a - \mu_a - c\delta_{a3}\Phi)\,f_a + (\gamma x + \gamma e - \nu)\,h\ . \tag{C.17}$$

If we use (C.11) instead, an additional term $-\gamma c\, v_3$ appears. Eliminating the auxiliary fields via

$$f_a \;=\; \tfrac{1}{2}\Phi^{-1}(\gamma v_a - \mu_a - c\delta_{a3}\Phi) \qquad \text{and} \qquad h \;=\; -\tfrac{1}{2}\widetilde{\Phi}^{-1}(\gamma x + \gamma e - \nu), \tag{C.18}$$

we arrive at

$$\mathcal{L}''_{1+3}\big| \;=\; \Phi\,\dot{x}^2 + \widetilde{\Phi}\,\dot{v}_a^2 - \tfrac{1}{4}\Phi^{-1}\big(\gamma v_a - \mu_a - c\delta_{a3}\Phi\big)^2 - \tfrac{1}{4}\widetilde{\Phi}^{-1}\big(\gamma(x{+}e) - \nu\big)^2\ , \tag{C.19}$$

where the Lagrange multiplier e only ensures that the zero mode $\overline{h}$ vanishes. Hence, its value is $e = -\overline{(\widetilde{\Phi}^{-1}x - \nu/\gamma)}/\overline{\widetilde{\Phi}^{-1}}$. Specializing to $\Phi = x^{-1}$ and $\widetilde{\Phi} = v^{-3}$, one gets

$$\mathcal{L}''_{1+3}\big| \;=\; x^{-1}\dot{x}^2 + v^{-3}\dot{v}_a^2 - \tfrac{1}{4}x\,\big(\gamma v_a - \mu_a - c\delta_{a3}x^{-1}\big)^2 - \tfrac{1}{4}v^3\big(\gamma(x{+}e) - \nu\big)^2\ . \tag{C.20}$$

In order to interpret this Lagrangian, we pass to standard kinetic terms (up to a factor of $\frac{1}{2}$) by changing the radial coordinates via

$$x = \tfrac{1}{4}\rho^2 \qquad \text{and} \qquad v = 4\sigma^{-2} \qquad \text{with} \quad [\rho] = [\sigma] = -\tfrac{1}{2} \tag{C.21}$$

and arrive at

$$\mathcal{L}^{\rm cone}_{1+3}\big| \;=\; \dot{\rho}^2 + \dot{\sigma}^2 + 4\widetilde{\ell}^2\sigma^{-2} - \tfrac{1}{16}\rho^2\big(\gamma\sigma^{-2}\vec{e}_\sigma - \vec{\mu} - 4c\rho^{-2}\vec{e}_3\big)^2 - \sigma^{-6}\big(\gamma(\rho^2{+}4e) - 4\nu\big)^2\ , \tag{C.22}$$

where $\widetilde{\ell}$ is the angular momentum in σ space, and $\vec{e}_\sigma$ and $\vec{e}_3$ denote unit vectors in the σ and 3 directions, respectively. We find a rather complicated potential in the four-dimensional target. If one employs the option (C.11), then linear terms $-\gamma c\, v_3$ and $-4\gamma c\,\sigma^{-2}\vec{e}_3$ have to be added to (C.20) and (C.22), respectively.

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