On Light-like Deformations of the Poincaré Algebra

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Abstract

We investigate the observational consequences of the light-like deformations of the Poincaré algebra induced by the jordanian and the extended jordanian classes of Drinfel'd twists. Twist-deformed generators belonging to a Universal Enveloping Algebra close non-linear W-algebras. In some cases the W-algebra is responsible for the existence of bounded domains of the deformed generators. The Hopf algebra coproduct implies associative non-linear additivity of the multi-particle states. A subalgebra of twist-deformed observables is recovered whenever the twist-deformed generators are either hermitian or pseudo-hermitian with respect to a common invertible hermitian operator.

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1 Introduction

This paper addresses the problem of the observational consequences of twist-deforming the Poincaré algebra. We work within a quantization scheme which has been previously applied to Drinfel'd twist deformations of quantum theories in a non-relativistic setting [1]-[4]. Open questions are investigated. We mention, in particular, the nature of the observables: which of them are consistently maintained in the deformed theory either as hermitian or pseudo-hermitian operators? To be specific, in the large class of deformed Poincaré theories (which include, e.g., κ-Poincaré theories [5]-[7], Deformed Special Relativity theories [8, 9] and many other examples [10]-[12]) we focus on the Drinfel'd twist [13, 14] deformations of a light-like direction. Due to this reason the deformations we consider here are based on the jordanian [15]-[17] and on the extended jordanian [18] twist (for physical applications of the Jordanian twist see [19, 20] and, for the extended Jordanian twist, [21]-[26]).

The deformations can be encoded in twist-deformed generators which, essentially, correspond to twist-covariant generalizations of the Bopp-shift [27]. The operators which in the undeformed case are associated with generators of the Lie algebra are, under a twist, mapped into given elements of the Universal Enveloping Algebra.

Observables in connection with twist-deformed generators were addressed in [28]. In our work we investigate different twists and present a different approach to the issue.

Different deformed theories are obtained from the original twist and its flipped version (obtained by a permutation of the tensor space).

A common feature of the deformed theories is that the deformed generators define a closed non-linear W-algebra (antisymmetry and Jacobi identities are respected, but the right hand side is a non-linear combination of the generators). Furthermore, the deformation modifies the domain of the physical parameters. For instance, in the simplest non-trivial case, the jordanian deformation implies the introduction of a maximal momentum along a light-cone direction. Induced by the coproduct (see [4]), non-linear addition formulas are obtained for multi-particle states. The non-linear addition formulas satisfy associativity. Their main *raison d'être* is that they allow to respect the domain of validity of the physical quantities (in the example above, the composite momentum along the light-cone direction is bounded by the maximal value). We postpone to the Conclusions a more detailed discussion of the implications of our results.

The scheme of the paper is as follows. The jordanian and the extended jordanian twists are recalled in Section **2**. In Section **3** twist-deformed generators are introduced. In Section **4** the (pseudo)-hermiticity property of twist-deformed generators is discussed. The arising of a non-linear W-algebra is investigated in Section **5**. The bounded domains of deformed physical observables and non-linear additive formulas are discussed in Section **6**. For completeness in the Appendix the (undeformed) Poincaré algebra in the light-cone basis is presented.

2 Jordanian versus extended Jordanian twist

We recall, see [29, 30] for details, that a Drinfel'd twist deformation of a Hopf Algebra A is induced by an invertible element $\mathcal{F} \in A \otimes A$ which satisfies the cocycle condition

$$(\mathbf{1} \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F} \;\; = \;\; (\mathcal{F} \otimes \mathbf{1})(\Delta \otimes id)\mathcal{F}. \tag{1}$$

In Sweedler's notation [31] $\mathcal{F}$ can be expanded according to

$$\mathcal{F} = f^{\beta} \otimes f_{\beta} \quad , \quad \mathcal{F}^{-1} = \overline{f}^{\beta} \otimes \overline{f}_{\beta}. \tag{2}$$

For $A = \mathcal{U}(\mathcal{G})$, the Universal Enveloping Algebra of a Lie algebra $\mathcal{G}$, the elements of $\mathcal{F}$ are taken from an even-dimensional subalgebra of $\mathcal{G}$. Therefore the simplest cases of twist are found for two-dimensional subalgebras.

There are (over $\mathbb{C}$) two inequivalent two-dimensional Lie algebras (we denote the generators as a, b):

i) the abelian algebra $[a, b] = 0$ and

ii) the non-abelian algebra $[a, b] = ib$.

The case *i*), the abelian twist, leads to constant non-commutativity (see, e.g., [32, 1]).

In this paper we focus on the second case. The non-abelian algebra *ii*) is, for example, the Borel subalgebra of $sl(2)$. The $sl(2)$ generators can be presented as D, H, K, satisfying the commutation relations

$$[D, H] = iH, \quad [D, K] = -iK, \quad [K, H] = 2iD. \tag{3}$$

We can identify $D \equiv a$ and $H \equiv b$. This leads, see [3], to non-commutativity of Snyder type.

We can also regard *ii*) as the subalgebra of a d-dimensional Poincaré algebra $\mathcal{P}(d)$. From its generators P_0, P_1, M_{01}, whose commutators are

$$[P_0, P_1] = 0, \quad [M_{01}, P_0] = -iP_1, \quad [M_{01}, P_1] = iP_0, \tag{4}$$

one can identify the *ii*) subalgebra from the positions $a \equiv -M_{01}$, $b \equiv P_+ = P_0 + P_1$.

The abelian algebra *i*) induces the abelian twist

$$\mathcal{F} \;\; = \;\; \exp(-i\alpha a \otimes b), \tag{5}$$

where α is the (dimensional) deformation parameter.

The non-abelian algebra *ii*) induces the non-abelian (jordanian) twist [15]-[17]

$$\mathcal{F} \;\; = \;\; \exp(-ia \otimes \ln(1 + \alpha b)), \tag{6}$$

where α is the (dimensional) deformation parameter.

Under the transposition operator τ ($\tau(v \otimes w) = w \otimes v$), the transposed twist

$$\mathcal{F}_{\tau} \;\; := \;\; \exp(-i \ln(1 + \alpha b) \otimes a), \tag{7}$$

still satisfies the cocycle condition (1). We are using both $\mathcal{F}$ and $\mathcal{F}_{\tau}$ in our paper.

The jordanian and the extended jordanian twist of the d-dimensional Poincaré algebra can be expressed, in terms of light-cone coordinates (see the Appendix for the $d = 4$ case) and Einstein's convention, through the position (see, e.g., [22])

$$\mathcal{F} = \exp\left(iM \otimes \ln(1 + \alpha P_+) + i\epsilon M_{+j} \otimes \ln(1 + \alpha P_+)\frac{P_j}{P_+}\right). \quad (8)$$

The jordanian case is recovered for $\epsilon = 0$; the extended jordanian case is recovered for $\epsilon = 1$. In two dimensions the two twists coincide since there are no transverse directions.

Under transposition, the $\mathcal{F}_\tau$ twist is given by

$$\mathcal{F}_\tau = \exp\left(i\ln(1 + \alpha P_+) \otimes M + i\epsilon \ln(1 + \alpha P_+)\frac{P_j}{P_+} \otimes M_{+j}\right). \quad (9)$$

The following four kinds of twist-deformations can be considered: the jordanian deformations ($\epsilon = 0$) based on $\mathcal{F}$ (case I) and $\mathcal{F}_\tau$ (case II) and the extended jordanian deformations ($\epsilon = 1$) based on $\mathcal{F}$ (case III) and $\mathcal{F}_\tau$ (case IV).

3 Twist-deformed generators

A twist deformation can be expressed in terms of the twist-deformed generators (see [32, 33, 1]). Under deformation, a Lie algebra generator $g \in \mathcal{G}$ is mapped into the Universal Enveloping Algebra element $g^{\mathcal{F}} \in \mathcal{U}(\mathcal{G})$, given by

$$g^{\mathcal{F}} = \overline{f}^{\beta}(g)\overline{f}_{\beta}, \quad (\mathcal{F}^{-1} = \overline{f}^{\beta} \otimes \overline{f}_{\beta}). \quad (10)$$

In this paper we consider a Lie algebra $\mathcal{G} = \{P_\mu, M_{\mu\nu}, x_\mu, \hbar\}$ containing the d-dimensional Poincaré algebra as a subalgebra (see the Appendix for the ordinary $d = 4$ spacetime).

We present the twist deformations for the previous Section cases I, II and IV, whose twist deformed generators can be presented in closed form.

In the case I ($\mathcal{F}$ twist with $\epsilon = 0$) we have

$$\begin{aligned}
P_+^{\mathcal{F}} &= P_+\frac{1}{1 + \alpha P_+}, \\
P_-^{\mathcal{F}} &= P_-(1 + \alpha P_+), \\
P_j^{\mathcal{F}} &= P_j, \\
M^{\mathcal{F}} &= M, \\
M_{+j}^{\mathcal{F}} &= M_{+j}\frac{1}{1 + \alpha P_+}, \\
M_{-j}^{\mathcal{F}} &= M_{-j}(1 + \alpha P_+), \\
N^{\mathcal{F}} &= N
\end{aligned} \quad (11)$$

and

$$\begin{aligned}
x_+^{\mathcal{F}} &= x_+\frac{1}{1 + \alpha P_+}, \\
x_-^{\mathcal{F}} &= x_-(1 + \alpha P_+), \\
x_j^{\mathcal{F}} &= x_j.
\end{aligned} \quad (12)$$

The undeformed generators can be expressed in terms of the deformed generators on the basis of inverse formulas. In particular we have

$$\begin{aligned} P_+ &= P_+^{\mathcal{F}} \frac{1}{1-\alpha P_+^{\mathcal{F}}}, \\ \frac{1}{1+\alpha P_+} = 1-\alpha P_+^{\mathcal{F}} &\quad , \quad 1+\alpha P_+ = \frac{1}{1-\alpha P_+^{\mathcal{F}}}. \end{aligned} \tag{13}$$

The twist-deformed generators for the transposed twists $\mathcal{F}_\tau$, with $\epsilon = 0, 1$, are

$$\begin{aligned} P_\bullet^{\mathcal{F}} &= P_\bullet, \\ M^{\mathcal{F}} &= \frac{1+2\alpha P_+}{1+\alpha P_+} M + \epsilon \left(\frac{\alpha P_j}{1+\alpha P_+} - \ln(1+\alpha P_+) \frac{P_j}{P_+} \right) M, \\ M_{+j}^{\mathcal{F}} &= M_{+j} + \epsilon \ln(1+\alpha P_+) M_{+j}, \\ M_{-j}^{\mathcal{F}} &= M_{-j} + \frac{2\alpha P_j}{1+\alpha P_+} M + \epsilon \left(\ln(1+\alpha P_+) \frac{P_j}{P_+} \delta_{jk} + \frac{2\alpha P_j P_k}{(1+\alpha P_+) P_+} - 2\ln(1+\alpha P_+) \frac{P_j P_k}{{P_+}^2} \right) M_{+k}, \\ N^{\mathcal{F}} &= N - \epsilon \epsilon_{jk} \ln(1+\alpha P_+) \frac{P_k}{P_+} M_{+j} \end{aligned} \tag{14}$$

and

$$\begin{aligned} x_+^{\mathcal{F}} &= x_+, \\ x_-^{\mathcal{F}} &= x_- + \frac{2\alpha\hbar}{1+\alpha P_+} M + 2\epsilon\hbar \left(\frac{\alpha}{1+\alpha P_+} - \frac{\ln(1+\alpha P_+)}{P_+} \right) \frac{P_j}{P_+} M_{+j}, \\ x_j^{\mathcal{F}} &= x_j - \epsilon \epsilon_{jk} \hbar \ln(1+\alpha P_+) \frac{1}{P_+} M_{+j}. \end{aligned} \tag{15}$$

The bullet $\bullet$ denotes $\pm, j$ (all translation generators are undeformed). The summation over repeated indices is understood.

4 The hermiticity and pseudo-hermiticity condition for twist-deformed generators

For an operator Ω the hermiticity condition is $\Omega^\dagger = \Omega$.

The pseudohermiticity condition [34] is

$$\Omega^\dagger = \eta \Omega \eta^{-1}, \tag{16}$$

for some invertible hermitian operator $\eta = \eta^\dagger$.

For the jordanian twist the pseudo-hermiticity properties of the deformed generators are the following.

In case I (jordanian $\mathcal{F}$ twist with $\epsilon = 0$):

$$\begin{aligned} P_{\bullet}^{\mathcal{F}\dagger} &= \eta^{\lambda} P_{\bullet}^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R}, \\ M^{\mathcal{F}\dagger} &= M^{\mathcal{F}}, \qquad i.e.\ \lambda = 0, \\ M_{+j}^{\mathcal{F}\,\dagger} &= \eta^{\lambda} M_{+j}^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R}, \\ M_{-j}^{\mathcal{F}\,\dagger} &= \eta M_{-j}^{\mathcal{F}} \eta^{-1}, \quad i.e.\ \lambda = 1, \\ N^{\mathcal{F}\dagger} &= \eta^{\lambda} N^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R}, \end{aligned} \tag{17}$$

together with

$$\begin{aligned} x_{+}^{\mathcal{F}\dagger} &= \eta^{\lambda} x_{+}^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R}, \\ x_{-}^{\mathcal{F}\dagger} &= \eta x_{-}^{\mathcal{F}} \eta^{-1}, \qquad i.e.\ \lambda = 1, \\ x_{j}^{\mathcal{F}\dagger} &= \eta^{\lambda} x_{j}^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R}, \end{aligned} \tag{18}$$

for the hermitian operator $\eta = 1 + \alpha P_{+}$.

One can observe that, in this case, the subset of hermitian ($\lambda = 0$) deformed operators is given by $\{P_{\pm}^{\mathcal{F}}, P_{j}^{\mathcal{F}}, M^{\mathcal{F}}, N^{\mathcal{F}}, M_{+j}^{\mathcal{F}}, x_{+}^{\mathcal{F}}, x_{j}^{\mathcal{F}}, \hbar^{\mathcal{F}}\}$, since the operators $M_{-j}^{\mathcal{F}}, x_{-}^{\mathcal{F}}$ are not hermitian.

In the case II (jordanian $\mathcal{F}_{\tau}$ twist with $\epsilon = 0$) we have

$$\begin{aligned} P_{\bullet}^{\mathcal{F}\dagger} &= \eta^{\lambda} P_{\bullet}^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R}, \\ M^{\mathcal{F}\dagger} &= \eta M^{\mathcal{F}} \eta^{-1}, \quad i.e.\ \lambda = 1, \\ M_{+j}^{\mathcal{F}\,\dagger} &= \eta^{\lambda} M_{+j}^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R}, \\ N^{\mathcal{F}\dagger} &= \eta^{\lambda} N^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R}, \end{aligned} \tag{19}$$

together with

$$\begin{aligned} x_{+}^{\mathcal{F}\dagger} &= \eta^{\lambda} x_{+}^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R}, \\ x_{j}^{\mathcal{F}\dagger} &= \eta^{\lambda} x_{j}^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R}, \end{aligned} \tag{20}$$

for the hermitian operator $\eta = \frac{1+\alpha P_{+}}{1+2\alpha P_{+}}$.

Unlike the other deformed generators, $M_{-j}^{\mathcal{F}}, x_{-}^{\mathcal{F}}$ do not satisfy the pseudo-hermiticity condition for any choice of η.

It is worth pointing out that, by taking the choice $\lambda = 1$, we formally obtain the same set of $P_{\pm}^{\mathcal{F}}, P_{j}^{\mathcal{F}}, M^{\mathcal{F}}, N^{\mathcal{F}}, M_{+j}^{\mathcal{F}}, x_{+}^{\mathcal{F}}, x_{j}^{\mathcal{F}}, \hbar^{\mathcal{F}}$ deformed generators as in the previous case. They are now, of course, different operators which satisfy a pseudo-hermiticity condition. As we shall see, they close a non-linear W-algebra.

For the extended jordanian twist, cases III and IV, one can verify by explicit computation through a Taylor expansion in the deformation parameter α, that most of the deformed generators are neither hermitian nor pseudo-hermitian.

5 Deformed algebras as non-linear W-algebras

In the case I (the jordanian $\mathcal{F}$ twist with $\epsilon = 0$) the deformed generators induce non-linear (at most quadratic) W-algebras. The basis of deformed generators, defining the $\mathcal{G}_{dfr}$ W-algebra is given by

$$\mathcal{G}_{dfr} \;:\; \{\hbar^{\mathcal{F}}, x_{\pm}^{\mathcal{F}}, x_j^{\mathcal{F}}, P_{\pm}^{\mathcal{F}}, P_j^{\mathcal{F}}, M^{\mathcal{F}}, N^{\mathcal{F}}, M_{\pm j}^{\mathcal{F}}\}. \tag{21}$$

Some of its relevant subalgebras are the deformed Poincaré subalgebra $\mathcal{P}_{dfr}$, with basis of deformed generators given by

$$\mathcal{P}_{drf} \;:\; \{P_{\pm}^{\mathcal{F}}, P_j^{\mathcal{F}}, M^{\mathcal{F}}, N^{\mathcal{F}}, M_{\pm j}^{\mathcal{F}}\}, \tag{22}$$

as well as the subalgebra of mutually consistent observables (generators with the same hermiticity/pseudo-hermiticity property). The $\mathcal{G}_{obs}$ and the $\mathcal{P}_{obs}$ subalgebras of observables are respectively given by

$$\mathcal{G}_{obs} \;:\; \{\hbar^{\mathcal{F}}, x_{+}^{\mathcal{F}}, x_j^{\mathcal{F}}, P_{\pm}^{\mathcal{F}}, P_j^{\mathcal{F}}, M^{\mathcal{F}}, N^{\mathcal{F}}, M_{+j}^{\mathcal{F}}\} \tag{23}$$

and

$$\mathcal{P}_{obs} \;:\; \{P_{\pm}^{\mathcal{F}}, P_j^{\mathcal{F}}, M^{\mathcal{F}}, N^{\mathcal{F}}, M_{+j}^{\mathcal{F}}\}. \tag{24}$$

The non-vanishing commutation relations of the $\mathcal{G}_{dfr}$ W-algebra are explicitly given by

$$\begin{aligned}
[P_{\pm}^{\mathcal{F}}, M^{\mathcal{F}}] &= \pm iP_{\pm}^{\mathcal{F}} \mp i\alpha P_{+}^{\mathcal{F}} P_{\pm}^{\mathcal{F}},\\
[P_{-}^{\mathcal{F}}, M_{+j}^{\mathcal{F}}] &= 2iP_j^{\mathcal{F}},\\
[P_{-}^{\mathcal{F}}, M_{-j}^{\mathcal{F}}] &= 2i\alpha P_{-}^{\mathcal{F}} P_j^{\mathcal{F}},\\
[P_j^{\mathcal{F}}, M_{\pm k}^{\mathcal{F}}] &= i\delta_{jk} P_{\pm}^{\mathcal{F}},\\
[M^{\mathcal{F}}, M_{\pm j}^{\mathcal{F}}] &= \mp iM_{\pm j}^{\mathcal{F}} \pm i\alpha M_{\pm j}^{\mathcal{F}} P_{+}^{\mathcal{F}},\\
[N^{\mathcal{F}}, M_{\pm j}^{\mathcal{F}}] &= i\epsilon_{jk} M_{\pm k}^{\mathcal{F}},\\
[M_{+j}^{\mathcal{F}}, M_{-k}^{\mathcal{F}}] &= -2i\delta_{jk} M^{\mathcal{F}} - 2i\epsilon_{jk} N^{\mathcal{F}} - 2i\alpha M_{+j}^{\mathcal{F}} P_k^{\mathcal{F}},\\
[M_{-j}^{\mathcal{F}}, M_{-k}^{\mathcal{F}}] &= 2i\alpha(M_{-j}^{\mathcal{F}} P_k^{\mathcal{F}} - M_{-k}^{\mathcal{F}} P_j^{\mathcal{F}}),\\
[P_{+}^{\mathcal{F}}, x_{-}^{\mathcal{F}}] &= 2i\hbar^{\mathcal{F}}(1 - \alpha P_{+}^{\mathcal{F}}),\\
[P_{-}^{\mathcal{F}}, x_{+}^{\mathcal{F}}] &= 2i\hbar^{\mathcal{F}},\\
[\mathcal{P}_{-}^{\mathcal{F}}, x_{-}^{\mathcal{F}}] &= 2i\alpha\hbar^{\mathcal{F}} P_{-}^{\mathcal{F}},\\
[\mathcal{P}_j^{\mathcal{F}}, x_k^{\mathcal{F}}] &= -i\delta_{jk}\hbar^{\mathcal{F}},\\
[M^{\mathcal{F}}, x_{\pm}^{\mathcal{F}}] &= \mp ix_{\pm}^{\mathcal{F}} \pm i\alpha x_{\pm}^{\mathcal{F}} P_{+}^{\mathcal{F}},\\
[N^{\mathcal{F}}, x_j^{\mathcal{F}}] &= i\epsilon_{jk} x_k^{\mathcal{F}},\\
[M_{+j}^{\mathcal{F}}, x_{-}^{\mathcal{F}}] &= -2ix_j^{\mathcal{F}} - 2i\alpha\hbar^{\mathcal{F}} M_{+j}^{\mathcal{F}},\\
[M_{-j}^{\mathcal{F}}, x_{+}^{\mathcal{F}}] &= -2ix_j^{\mathcal{F}} + 2i\alpha x_{+}^{\mathcal{F}} P_j^{\mathcal{F}},\\
[M_{-j}^{\mathcal{F}}, x_{-}^{\mathcal{F}}] &= 2i\alpha(\hbar^{\mathcal{F}} M_{-j}^{\mathcal{F}} - x_{-}^{\mathcal{F}} P_j^{\mathcal{F}}),\\
[M_{\pm j}^{\mathcal{F}}, x_k^{\mathcal{F}}] &= -i\delta_{jk} x_{\pm}^{\mathcal{F}},\\
[x_{+}^{\mathcal{F}}, x_{-}^{\mathcal{F}}] &= -2i\alpha\hbar^{\mathcal{F}} x_{+}^{\mathcal{F}}.
\end{aligned} \tag{25}$$

From the above formulas one can check that $\mathcal{P}_{dfr}$, $\mathcal{G}_{obs}$ and $\mathcal{P}_{obs}$ close as non-linear W-subalgebras.

In case II (the jordanian $\mathcal{F}_\tau$ twist) the algebra of the deformed observables, given by the operators $\hbar^{\mathcal{F}}, x_+^{\mathcal{F}}, x_j^{\mathcal{F}}, P_\pm^{\mathcal{F}}, P_j^{\mathcal{F}}, M^{\mathcal{F}}, N^{\mathcal{F}}, M_{+j}^{\mathcal{F}}$ obtained from formulas (14) and (15) with $\epsilon = 0$, also closes as a non-linear W-algebra. In this class of operators the only one which differs from its undeformed counterpart is $M^{\mathcal{F}}$, given by

$$M^{\mathcal{F}} = ZM \quad , \quad Z = \frac{1+2\alpha P_+}{1+\alpha P_+} = \frac{1+2\alpha P_+^{\mathcal{F}}}{1+\alpha P_+^{\mathcal{F}}}. \tag{26}$$

It turns out that the commutators with non-linear right hand side are the following ones:

$$\begin{aligned} [M^{\mathcal{F}}, P_\pm^{\mathcal{F}}] &= \mp iZP_\pm^{\mathcal{F}}, \\ [M^{\mathcal{F}}, M_{+j}^{\mathcal{F}}] &= -iZM_{+j}^{\mathcal{F}}, \\ [M^{\mathcal{F}}, x_+^{\mathcal{F}}] &= -iZx_+^{\mathcal{F}}. \end{aligned} \tag{27}$$

All the remaining commutators are linear and coincide with the ones (see the Appendix) for the undeformed generators.

6 Bounded domains and non-linear additivity in deformed systems

The second order (mass-term) Casimir of the Poincaré algebra

$$C_2 = P_+P_- - (P_j)^2 \tag{28}$$

remains undeformed under the twist-deformations I, II, IV introduced in Section **2**. We get

$$C_2 = P_+^{\mathcal{F}}P_-^{\mathcal{F}} - (P_j^{\mathcal{F}})^2. \tag{29}$$

All ten twisted generators (collectively denoted as $g_I^{\mathcal{F}}$, $I = 1, \ldots, 10$) entering their respective deformed Poincaré algebras, commute with C_2:

$$[g_I^{\mathcal{F}}, C_2] = 0, \quad \forall g_I^{\mathcal{F}}. \tag{30}$$

To analyze the physical consequences of the deformation we consider here the simplest setting, namely the case I (the $\mathcal{F}$ jordanian twist with $\epsilon = 0$). We will discuss the properties of the deformed momenta and their non-linear addition formulas.

Let's set, for simplicity, the transverse momenta $P_j \equiv 0$. For convenience we introduce new variables, defined as

$$\begin{aligned} r = P_+, \quad & l = P_-, \\ e = P_0, \quad & p = P_1. \end{aligned} \tag{31}$$

Since $P_\pm = P_0 \pm P_1$, we have

$$e = \frac{1}{2}(r+l), \qquad p = \frac{1}{2}(r-l), \tag{32}$$

with e representing the energy of the system.

For a massive representation we get, on shell,

$$rl = m^2, \qquad e^2 - p^2 = m^2. \tag{33}$$

Let's set, without loss of generality, $m = 1$. Therefore

$$\begin{aligned} rl &= 1, \\ e^2 - p^2 &= 1, \\ l &= \frac{1}{r}, \\ e &= \frac{1}{2}(r + \frac{1}{r}), \\ p &= \frac{1}{2}(r - \frac{1}{r}). \end{aligned} \tag{34}$$

In order to have a positive energy e, r should be non-negative. The observables are therefore bounded in the domains

$$\begin{aligned} r &\in]0, +\infty], \\ l &\in]0, +\infty], \\ e &\in [1, +\infty], \\ p &\in [-\infty, +\infty]. \end{aligned} \tag{35}$$

The rest condition for the P_1 momentum corresponds to $p = 0$ obtained at $r = 1$. For this value the energy is minimal ($e = 1$).

The twist-deformed variables will be denoted with a bar, the deformation parameter being α. We have

$$\overline{r} = \frac{r}{1 + \alpha r}, \qquad \bar{l} = l(1 + \alpha r). \tag{36}$$

The condition

$$\alpha \geq 0 \tag{37}$$

has to be imposed to avoid singularities.

On-shell we have

$$\begin{aligned} \bar{l} &= \frac{1}{r}(1 + \alpha r), \\ \overline{e} &= \frac{1}{2}(\overline{r} + \bar{l}) = \frac{1}{2}\left(\frac{r}{1 + \alpha r} + \frac{1}{r}(1 + \alpha r)\right), \\ \overline{p} &= \frac{1}{2}(\overline{r} - \bar{l}) = \frac{1}{2}\left(\frac{r}{1 + \alpha r} - \frac{1}{r}(1 + \alpha r)\right). \end{aligned} \tag{38}$$

The rest condition $\overline{p} = 0$ for the deformed P_1 momentum is obtained for

$$r = \frac{1}{1 - \alpha}. \tag{39}$$

Since r should be positive, one finds the restriction $\alpha < 1$. Therefore, the range of the deformation parameter α is

$$0 \leq \alpha < 1. \tag{40}$$

As a consequence, the domains of the deformed operators are modified with respect to their undeformed counterparts. We get

$$\begin{aligned} \overline{r} &\in]0, \frac{1}{\alpha}[, \\ \bar{l} &\in]\alpha, +\infty], \\ \overline{e} &\in [1, +\infty], \\ \overline{p} &\in [-\infty, \frac{1}{2}(\alpha - \frac{1}{\alpha})[. \end{aligned} \tag{41}$$

Following [4], the additive formulas of the deformed momenta are induced by their coproduct. In particular, the 2-particle addition formula for the deformed P_+ momenta $\overline{r}_1, \overline{r}_2$ of, respectively, the first and the second particle reads, in terms of the undeformed momenta, as

$$\overline{r}_{1+2} = \frac{r_1 + r_2}{1 + \alpha(r_1 + r_2)}. \tag{42}$$

Closely expressed in terms of the deformed momenta it is given by

$$\overline{r}_{1+2} = \frac{\overline{r}_1 + \overline{r}_2 - 2\alpha\overline{r}_1\overline{r}_2}{1 - \alpha^2\overline{r}_1\overline{r}_2}. \tag{43}$$

The binary operation defined by Eq. (43) is a group operation with $\overline{r} = 0$ as identity element and inverse element given by $\overline{r}^{-1} = -\frac{\overline{r}}{1-2\alpha\overline{r}}$. The associativity is satisfied due to the relation

$$\overline{r}_{(1+2)+3} = \overline{r}_{1+(2+3)} = \frac{\overline{r}_1 + \overline{r}_2 + \overline{r}_3 - 2\alpha(\overline{r}_1\overline{r}_2 + \overline{r}_2\overline{r}_3 + \overline{r}_3\overline{r}_1) + 3\alpha^2\overline{r}_1\overline{r}_2\overline{r}_3}{1 - \alpha^2(\overline{r}_1\overline{r}_2 + \overline{r}_2\overline{r}_3 + \overline{r}_3\overline{r}_1) + 2\alpha^3\overline{r}_1\overline{r}_2\overline{r}_3}. \tag{44}$$

It should be noted, however, that the physical requirement of $\overline{r}$ belonging to the domain in Eq. (41) excludes the identity and the inverse element from the physical values. Thus, on physical states, the addition formula (43) only satisfies the properties of a semigroup operation.

It is useful to compare the formula (43) with the non-linear addition of velocities in special relativity. Let us change variables once more and set $\overline{r}_{1,2} = v_{1,2}$, $\alpha = \frac{1}{c}$.

In special relativity we have

$$v_{1+2,s.r.} = \frac{v_1 + v_2}{1 + \frac{1}{c^2}v_1v_2}. \tag{45}$$

In the above jordanian deformation we get

$$v_{1+2,jd.} = \frac{v_1 + v_2 - \frac{2}{c}v_1v_2}{1 - \frac{1}{c^2}v_1v_2}. \tag{46}$$

Both non-linear addition formulas are symmetric in the $v_1 \leftrightarrow v_2$ exchange and associative.

They can also be defined in the interval $0 \leq v_{1,2} \leq c$, so that the non-linear additive velocities belong to the $[0, c]$ range (in both cases if $v_1 = 0$, then $v_{1+2} = v_2$ and, if $v_1 = c$, $v_{1+2} = c$).

The main difference is that in special relativity the formula can be nicely extended to negative velocities belonging to the $-c \leq v_{1,2} \leq c$ interval. This is not the case for the non-linear addition formula based on the jordanian deformation.

It is worth to point out that an important feature of the non-linear additivity consists in respecting the physical domain of the variables defining the multi-particle state.

7 Conclusions

In this paper we investigated the observational consequences of the light-like deformations of the Poincaré algebra induced by the jordanian and the extended jordanian twists (which both belong to the class of Drinfel'd twist deformations). We used the framework of [1]-[4] where, in particular, the deformed quantum theory is recovered from deformed twist-generators belonging to a Universal Enveloping Algebra. The Hopf algebra structure of the twist-deformation controls the physical properties of the theory. The deformed generators induce non-linear W-algebras, while the coproduct implies associative non-linear additivity of the multi-particle states. In the simplest setting of the jordanian deformation along a light-cone direction, these non-linearities consistently imply the existence of a maximal light-cone momentum. The undeformed theory is recovered by allowing the maximal light-cone momentum to go to infinity. This situation finds a parallelism, as we noted, with the light velocity as the maximal speed in special relativity.

A question that we raised regards the status, as observables of a quantum theory, of the deformed generators. It is rewarding that for the jordanian deformation a large subset of the deformed Poincaré generators are pseudo-hermitian for the same choice of an invertible hermitian η operator, see e.g. formula (19). Therefore, they are observable with respect to the η-modified inner product [34]. This subset of observables close a non-linear W-algebra as a consistent deformation of a Poincaré subalgebra.

The picture is different for the extended jordanian twist. One can explicitly check, by Taylor-expanding in power series of the deformation parameter, that most of the Poincaré deformed generators are neither hermitian nor pseudo-hermitian. It is still an open question the status as observables of the extended jordanian twist-deformed generators.

Appendix: Poincaré algebra in the light-cone basis.

We work with the metric $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ $(\mu, \nu = 0, 1, 2, 3)$.

The Poincaré algebra is given by the commutators

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, P_\rho] &= i\eta_{\mu\rho}P_\nu - i\eta_{\nu\rho}P_\mu, \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\mu\sigma}M_{\nu\rho} - i\eta_{\nu\rho}M_{\mu\sigma} + i\eta_{\nu\sigma}M_{\mu\rho}. \end{aligned} \tag{47}$$

In terms of the spacetime coordinates x^μ (x_μ), with $\mu = 0, j$ and $j = 1, 2, 3$, the Poincaré generators can be realized through the positions

$$\begin{aligned} P_0 &= i\hbar\frac{\partial}{\partial x_0}, \\ P_j &= -i\hbar\frac{\partial}{\partial x_j}, \\ M_{0j} &= -ix_0\frac{\partial}{\partial x_j} - ix_j\frac{\partial}{\partial x_0}, \\ M_{jk} &= -ix_j\frac{\partial}{\partial x_k} + ix_k\frac{\partial}{\partial x_j}. \end{aligned} \tag{48}$$

$P_\mu, M_{\mu\nu}$, together with the coordinates x_μ and the central charge $\hbar$ close a Lie algebra $\mathcal{G} = \{P_\mu, M_{\mu\nu}, x_\mu, \hbar\}$ whose remaining non-vanishing commutators are given by

$$\begin{aligned} [P_0, x_0] &= i\hbar, \\ [P_j, x_k] &= -i\delta_{jk}\hbar, \\ [M_{0j}, x_0] &= -ix_j, \\ [M_{0j}, x_k] &= -ix_0\delta_{jk}, \\ [M_{jk}, x_r] &= -ix_j\delta_{kr} + ix_k\delta_{jr}. \end{aligned} \tag{49}$$

The light-cone coordinates mix the time-direction x_0 and the space direction x_1; the remaining space coordinates are the transverse directions. The light-cone generators are labeled by the indices $\pm$ and $j = 1, 2$. The latter is used for the transverse coordinates.

In the light-cone basis the generators are expressed through the positions

$$\overline{x}_\pm = x_0 \pm x_1, \qquad \overline{x}_j = x_{j+1} \tag{50}$$

and

$$\begin{aligned} \overline{P}_\pm = P_0 \pm P_1, \qquad & \overline{P}_j = P_{j+1}, \\ \overline{M} &:= \overline{M}_{+-} = M_{01}, \\ \overline{M}_{\pm j} &= M_{0,j+1} \pm M_{1,j+1}, \\ \overline{N} &= M_{23}. \end{aligned} \tag{51}$$

In the above formulas relating the two presentations of space-time coordinates and Poincaré generators, for reason of clarity, the Poincaré generators and space-time coordinates expressed in the light-cone variables are overlined. Since however in the main text we only work with light-cone quantities and no confusion will arise, for simplicity, these quantities will not be overlined.

In the light-cone basis the non-vanishing commutators of the Poincaré algebra read as

follows

$$\begin{aligned}
[M, P_\pm] &= \mp i P_\pm, \\
[M, M_{\pm j}] &= \mp i M_{\pm j}, \\
[M_{\pm j}, P_\mp] &= -2i P_j, \\
[M_{\pm j}, P_k] &= -i\delta_{jk} P_\pm, \\
[M_{+j}, M_{-k}] &= -2i\delta_{jk} M - 2i\epsilon_{jk} N, \\
[M_{\pm j}, N] &= -i\epsilon_{jk} M_{\pm k}, \\
[N, P_j] &= i\epsilon_{jk} P_k
\end{aligned} \tag{52}$$

(the constant antisymmetric tensor is $\epsilon_{12} = -\epsilon_{21} = 1$).

The non-vanishing commutators of the Poincaré generators with the light-cone coordinates $x_\pm, x_j$ are

$$\begin{aligned}
[P_\pm, x_\mp] &= 2i\hbar, \\
[P_j, x_k] &= -i\delta_{jk}\hbar, \\
[M, x_\pm] &= \mp i x_\pm, \\
[M_{\pm j}, x_\mp] &= -2i x_j, \\
[M_{\pm j}, x_k] &= -i x_\pm \delta_{jk}, \\
[N, x_j] &= i\epsilon_{jk} x_k.
\end{aligned} \tag{53}$$

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