An Angular Dependent Supersymmetric Quantum Mechanics with a $\mathbf{Z}_2$-invariant Potential

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Abstract

We generalize the conformally invariant topological quantum mechanics of a particle propagating on a punctured plane by introducing a potential that breaks both the rotational and the conformal invariance down to a $\mathbf{Z}_2$ angular-dependent discrete symmetry. We derive a topological quantum mechanics whose localization gauge functions give interesting self-dual equations. The model contains an order parameter and exhibits a spontaneous symmetry breaking with two ground states above a critical scale. Unlike the ordinary $O(2)$-invariant Higgs potential, an angular-dependence is found and saddle points, instead of local maxima, appear, posing subtle questions about the existence of instantons. The supersymmetric quantum mechanical model is constructed in both the path integral and the operatorial frameworks.

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1 Introduction

In [1] a solvable model for a superconformal quantum mechanical system was introduced, giving an example of a quantum topological theory with no ground state and no mass gap. In [2] this model was extended to the case of a possibly infinite chain. This toy model was motivated by the search of an abstract definition of mechanical systems possessing epicycle descriptions, in an attempt to encode the robotic movements of rods presented in [3].

More refined models are expected to have a breaking of conformal symmetry. It is tempting to look for new classes of potentials that break other types of symmetries as well, like the rotational invariance, while keeping some of the properties of the topological models [1, 2]. Here we generalize the topological invariant $\int_\gamma d\theta$ that defines the topological quantum mechanics of [1] into another invariant, which has a further dependence on a point A distinct from the singular origin O of the punctured plane; this allows us to introduce a length $a = |\overrightarrow{OA}|$ and a dimensionless order parameter. The model is defined by a potential which breaks both the conformal and the rotational symmetry, but keeps a discrete $\mathbf{Z}_2$ symmetry. This discrete symmetry is reminiscent of the T-duality transformation in string theory, see [4]. It carries, however, an extra angular dependence.

The introduction of the length scale allows for the existence of a discrete spectrum and a normalizable ground state. A spontaneous symmetry breaking is found. Indeed, above a critical value for the order parameter, two ground states are found. Two main properties single out this potential with respect to the standard Higgs potential. The first one is the presence of an angular-dependence; the second one is the fact that, above the critical scale, the potential does not possess any local maximum, but only (two) saddle points. The possibility, following [5], of the existence of instantons is investigated.

2 A reminder of conformal topological mechanics

The conformally invariant topological quantum mechanics of [1] is defined by the topological gauge-fixing following the BRSTQFT, i.e. localization, scheme of [6] for the topological invariant $\int_\gamma d\theta$ on a punctured plane with real coordinates q^i ($i = 1, 2, x \equiv q^1, y \equiv q^2$) or complex coordinates $q^1 + iq^2 = \sqrt{|\vec{q}|^2} \exp i\theta$.

By introducing a coupling constant g, the topological classical action [1] reads

$$2\pi N g = g \int_\gamma d\theta = g \int_\gamma dt \, \dot\theta = g \int_\gamma dt \epsilon_{ij} \frac{q^i \dot{q}^j}{2|\vec{q}|^2}, \tag{2.1}$$

where t is a Euclidean time and the dot means the time derivative $\frac{d}{dt}$.

This action must be gauge-fixed in a topological BRST invariant way. Using the complex coordinate notation $z(t) \equiv q^1(t) + iq^2(t)$ ($\bar{z} = z^*$), the topological gauge-fixing equation was obtained in [1] as:

$$\dot{z} - \frac{ig}{\bar{z}} = 0. \tag{2.2}$$

The BRSTQFT gauge invariant procedure provides a conformally invariant supersymmetric action which, once untwisted, gives a $\mathcal{N} = 2$ supersymmetry.

The solutions of (2.2) are periodic instantons $z_N(t)$, where N is an integer. We have

$$z_N = \frac{1}{\sqrt{2\pi N g}} \exp(2\pi i N t). \tag{2.3}$$

They minimize (to the zero value) the non-negative bosonic action

$$\int_\gamma dt(|\dot{\vec{q}}|^2 + \frac{g^2}{\vec{q}^2} + g\dot\theta). \tag{2.4}$$

The instantons are pseudo-particles that make N cycles per unit time at constant angular velocity around the singular origin on circles with radius $|z_N| = \frac{1}{\sqrt{2\pi N g}}$. The area Σ_N of the surface that the instanton z_N circles around is independent on the value of N:

$$\Sigma_N = N\pi(\frac{1}{\sqrt{2\pi N g}})^2 = \frac{g}{2}. \tag{2.5}$$

The instanton structure follows from the identity

$$|\dot{\vec{q}}|^2 + \frac{g^2}{|\vec{q}|^2} = \frac{1}{2}(\dot{q}_i - g\epsilon_{ij}\frac{q^j}{\sqrt{2}|\vec{q}|^2})^2 + \frac{1}{2}(\dot{q}_i + g\epsilon_{ij}\frac{q^j}{\sqrt{2}|\vec{q}|^2})^2, \tag{2.6}$$

where the vector indices i, j are raised/lowered by the Euclidean metric δ_{ij}.

The equation of motion, derived from the supersymmetric action, of the fermionic partner $\Psi = \Psi^1 + i\Psi^2$ of z, corresponds to the supersymmetric variation Q of Eq. (2.2). We have

$$Q(\dot{z} - \frac{ig}{\bar{z}}) = ig\dot{\Psi}^* + \frac{g}{|\vec{q}|^2}U_\theta\Psi = 0. \tag{2.7}$$

Here $U_\theta = R_\theta C R_{-\theta}$, where R_θ represents an $SO(2)$ rotation ($\det R_\theta = 1$) with angle θ and C is a conjugation matrix with $\det C = -1$; therefore $\det U_\theta = -1$.

The representations of the matrices acting on vectors are

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

$$U_\theta = |\vec{q}|^2 \frac{\delta^2\theta}{\delta q_i \delta q_j} = |\vec{q}|^2 \frac{\delta^2 \log\sqrt{|\vec{q}|^2}}{\delta q_i \delta q_j} = R_\theta C R_{-\theta} = \begin{pmatrix} -\sin 2\theta & \cos 2\theta \\ \cos 2\theta & \sin 2\theta \end{pmatrix}. \tag{2.8}$$

In complex notation one has $U_\theta z = i\exp(-2i\theta)z^*$.

The action of the rotation $R_{\pm\theta}$ on the instanton z_N is a multiplication by $\exp(\pm 2\pi iNt)$. Therefore, the solution of the fermion equation (2.7) in the z_N background is

$$\Psi(t) = \Psi_0 \exp(\pm i2\pi Nt), \tag{2.9}$$

where Ψ_0 is time independent and

$$\Psi_0^* \pm C\Psi_0 = 0. \tag{2.10}$$

It follows that Ψ_0 is just an eigenvector of the operator C which runs attached to the instanton. One has

$$\Psi_0 = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}. \tag{2.11}$$

The fermionic zero modes can be thought as tangent vectors cycling at constant frequencies $2\pi Ng$ around the circles of radius $\sqrt{2\pi Ng}$. Their existence implies a non-trivial cohomology for the BRST symmetry of the theory. The paper [1] discusses in detail the model and the way it allows to compute some (simple) topological invariants of the punctured plane.

3   A new model and its topological gauge-fixing

Let us introduce a point $A \neq O$ on the punctured plane with singular origin O. We define the orthonormal coordinates $(x = q^1, \ y = q^2)$ such that $\overrightarrow{OA} = (a, 0)$. We also use the complex and polar coordinates $z = q^1 + iq^2 = r\exp(i\theta)$ when needed.

Given any contour γ in the punctured plane, we add to the closed and not-exact form $gd\theta$ an exact-term that depends on the distance of the particle to the point A. The simplest possibility is offered by $gd\theta + \frac{\omega}{2}d(|\vec{q} - \overrightarrow{OA}|^2)$. We therefore define the classical topological action $I_\gamma^{cl}(g, \omega)$ of the new model as

$$I_\gamma^{cl}(g, \omega) \;=\; \int_\gamma \left(gd\theta + \frac{\omega}{2}d(|\vec{q} - \overrightarrow{OA}|^2) \right). \tag{3.1}$$

The action $I_\gamma^{cl}(g, \omega)$ is independent of any infinitesimal shift of coordinates

$$\delta r = \epsilon_r(t), \qquad \delta\theta = \epsilon_\theta(t), \tag{3.2}$$

with appropriate boundary conditions.

Following the general BRSTQFT scheme of [6], we are interested in computing observables that satisfy the Ward identities corresponding to the above gauge symmetry (3.2). We will come back on this later, by showing the possibility of fermionic zero modes. Comparing the actions (2.1) and (3.1) one understands that the topological gauge-fixing (2.2) must be improved into

$$\dot{z} \;=\; \frac{ig}{\overline{z}} + \omega(z - a). \tag{3.3}$$

The above equation reads, in polar coordinates, as

$$\dot{r} = \omega(r - a\cos\theta), \qquad r\dot{\theta} = \frac{g}{r} + \omega a \sin\theta \tag{3.4}$$

and, in Cartesian coordinates, as

$$\dot{x} = \frac{gy}{x^2 + y^2} + \omega(x - a), \qquad \dot{y} = -\frac{gx}{x^2 + y^2} + \omega y. \tag{3.5}$$

The parametrization expressed by a, ω, g can be simplified through redefinitions, as we will shortly see. On the other hand, to have the possibility of discussing several interesting limits, it is better to keep track of the explicit dependence on these parameters.

The supersymmetric Q-exact action that localizes the topological gauge fixing (3.3) is easier to compute in Cartesian coordinates. It transforms the topological term into a supersymmetric action. It gives, on the same time, a consistent (that is, BRSTQFT-invariant) information about the g, ω, a dependence of the theory. We get

$$
\begin{aligned}
I_\gamma^{cl}(g, \omega) &= \int_\gamma \Big(gd\theta + \frac{\omega}{2}d(|\vec{q} - \overrightarrow{OA}|^2)\Big) \mapsto \\
\mapsto I_\gamma^{susy}(g, \omega) &= \int_\gamma \Big(gd\theta + \frac{\omega}{2}d(|\vec{q} - \overrightarrow{OA}|^2) + Q\Big(\overline{\Psi}_x(\dot{x} - \frac{gy}{x^2 + y^2} - \omega(x - a) + \frac{1}{2}H_x) + \\
&\qquad + \overline{\Psi}_y(\dot{y} + \frac{gx}{x^2 + y^2} - \omega y + \frac{1}{2}H_y)\Big)\Big).
\end{aligned} \tag{3.6}
$$

The time-dependent fields $\Psi_x(t), \Psi_y(t), \overline{\Psi}_x(t), \overline{\Psi}_y(t)$ are fermion (that is, Grassmann coordinates), while the time-dependent fields $H_x(t), H_y(t)$ are auxiliary bosonic fields satisfying algebraic equations of motion. The nilpotent operator Q ($Q^2 = 0$, which also satisfies $[Q, d/dt] = 0$) acts as follows on the fields

$$
\begin{aligned}
Qx &= -\Psi_x, & Qy &= -\Psi_y, \\
Q\overline{\Psi}_x &= H_x, & Q\overline{\Psi}_y &= H_y
\end{aligned} \tag{3.7}
$$

(and vanishing otherwise).

The computation of the Q-transformation of the topological gauge function provides a Dirac-type operator acting on fermions which, in Cartesian coordinates, reads as

$$
\mathcal{D}_{ij} = \delta_{ij}(\frac{d}{dt} - \omega) - g\frac{\epsilon_{ik}}{|\vec{q}|^2}(\delta_{jk} - 2\frac{q^j q^k}{|\vec{q}|^2}). \tag{3.8}
$$

Therefore

$$
\mathcal{D}_{ij} = \delta_{ij}(\frac{d}{dt} - \omega) + \frac{g}{|\vec{q}|^2}\begin{pmatrix} -\sin 2\theta & \cos 2\theta \\ \cos 2\theta & \sin 2\theta \end{pmatrix}_{ij} = \delta_{ij}(\frac{d}{dt} - \omega) + \frac{g}{|\vec{q}|^2}(R_\theta C R_{-\theta})_{ij},
$$

$$
\tag{3.9}
$$

where R_θ and C have been introduced in Eq. (2.8).

We postpone to Section **7** some discussion about the supersymmetry of the model given by the action $I_\gamma^{susy}(g, \omega)$.

4 The bosonic part of the action

We are interested at first in the vacuum and therefore in the bosonic part of the Euclidean action $I_\gamma^{susy}(g,\omega)$. After expanding the Q-exact term in (3.6), eliminating the auxiliary fields H_x, H_y by their algebraic equations of motion and discarding the terms involving fermions, one obtains the following Euclidean bosonic action (written, for convenience, in polar coordinates). One has

$$I_{bosonic} = \int_\gamma dt(\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + V(r,\theta)), \qquad (4.1)$$

where the potential $V(r,\theta)$ is non-negative, being the sum of squares:

$$\begin{aligned} V(r,\theta) &= \frac{1}{2}\omega^2(r - a\cos\theta)^2 + \frac{1}{2}(\frac{g}{r} + \omega a\sin\theta)^2 = \\ &= \frac{1}{2}\frac{g^2}{r^2} + \frac{1}{2}\omega^2 r^2 + \frac{ag\omega}{r}\sin\theta - \omega^2 ar\cos\theta + \frac{1}{2}\omega^2 a^2. \end{aligned} \qquad (4.2)$$

The potential possesses the interesting $\mathbf{Z}_2$ discrete symmetry T ($T^2 = \mathbf{I}$), given by

$$\frac{g}{r} \leftrightarrow \omega r, \qquad \cos\theta \leftrightarrow -\sin\theta. \qquad (4.3)$$

The symmetry can be expressed as

$$T \quad : \quad r \mapsto r' = \frac{r_0^2}{r}, \qquad \theta \mapsto \theta' = -\theta - \frac{\pi}{2}, \qquad (4.4)$$

where r_0, given by

$$r_0 = \sqrt{\frac{g}{\omega}}, \qquad (4.5)$$

can be set as a unit of length. One can thus define the dimensionless parameter

$$\alpha \equiv \frac{a}{r_0} = \sqrt{\frac{a^2\omega}{g}}. \qquad (4.6)$$

The potential (4.2) can be expressed as the product of the dimensional factor $g\omega$ times a dimensionless function:

$$V(r,\theta) = g\omega\left(\frac{1}{2}\left(\frac{r_0}{r}\right)^2 + \frac{1}{2}\left(\frac{r}{r_0}\right)^2 + \alpha\left(\frac{r_0}{r}\sin\theta - \frac{r}{r_0}\cos\theta\right) + \frac{1}{2}\alpha^2\right). \qquad (4.7)$$

The action (4.1) is both conformally and rotationally invariant at $\alpha = 0$. Both symmetries are broken for $\alpha \neq 0$. On the other hand the discrete symmetry T in (4.4) of the potential is maintained. We will show that the potential exhibits a transition between a simple and a double well, with a critical value for the dimensionless parameter α.

If we use r_0 as the unit of length the only relevant parameter of the model is $\alpha = a$. Therefore, the parameter of the conformal symmetry breaking is the distance between the rest point of the harmonic oscillator and the singularity of the potential $1/r^2$ expressed in units of r_0. At vanishing α the restored conformal symmetry is the well-known hidden $SL(2)$ symmetry [7, 8, 9] of quantum theories with potential $r^2 + 1/r^2$. The $SL(2)$ conformal symmetry without the oscillatorial term was introduced in [10] (see [1] and [11] for its formulation in the topological setting).

5 The shape of the potential. Are there instantons?

We follow the method explained by Coleman in [5]. The potential (4.2) is a sum of squares. The instantons are solutions of the Euclidean equations of motion giving a finite value to the action. Since they must make it extremal, and since the action (4.1) is non-negative, they are solutions of Eq. (3.3). Avoiding the possibility of bounces, they should connect pairs of local maxima $M^{\pm}$ of the Euclidean potential $-V(x, \theta)$, with $V(r^+, \theta^+) = V(r^-, \theta^-)$, leaving and reaching $M^{\pm}$ at zero velocity.

In quantum mechanics with more than one degree of freedom one is often concerned with the stability of the Euclidean solutions, even when they are energetically possible for connecting two vacua. We should point out that in the present case the potential (4.7), whose introduction is justified by geometrical considerations, leads to self-dual equations (3.4) that cannot be solved analitically. We also point out, depending on the order parameter (the coupling constant) of the model, that extrema of the potentials are local minima, local maxima and saddle points. Despite these difficulties, it is instructive to analyze the possibility for instantons and to draw conclusions about the classical solutions by analyzing the shape of the potential.

Therefore we check at first under which condition two $M^{\pm}$ maxima are obtained. The Euclidean potential $-V(r, \theta)$ is always negative or null, with $V \to \infty$ when $r \to \infty$ and $r \to 0$. It is obvious that $-V(r, \theta)$, for continuity reasons due to the behaviour of the potential at $r = \infty$ and $r = 0$, has at least one absolute maximum. The two equations from Eq. (3.3) give, at $\dot{r} = \dot{\theta} = 0$, the condition for an instanton to start at zero velocity from a local maximum of the potential:

$$\omega(r - a \cos \theta) = 0, \qquad \frac{g}{r} + \omega a \sin \theta = 0. \tag{5.1}$$

At $\omega \neq 0$, $a \neq 0$, these equations are equivalent to

$$r = a \cos \theta, \qquad \sin 2\theta = -\frac{2g}{\omega a^2}, \qquad \tan \theta = -\frac{g}{\omega r^2}. \tag{5.2}$$

It is convenient to express all results in terms of the two fundamental parameters, the unit length r_0 introduced in (4.5) and the non-negative dimensionless constant α ($\alpha \geq 0$) introduced in (4.6).

By using this parametrization, the second equation in (5.2), since $|\sin 2\theta| \neq 1$, admits solution only in the range

$$\alpha \geq \sqrt{2}. \tag{5.3}$$

Furthermore, only two maxima $M^{\pm}$ of the potential $-V(r, \theta)$, with $V(M^{\pm}) = 0$, can exist above the critical value $\alpha_c = \sqrt{2}$ for α.

We give the explicit expression of the two maxima $M^{\pm}$, both in polar $(r^{\pm}, \theta^{\pm})$ and in Cartesian $(x^{\pm}, y^{\pm})$ coordinates. It is convenient to introduce the angle φ,

$$\varphi = \frac{1}{2} \arcsin(\frac{2}{\alpha^2}), \qquad 0 < \varphi \leq \frac{\pi}{4}. \tag{5.4}$$

In terms of φ we can write, in polar coordinates,

$$r^{+} = \alpha r_0 \cos \varphi, \qquad \theta^{+} = -\varphi,$$

$$r^{-} = \alpha r_0 \sin \varphi, \qquad \theta^{-} = \varphi - \frac{\pi}{2}. \tag{5.5}$$

In Cartesian coordinates we have

$$x^{+} = \alpha r_0 \cos^2 \varphi, \qquad y^{+} = -\frac{1}{2}\alpha r_0 \sin(2\varphi) = -\frac{r_0}{\alpha},$$

$$x^{-} = \alpha r_0 \sin^2 \varphi, \qquad y^{-} = -\frac{1}{2}\alpha r_0 \sin(2\varphi) = -\frac{r_0}{\alpha}. \tag{5.6}$$

Due to the discrete T symmetry of the potential, the following relations are found:

$$x^+ + x^- = \alpha r_0, \qquad\qquad y^+ = y^-, \tag{5.7}$$

as well as

$$\sqrt{r^+ r^-} = r_0, \qquad\qquad \theta^+ + \theta^- = -\frac{\pi}{2}. \tag{5.8}$$

In order to further clarify the structure of the theory we compute the extremal points of the potential (4.7), obtained by solving the coupled system of equations

$$\frac{\partial V}{\partial \theta} = 0 \quad \rightarrow \quad \frac{r}{r_0}\cos\theta + \frac{r}{r_0}\sin\theta = 0,$$

$$\frac{\partial V}{\partial r} = 0 \quad \rightarrow \quad r^4 - r_0^4 - \alpha r_0 r(r_0^2 \sin\theta + r^2 \cos\theta) = 0. \tag{5.9}$$

At the extrema, we check the sign of the second-order derivatives to determine local minima, maxima and saddle points. We present the complete set of solutions in the different ranges of the order parameter α:

i) at $\alpha = 0$ we recover the $O(2)$ rotational invariant theory. The case is analogous to the one investigated in [1], with the important difference that the harmonic term allows for a discrete spectrum with a normalizable vacuum. The unbroken $O(2)$ symmetry implies a $\mathbf{S}^1$ circle of degenerate maxima. It is given by the points at distance r_0 from the origin and $-V(r_0, \theta) < 0$;

ii) in the $0 < \alpha < \sqrt{2}$ range two extremal points are found (we call them M^{0a} and M^{0b}). Their polar coordinates (which do not depend on α) are

$$M^{0a} \equiv (r_0, -\frac{\pi}{4}), \qquad M^{0b} \equiv (r_0, \frac{3\pi}{4}). \tag{5.10}$$

By inspecting the second-order derivatives in M^{0a}, M^{0b}, one proves that M^{0a} is the unique maximum of $-V(r, \theta)$, while M^{0b} is a saddle point. We have

$$-V(M^{0b}) < -V(M^{0a}) < 0; \tag{5.11}$$

iii) at $\alpha = \sqrt{2}$, the critical case, we have two extremal points as before. This time

$-V(M^{0\alpha}) = 0;$

iv) in the range $\alpha > \sqrt{2}$ we have four extremal points, the two maxima $M^{\pm}$ given by Eq. (5.5) and the two points M^{0a}, M^{0b} given by Eq. (5.10). The two maxima are such that $-V(M^{\pm}) = 0$, while in this range of α both M^{0a}, M^{0b} are saddle points.

The value of the potential $V(r, \theta)$ from Eq. (4.7), computed at M^{0a}, M^{0b}, reads

$$V(M^{0a,b}) = g\omega(1 + \frac{1}{2}\alpha^2 \pm \alpha\sqrt{2}), \tag{5.12}$$

where the sign $-$ $(+)$ corresponds to M^{0a} (M^{0b}).

Energetically one has the possibility of a finite action trajectory linking $-V(M^{\pm})$ and passing through M^{0a} and/or M^{0b}. It is an open question, left for future investigation, whether such a trajectory indeed exists.

6 The fermionic part of the action

The fermionic part of the $I_\gamma^{susy}(g, \omega)$ action (3.6) is

$$I_{fermion} = -\int dt \, (\, \overline{\Psi}_x \quad \overline{\Psi}_y \,) \, Q \begin{pmatrix} \dot{x} - \frac{gy}{x^2+y^2} - \omega(x-a) \\ \dot{y} + \frac{gx}{x^2+y^2} - \omega y \end{pmatrix}. \tag{6.1}$$

By taking into account Eq. (3.7) for the operator Q one has

$$I_{fermion} = \int dt \, (\, \overline{\Psi}_x \quad \overline{\Psi}_y \,) \begin{pmatrix} \frac{d}{dt} + g\frac{2xy}{(x^2+y^2)^2} - \omega & -g\frac{x^2-y^2}{(x^2+y^2)^2} \\ -g\frac{x^2-y^2}{(x^2+y^2)^2} & \frac{d}{dt} - g\frac{2xy}{(x^2+y^2)^2} - \omega \end{pmatrix} \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix}. \tag{6.2}$$

We call this expression the "classical fermionic action"; the fermions $\Psi_x, \Psi_y, \overline{\Psi}_x, \overline{\Psi}_y$ are time-dependent anticommuting Grassmann variables.

The fermionic zero modes are defined as the solutions of the following equation,

$$\begin{pmatrix} \dot{\Psi}_x \\ \dot{\Psi}_y \end{pmatrix} = \frac{1}{r^2} \begin{pmatrix} -g\sin 2\theta + \omega r^2 & g\cos 2\theta \\ g\cos 2\theta & g\sin 2\theta + \omega r^2 \end{pmatrix} \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix} =$$

$$= \frac{1}{r^2} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \omega r^2 & g \\ g & \omega r^2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix}. \tag{6.3}$$

By setting $\Psi = \Psi_x + i\Psi_y$ one can also write

$$\dot{\Psi} = i\frac{g}{r^2} \exp(i2\theta) \, \Psi^* + \omega\Psi. \tag{6.4}$$

In the above equation the functions $r(t)$ and $\theta(t)$ solve the self-duality equations (3.4).

By redefining $\Psi \to \tilde{\Psi} = \exp(-i\theta)\Psi$, one gets the equation

$$\dot{\tilde{\Psi}} + i\dot{\theta}\tilde{\Psi} = i\frac{g}{r^2}\tilde{\Psi}^* + \omega\tilde{\Psi}. \tag{6.5}$$

By expressing $\dot{\theta}$ in terms of the second equation of (3.4), one has

$$\dot{\theta} = \frac{g}{r^2} + \frac{\omega a}{r}\sin\theta, \tag{6.6}$$

so that one gets the following zero mode equation

$$\dot{\tilde{\Psi}} + (i\frac{g}{r^2} + i\frac{\omega a}{r}\sin\theta - \omega)\tilde{\Psi} = i\frac{g}{r^2}\tilde{\Psi}^*. \tag{6.7}$$

This first order differential equation indicates that, most likely, fermionic zero modes exist in the background of the $r(t), \theta(t)$ solutions of the self-duality equations (3.4).

7 The supersymmetric Hamiltonian

The passage from the supersymmetric Lagrangian entering (3.6) to the classical Hamiltonian formulation is done following the standard prescription, by performing a Legendre transformation for both bosonic and fermionic coordinates. One introduces the conjugate momenta p_i to the coordinates q_i, while the fermionic Grassmann fields $\overline{\Psi}_i$ turn out to be conjugate to the fermionic fields Ψ_i. The Poisson brackets are $\mathbf{Z}_2$-graded. The non-vanishing Poisson brackets are

$$\{p_i, q_j\} = \delta_{ij}, \qquad \{\overline{\Psi}_i, \Psi_j\} = \delta_{ij}. \tag{7.1}$$

The classical Hamiltonian H is $\mathcal{N} = 2$ supersymmetric. It corresponds to the Morse function $S = g\theta + \frac{\omega}{2}|\vec{q} - \overrightarrow{OA}|^2$. The supersymmetric charges $Q, \overline{Q}$ are given by

$$Q = \Psi^i(p_i - S_i), \qquad \overline{Q} = \overline{\Psi}^i(p_i + S_i), \tag{7.2}$$

where

$$S_i = \frac{\delta S}{\delta q_i} = g\epsilon_{ij}\frac{q^j}{|\vec{q}|^2} + \omega(q_i - a_i), \qquad (a_1 = a, \ a_2 = 0). \tag{7.3}$$

One has

$$H = \frac{1}{2}\{Q, \overline{Q}\} = \frac{1}{2}p_i^2 - \frac{1}{2}S_i^2 - \overline{\Psi}^i\,\frac{\partial S_i}{\partial q^j}\,\Psi^j. \tag{7.4}$$

A quick computation shows that the classical fermionic part of the Hamiltonian is

$$\overline{\Psi}^i\Big(\frac{\partial S_i}{\partial q^j}\Big)\Psi^j = \omega(R_{-\theta}\overline{\Psi})^i\Big(\mathbf{I}_2 + \frac{r_0^2}{r^2}C\Big)_{ij}(R_{-\theta}\Psi)^j, \tag{7.5}$$

where $\mathbf{I}_2$ is the 2×2 identity matrix, while $R_{-\theta}, C$ have been introduced in (2.8). At this level Ψ^i and $\overline{\Psi}^i$ are Grassmann variables and one can compute correlation functions within the functional integral approach.

The quantum Hamiltonian $\mathbf{H}$ is obtained by realizing the $\mathbf{Z}_2$-graded Poisson brackets (7.1) in terms of (anti)commutators. This implies, in particular, that the quantum fermionic operators Ψ^i, $\overline{\Psi}^i$ are given by constant matrices satisfying Clifford algebra relations; for this reason extra terms are present with respect to the (7.4) expression of the Hamiltonian.

By performing an analytic continuation and a Wick rotation of the time coordinate, we can go back and forth from the Euclidean time to the real time formulation of the theory. We present here the results for the real time formulation. This means that the quantum Hamiltonian $\mathbf{H}$ is hermitian and the quantum supercharges $\mathbf{Q}$, $\overline{\mathbf{Q}}$ are hermitian conjugates:

$$\mathbf{H} = \mathbf{H}^\dagger, \qquad \mathbf{Q}^\dagger = \overline{\mathbf{Q}}. \tag{7.6}$$

A convenient presentation of the quantum fermionic operators $\Psi^i, \overline{\Psi}^i$ is via the 4×4 matrices $\xi^{\pm i}$, through the positions

$$\xi^{\pm i} = \Psi^i \pm \overline{\Psi}^i, \tag{7.7}$$

where $\xi^{\pm i}$ are given by

$$\xi^{+1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \xi^{+2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\xi^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \xi^{-2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{7.8}$$

The matrices $\xi^{\pm i}$ generate the $Cl(2,2)$ Clifford algebra. Indeed, by setting

$$\Gamma^\mu \equiv (\xi^{+1}, \xi^{+2}, \xi^{-1}, \xi^{-2}), \tag{7.9}$$

for $\mu = 1, 2, 3, 4$, we end up with the basic relations, for their anticommutators,

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{I}_4, \tag{7.10}$$

where $\eta^{\mu\nu}$ is the flat metric with diagonal entries $(1, 1, -1, -1)$.

The hamiltonian $\mathbf{H}$ and the supercharges $\mathbf{Q}, \overline{\mathbf{Q}}$ can be expressed as 4×4 differential matrix operators. In polar coordinates the hamiltonian is

$$\mathbf{H} = \left(-\frac{1}{2}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2) + V_0(r, \theta) \right) \cdot \mathbf{I}_4 + \mathbf{V}_1(r, \theta), \tag{7.11}$$

where $V_0(r, \theta)$ coincides with the potential in Eq. (4.2), namely

$$V_0(r, \theta) = \frac{1}{2}\frac{g^2}{r^2} + \frac{1}{2}\omega^2 r^2 + \frac{ag\omega}{r}\sin\theta - \omega^2 ar\cos\theta + \frac{1}{2}\omega^2 a^2, \tag{7.12}$$

while

$$\mathbf{V}_1(r, \theta) = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 \\ 0 & 0 & -\frac{g}{r^2}\sin(2\theta) & -\frac{g}{r^2}\cos(2\theta) \\ 0 & 0 & -\frac{g}{r^2}\cos(2\theta) & \frac{g}{r^2}\sin(2\theta) \end{pmatrix}. \tag{7.13}$$

The supercharge $\mathbf{Q}$ is explicitly given by

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 & e_{13} & e_{14} \\ 0 & 0 & 0 & 0 \\ 0 & e_{32} & 0 & 0 \\ 0 & e_{42} & 0 & 0 \end{pmatrix}, \tag{7.14}$$

with

$$e_{13} = -e_{42} = -i(\cos\theta\partial_r - \frac{1}{r}\sin\theta\partial_\theta) - i\omega(r\cos\theta - a) + \frac{ig\sin\theta}{r},$$

$$e_{14} = e_{32} = i(\sin\theta\partial_r + \frac{1}{r}\cos\theta\partial_\theta) + i\omega r\sin\theta + \frac{ig\cos\theta}{r}. \tag{7.15}$$

The supercharge $\overline{\mathbf{Q}}$ is the hermitian conjugate of the matrix given in Eq. (7.14).

The operators $\mathbf{H}$, $\mathbf{Q}$, $\overline{\mathbf{Q}}$ satisfy the $\mathcal{N} = 2$ superalgebra

$$\{\mathbf{Q}, \overline{\mathbf{Q}}\} = 2\mathbf{H}, \quad [\mathbf{H}, \mathbf{Q}] = [\mathbf{H}, \overline{\mathbf{Q}}] = \{\mathbf{Q}, \mathbf{Q}\} = \{\overline{\mathbf{Q}}, \overline{\mathbf{Q}}\} = \mathbf{0}. \tag{7.16}$$

The Fermion Parity Operator $\mathbf{N_F}$ is given by the 4×4 diagonal matrix

$$\mathbf{N_F} \;=\; diag(1, 1, -1, -1). \tag{7.17}$$

Bosons (fermions) are the eigenvectors of $\mathbf{N_F}$ with eigenvalues $+1$ (-1).

We note that the Hamiltonian (7.11) cannot be diagonalized. Indeed, a rotation, acting on fermions alone, which diagonalizes the potential $\mathbf{V}_1(r, \theta)$, is θ-dependent, so that it produces non-diagonal terms when applied to the Laplacian. By realizing that the lower two-dimensional block in $\mathbf{V}_1(r, \theta)$ can be expressed as $-\frac{g}{r^2} R_{-\theta} C R_\theta$, where R_θ, C have been introduced in (2.8), a simpler expression for the quantum dynamics is given by the hamiltonian $\widehat{\mathbf{H}}$, unitarily equivalent to $\mathbf{H}$ via the transformation

$$\widehat{\mathbf{H}} \;=\; \begin{pmatrix} \mathbf{I}_2 & 0 \\ 0 & R_{-\theta} \end{pmatrix} \mathbf{H} \begin{pmatrix} \mathbf{I}_2 & 0 \\ 0 & R_\theta \end{pmatrix}. \tag{7.18}$$

One has

$$\widehat{\mathbf{H}} \;=\; \left(-\frac{1}{2}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2) + V_0(r, \theta) \right) \cdot \mathbf{I}_4 + \widehat{\mathbf{V}}_1(r, \theta), \tag{7.19}$$

where the 4×4 matrix $\widehat{\mathbf{V}}_1(r, \theta)$ is decomposed in 2×2 blocks according to

$$\widehat{\mathbf{V}}_1(r, \theta) \;=\; \begin{pmatrix} \omega\sigma_3 & 0 \\ 0 & -\frac{1}{2r^2}(\mathbf{I}_2 + 2(\partial_\theta + g)C) \end{pmatrix}. \tag{7.20}$$

σ_3 is the diagonal Pauli matrix, while $C = \sigma_1$ is the conjugation matrix introduced in (2.8). The non-diagonal terms correspond to a first-order differential operator.

8   Conclusions

We introduced a deformation of the conformally invariant topological quantum model [1] of a particle moving on the punctured plane. The modified potential depends on a length

scale r_0 and on a dimensionless parameter $\alpha \geq 0$. It explicitly breaks the rotational and the conformal invariance, but it preseves a discrete $\mathbf{Z}_2$ simmetry given by the idempotent map $r/r_0 \to r_0/r$ and $\theta \to -\theta - \pi/4$. The role of α is that of an order parameter with a critical value $\alpha_c = \sqrt{2}$. Below that value (for $0 < \alpha \leq \sqrt{2}$) the ground state is unique. Above that value ($\alpha > \sqrt{2}$) the ground state is doubly degenerate.

Two key features distinguish this potential with respect to the well-known Higgs potential $\lambda(|\vec{q}|^2 - \frac{m}{\lambda})^2$ in the plane. The first one is the presence of an angular dependence (since the model is not rotationally invariant). The second relevant feature is that, above the critical value, the extremal points are the two ground states and two saddle points (therefore, the theory does not possess local maxima).

We proved (with techniques similar as those applied in [5], [12]) that, in the Euclideanized version of the model, instantons as defined by Coleman are energetically possible. The actual existence of the instantons, due to the fact that the self-duality equations are not analitically solvable and the potential has a complicated structure for the presence of saddle points, is an open question which deserves further investigations.

The BRSTQFT (see [6]) gauge-fixing method induces a $\mathcal{N} = 2$ supersymmetric quantum mechanics that we constructed both in the path-integral (the fermionic time-dependent fields are Grassmann coordinates) and in the operatorial (the fermionic degrees of freedom are realized by Clifford matrices) frameworks.

In the operatorial (and real time) formulation the Hamiltonian is a 4×4 hermitian differential operator. By construction the Hamiltonian, for any $\alpha \geq 0$, has a well-defined, discrete and bounded from below, spectrum. The Hamiltonian is non-diagonalizable and this is another indication of the richness and the non-triviality of the model.

We leave for a future paper the computation of the spectrum (numerical, if not analytical) of this quantum theory and of its further properties.

As mentioned in the Introduction the construction of this model with such a critical phenomenon was found by searching for a potential that can possibly simulate the constraints of a mechanical device that acts as a series of locks, allowing the propagation of

signals from one side to the other of a chain, reproducing the behaviour of some of the machines described in [3].

Acknowledgments

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