Higher Spin Symmetries and Deformed Schrödinger Algebra in Conformal Mechanics

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Abstract

The dynamical symmetries of $1+1$-dimensional Matrix Partial Differential Equations with a Calogero potential (with/without the presence of an extra oscillatorial De Alfaro-Fubini-Furlan, DFF, damping term) are investigated. The first-order invariant differential operators induce several invariant algebras and superalgebras. Besides the $sl(2)\oplus u(1)$ invariance of the Calogero Conformal Mechanics, an $osp(2|2)$ invariant superalgebra, realized by first-order and second-order differential operators, is obtained. The invariant algebras with an infinite tower of generators are given by the universal enveloping algebra of the deformed Heisenberg algebra, which is shown to be equivalent to a deformed version of the Schrödinger algebra. This vector space also gives rise to a higher spin (gravity) superalgebra. We furthermore prove that the pure and DFF Matrix Calogero PDEs possess isomorphic dynamical symmetries, being related by a similarity transformation and a redefinition of the time variable.

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1 Introduction

In this paper we investigate the dynamical symmetries of $1+1$-dimensional Matrix Partial Differential Equations with a Calogero [1] potential (with/without the presence of an extra oscillatorial De Alfaro-Fubini-Furlan, DFF, [2] damping term). The first-order invariant differential operators induce several invariant algebras and superalgebras. Besides the $sl(2) \oplus u(1)$ invariance of the Calogero Conformal Mechanics, an $osp(2|2)$ invariant superalgebra, realized by first-order and second-order differential operators, is obtained. The invariant (super)algebras with an infinite tower of generators are given by the universal enveloping algebra of a deformed Heisenberg algebra [3, 4, 5, 6]. It also contains a deformed version of the Schrödinger algebra (in both cases the deformation stems from the presence of a Klein operator). As a vector space the universal enveloping algebra carries the $\mathbb{Z}_2$-graded higher spin superalgebra $q(2;\nu)$.

The higher spin superalgebra $q(2;\nu)$ was introduced in [6] and applied to the construction of a Chern-Simons higher spin gravity. In [6] it was also shown that $q(2;\nu)$ is provided by the universal enveloping algebra of the $osp(2|2)$ superalgebra. The $q(2;\nu)$ algebra was also employed to extend the Chern-Simons higher spin gravity model [6] with fractional spin fields [7, 8].

We will show that these algebraic structures appear in the different context of non-relativistic quantum mechanics in the presence of a Calogero potential, either with or without an extra harmonic potential term. We highlight the main features and results of our approach.

We use the standard tools (see, e.g., [9]) of dynamical symmetries of (matrix) PDEs. The operators under consideration are *local* (matrix) differential operators, in contrast with the non-local realizations given in [3, 4, 5] (see also [10]).

The construction based on an invariant PDE allows us to detect an invariant deformed Schrödinger algebra which has not been discussed in the existing literature. This deformation can be regarded as a deformation of the mass-central-charge of the algebra of Galileo boosts and translations by a term proportional to the Klein operator. This algebra is connected with Vasiliev's $q(2;\nu)$ superalgebra. As a vector space it is spanned by the same infinite set of differential operators which, on the other hand, are endowed with different brackets (ordinary commutators versus $\mathbb{Z}_2$-graded commutators). We furthermore prove that the models with pure Calogero potentials and those with the extra DFF damping term possess isomorphic dynamical symmetry (super)algebras. This results from the fact that a triplet of Matrix PDEs close an $sl(2)$ algebra. In a given canonical form, the Matrix PDE associated with the positive root of $sl(2)$ corresponds to the Schrödinger equation of the pure Matrix Calogero system (it would have been tantamount to work with the negative root of $sl(2)$), while a canonical form exists such that the $sl(2)$ Cartan generator is associated with the Schrödinger equation for the DFF Matrix Calogero system. The two canonical forms are related by a similarity transformation coupled with a redefinition of the time variable.

The pure Matrix Calogero Hamiltonian is an $\mathcal{N}=2$ Supersymmetric Quantum Mechanical System. On the other hand the DFF Matrix Calogero Hamiltonian is not supersymmetric (this is a common feature of superconformal models possessing an oscillatorial damping term, see [11]); it is, nevertheless, related to supersymmetric Hamiltonians via a shift realized by a diagonal operator; following [12] we can state that, in this case, the system enjoys a "soft" supersymmetry.

In the last part of the paper we discuss the appearance of the higher spin superalgebra, which is now interpreted as a dynamical symmetry of the models under consideration (the higher-spin symmetry of the free non-relativistic particle case was pointed out in [13]).

The scheme of the paper is as follows. In Section **2** the Matrix Calogero PDEs (with/without the DFF oscillatorial damping potential term) are introduced. It is explained why, in the presence of a Calogero potential, a Matrix PDE is needed to get (deformed) Heisenberg symmetry generators. The dynamical symmetry generators of these systems are computed. In Section **3** their dynamical symmetry subalgebras are investigated. In particular the $osp(2|2)$ superalgebra (realized by first-order and second-order differential symmetry operators) is derived, as well as the deformed Heisenberg algebra. The existence of an $\mathcal{N}=2$ supersymmetry for the pure Matrix Calogero Hamiltonian and a "soft" supersymmetry for the DFF Matrix Calogero Hamiltonian is pointed out. The connection between pure and DFF Matrix Calogero PDEs is presented in Section **4**. In Section **5** it is shown that both a Deformed Schrödinger algebra and a higher spin superalgebra appear as dynamical symmetries of (pure and DFF) Matrix Calogero PDEs. In the Conclusions we comment our results and sketch some directions for future investigations.

2 Matrix PDEs with Calogero potentials

It is convenient to systematically review the arising of higher-spin (super)algebras in Calogero systems by analyzing the symmetry, realized by first-order differential operators, of the (matrix) Partial Differential Equations containing Calogero potentials. This analysis uses well-known methods.

To fix our conventions, if $\Omega=\Omega^\dagger$ is a hermitian (matrix) PDE, a first-order differential operator Σ is a symmetry operator if it satisfies the equation

$$[\Sigma,\Omega] \;\;=\;\; \Phi_\Sigma\cdot\Omega, \tag{2.1}$$

for a given matrix-valued function Φ_Σ.

In our analysis we do not need to introduce symmetry operators which, unlike (2.1), are defined by a "right" convention, i.e. $[\Sigma,\Omega]=\Omega\cdot\Phi'_\Sigma$. One should note, on the other hand, that the symmetry operator Σ is not necessarily hermitian (in that case its hermitian conjugate $\Sigma^\dagger$ satisfies the "right" equation with $\Phi'_{\Sigma^\dagger}=-\Phi^*_\Sigma$).

Before investigating matrix PDEs (our results require a 2×2 matrix whose entries are differential operators) we recall some basic features of the standard Partial Differential Equations. The Schrödinger equation in $1+1$ dimension induces the maximal 6-generator Schrödinger algebra of invariant operators for three choices (up to normalization and trivial shift) of the potential $V(x)$ [14, 15, 16, 17]. $V(x)$ is either constant (corresponding to the free equation), linear (giving rise to Airy's functions) or quadratic (corresponding to the harmonic oscillator). For all other choices of the potential the symmetry algebra possesses fewer generators. This is particularly the case for the Calogero potential.

2.1 The ordinary Calogero PDEs

The pure Calogero PDE is defined by the operator

$$\Omega_{Cal} \;\;=\;\; i\partial_t+\frac{1}{2}\partial_x^2-\frac{g}{x^2}, \tag{2.2}$$

where g is the coupling constant.

The symmetry algebra for the Calogero potential is $sl(2)\oplus(1)$. It contains four generators. Two extra generators, corresponding to the Heisenberg subalgebra of the Schrödinger algebra, are no longer encountered in the presence of a non-vanishing, $g\neq 0$, Calogero potential. We have in this case, as suitably normalized symmetry generators,

$$\begin{aligned} c &= 1, \\ z_+ &= \partial_t, \\ z_0 &= -2t\partial_t - x\partial_x - \frac{1}{2}, \\ z_- &= -t^2\partial_t - tx\partial_x + i\frac{x^2}{2} - \frac{t}{2}. \end{aligned} \tag{2.3}$$

They satisfy the non-vanishing commutators

$$[z_0, z_\pm] = \pm 2z_\pm, \qquad [z_+, z_-] = z_0, \tag{2.4}$$

while c is a central charge.

The non-vanishing commutators involving Ω_{Cal} are

$$[z_0, \Omega_{Cal}] = 2\Omega_{Cal}, \qquad [z_-, \Omega_{Cal}] = 2t\Omega_{Cal}. \tag{2.5}$$

Based on (2.1), this means that

$$\Phi_{z_0} = 2, \qquad \Phi_{z_-} = 2t. \tag{2.6}$$

The conveniently chosen symmetry operators

$$\overline{z}_+ = iz_+, \qquad \overline{z}_0 = i(z_0 - c) \equiv i(z_0 - 1), \tag{2.7}$$

are hermitian.

One should note that the four (2.3) symmetry operators do not depend on the Calogero coupling constant g.

In the case of the Calogero potential with the extra addition, see [2], of the DFF quadratic damping term, we can introduce without loss of generality (by suitably normalizing the frequency of the harmonic oscillator), the Ω_{DFF} operator given by

$$\Omega_{DFF} \;=\; i\partial_t + \frac{1}{2}\partial_x^2 - \frac{g}{x^2} - \frac{1}{2}x^2. \tag{2.8}$$

In this case as well we obtain a $sl(2) \oplus u(1)$ symmetry algebra. Its four generators are

$$\begin{aligned} \widehat{c} &= 1, \\ \widehat{z}_+ &= \frac{1}{2}e^{-2it}(\partial_t - ix\partial_x + ix^2 - \frac{i}{2}), \\ \widehat{z}_0 &= i\partial_t, \\ \widehat{z}_- &= \frac{1}{2}e^{2it}(\partial_t + ix\partial_x + ix^2 + \frac{i}{2}). \end{aligned} \tag{2.9}$$

The non-vanishing commutators are

$$[\widehat{z}_0, \widehat{z}_\pm] = \pm 2\widehat{z}_\pm, \qquad [\widehat{z}_+, \widehat{z}_-] = \widehat{z}_0. \tag{2.10}$$

It is worth pointing out that, when $g \neq 0$, for no other non-constant $w(x)$, the potential $\frac{g}{x^2} + w(x)$ produces a 4-generator symmetry algebra. In particular, for a linear $w(x)$ ($w(x) = kx$, $k \neq 0$), we only recover a 2-generator symmetry algebra.

The arising of a (generalized) Heisenberg and of higher-spin (super)algebras requires a matrix Calogero PDE. For this purpose a 2×2 matrix system is sufficient.

2.2 The 2×2 -matrix Calogero PDEs

We check under which conditions we can obtain off-diagonal symmetry generators for a 2×2-matrix differential operator containing Calogero potentials.

Let us denote with e_{ij} $(i,j=1,2)$ the matrix with entry $+1$ at the i-th column and j-th row and 0 otherwise. We are looking for symmetry generators of the matrix PDE defined by the diagonal operator

$$\Omega \;=\; (e_{11}+e_{22})(i\partial_t+\frac{1}{2}\partial_x^2)-v_1(x)e_{11}-v_2(x)e_{22}. \qquad (2.11)$$

Since Ω is a diagonal operator, from either (2.3) or (2.9), we obtain $4+4$ diagonal symmetry generators (4 of them are associated with the upper diagonal element e_{11}; the remaining 4 generators are associated with the lower diagonal element e_{22}).

Concerning the non-diagonal symmetry operators, they should be expressed as

$$\begin{aligned}\Sigma_{up} &= (f(x,t)\partial_t+g(x,t)\partial_x+h(x,t))e_{12},\\ \Sigma_{down} &= (\overline{f}(x,t)\partial_t+\overline{g}(x,t)\partial_x+\overline{h}(x,t))e_{21},\end{aligned} \qquad (2.12)$$

for appropriate functions $f(x,t),g(x,t),h(x,t),\overline{f}(x,t),\overline{g}(x,t),\overline{h}(x,t)$.

We assume the potentials $v_1(x),v_2(x)$ to be

$$v_i(x) \;=\; \frac{a_i}{x^2}+b_ix^2+c_i, \qquad (2.13)$$

with $a_2\neq a_1$. Without loss of generality we can set, via similarity transformations, $c_1=c_2=0$.

It is easily realized that the requirement

$$b_2 \;=\; b_1 \qquad (2.14)$$

is implied by the existence of a non-vanishing solution Σ_{up} (Σ_{down}) for the respective equations

$$[\Sigma_{up},\Omega]=0, \qquad [\Sigma_{down},\Omega]=0. \qquad (2.15)$$

The constraint on the functions entering Σ_{up}, Σ_{down} are derived with standard techniques. We need to make vanishing the coefficients of the operators entering the r.h.s. of equations (2.15). We get, for Σ_{up},

$$\begin{aligned}\partial_x\partial_t &: \;\; f_x=0,\\ \partial_x^2 &: \;\; g_x=0,\\ \partial_t &: \;\; -i\dot{f}-\frac{1}{2}f_{xx}+(v_1-v_2)f=0 \quad \Rightarrow f=0 \quad for \quad v_2\neq v_1,\\ \partial_x &: \;\; -i\dot{g}-h_x+(v_1-v_2)=0,\\ 1 &: \;\; -i\dot{h}-\frac{1}{2}h_{xx}+(v_1-v_2)h-gv_{2x}=0.\end{aligned} \qquad (2.16)$$

Similar equations are obtained for $\overline{f}(x,t),\overline{g}(x,t),\overline{h}(x,t)$ entering Σ_{down}.

The above system of equations tells us that, in particular, the condition

$$a_1+a_2-a_1^2-a_2^2+2a_1a_2 \;=\; 0 \qquad (2.17)$$

has to be fulfilled. After setting $\nu=2(a_2-a_1)$, it is solved by

$$a_1=\frac{1}{8}\nu(\nu-2), \qquad a_2=\frac{1}{8}\nu(\nu+2). \qquad (2.18)$$

We can therefore write the most general (2.11) operator admitting off-diagonal symmetry operators. It depends on the parameter ϵ, whose values are either $\epsilon = 0$ for the pure Calogero system, or $\epsilon = 1$ for the the Calogero system with the extra oscillatorial damping term. Due to (2.14), the oscillatorial term is the same in both upper and lower diagonal entries. We can write, with a proper normalization,

$$\Omega_\epsilon = (e_{11}+e_{22})(i\partial_t + \frac{1}{2}\partial_x^2 - \frac{1}{2}\epsilon x^2 - \frac{\nu^2}{8x^2}) + (e_{11}-e_{22})\frac{\nu}{4x^2}. \quad (2.19)$$

The operator K,

$$K = e_{11} - e_{22}, \quad (2.20)$$

plays different roles, depending on the context. It is either the Fermion Parity Operator (in Supersymmetric Quantum Mechanics) or the Klein Operator (entering the definition of the deformed oscillators). We note that, in particular, $K^2 = \mathbb{I}_2$.

In both $\epsilon = 0, 1$ cases we get two upper triangular and two lower triangular symmetry operators.

At $\epsilon = 0$ we have, up to normalization,

$$\begin{aligned}\Sigma_1 &= e_{12}(\partial_x + \frac{\nu}{2x}),\\ \Sigma_2 &= e_{12}(t\partial_x + t\frac{\nu}{2x} - ix),\\ \Sigma_3 &= e_{21}(\partial_x - \frac{\nu}{2x}),\\ \Sigma_4 &= e_{21}(t\partial_x - t\frac{\nu}{2x} - ix).\end{aligned} \quad (2.21)$$

At $\epsilon = 1$ we have, up to normalization,

$$\begin{aligned}\Xi_1 &= e_{12}e^{it}(\partial_x + \frac{\nu}{2x} + x),\\ \Xi_2 &= e_{12}e^{-it}(\partial_x + \frac{\nu}{2x} - x),\\ \Xi_3 &= e_{21}e^{it}(\partial_x - \frac{\nu}{2x} + x),\\ \Xi_4 &= e_{21}e^{-it}(\partial_x - \frac{\nu}{2x} - x).\end{aligned} \quad (2.22)$$

3 Symmetry superalgebra of the pure Calogero Matrix system

For the 2×2 pure Calogero Matrix (2.19) $\Omega_{\epsilon=0}$ operator we can introduce the basis of 4 off-diagonal hermitian operators. They are given by

$$Q_1 = \frac{i}{\sqrt{2}}(\Sigma_1 + \Sigma_3), \quad (3.23)$$

$$Q_2 = iKQ_1 = \frac{1}{\sqrt{2}}(\Sigma_3 - \Sigma_1), \quad (3.24)$$

$$\widetilde{Q}_1 = \frac{i}{\sqrt{2}}(\Sigma_2 + \Sigma_4), \quad (3.25)$$

$$\widetilde{Q}_2 = iK\widetilde{Q}_1 = \frac{1}{\sqrt{2}}(\Sigma_4 - \Sigma_2). \quad (3.26)$$

By construction

$$Q_i^\dagger = Q_i \quad , \quad \widetilde{Q}_i^\dagger = \widetilde{Q}_i \qquad (i=1,2). \tag{3.27}$$

Since, for K given by (2.20), we have

$$\{K, Q_i\} = \{K, \widetilde{Q}_i\} = 0, \tag{3.28}$$

the quantum mechanical system defined by $\Omega_{\epsilon=0}$ is $\mathcal{N}=2$ supersymmetric. Indeed,

$$\{Q_i, Q_j\} \;=\; 2\delta_{ij}\mathbf{H}, \tag{3.29}$$

where

$$\mathbf{H} \;=\; \mathbb{I}_2(-\frac{1}{2}\partial_x^2 + \frac{\nu^2}{8x^2}) + \frac{\nu}{4x^2}K. \tag{3.30}$$

The following non-vanishing anticommutators are recovered:

$$\begin{aligned} \{Q_i, Q_j\} &= 2\delta_{ij}\mathbf{H}, \\ \{\widetilde{Q}_i, \widetilde{Q}_j\} &= 2\delta_{ij}\mathbf{K}, \\ \{Q_i, \widetilde{Q}_j\} &= \delta_{ij}\mathbf{D} + \epsilon_{ij}\mathbf{J}, \end{aligned} \tag{3.31}$$

where, besides $\mathbf{H}$ given in (3.30), we have

$$\mathbf{D} \;=\; \mathbb{I}_2(-t\partial_x^2 + ix\partial_x + \frac{i}{2} + t\frac{\nu^2}{4x^2}) - Kt\frac{\nu}{2x^2}, \tag{3.32}$$

$$\mathbf{K} \;=\; \frac{1}{2}\mathbb{I}_2(-t^2\partial_x^2 + 2itx\partial_x + x^2 + it + t^2\frac{\nu^2}{4x^2}) - \frac{1}{4}Kt^2\frac{\nu}{x^2}, \tag{3.33}$$

$$\mathbf{J} \;=\; \frac{1}{2}(K + \nu\mathbb{I}_2). \tag{3.34}$$

The four odd (fermionic) operators $Q_i, \widetilde{Q}_i$, together with the four even (bosonic) operators $\mathbf{H}, \mathbf{D}, \mathbf{K}, \mathbf{J}$, close the finite simple Lie superalgebra $osp(2|2)$, as it can be easily checked. By construction $\mathbf{H}, \mathbf{D}, \mathbf{K}, \mathbf{J}$ commute with $\Omega_{\epsilon=0}$. Therefore, $osp(2|2)$ is an invariant subalgebra of the 2×2 pure Matrix Calogero system.

We have indeed, for any generator $\mathfrak{g} \in osp(2|2)$,

$$[\mathfrak{g}, \Omega_{\epsilon=0}] \;=\; 0. \tag{3.35}$$

The set of generators $Q_1, \widetilde{Q}_1, \mathbf{H}, \mathbf{D}, \mathbf{K}$ (and, similarly, $Q_2, \widetilde{Q}_2, \mathbf{H}, \mathbf{D}, \mathbf{K}$) close an $osp(1|2)$ superalgebra. $Q_1, \widetilde{Q}_1$, (respectively $Q_2, \widetilde{Q}_2$) are odd generators. If, nevertheless, we compute their commutators, we obtain

$$[Q_1, \widetilde{Q}_1] = [Q_2, \widetilde{Q}_2] = \tfrac{i}{2}(\mathbb{I}_2 + \nu K), \tag{3.36}$$

therefore recovering the (ν-)deformed Heisenberg algebra.

For future convenience we also consider the non-hermitian linear combinations

$$\overline{A} = i(Q_1 - \widetilde{Q}_1), \qquad \overline{A}^\dagger = Q_1 + \widetilde{Q}_1, \tag{3.37}$$

which satisfy

$$[\overline{A}, \overline{A}^\dagger] \;=\; \mathbb{I}_2 + \nu K. \tag{3.38}$$

4 Connection between pure and DFF Calogero Matrix systems

It is convenient to normalize the $sl(2)$ root symmetry generators of the pure Calogero system Ω_{Cal}, given in (2.2), according to $z'_\pm = \mp i z_\pm$, with $z_\pm$ entering (2.3). The corresponding symmetry operators for the pure Calogero Matrix system $\Omega_{\epsilon=0}$, given in (2.19), are $z'_\pm \mathbb{I}_2$.

It is convenient to redefine $\Omega_{\epsilon=0}$ as

$$\Omega_{+1} \quad := \quad \Omega_{\epsilon=0}. \tag{4.39}$$

Indeed, a triplet of operators, $\Omega_{\pm1}, \Omega_0$, carrying a $sl(2)$ representation generated by $z'_\pm \mathbb{I}_2, z_0 \mathbb{I}_2$, is encountered. We have

$$[z'_- \mathbb{I}_2, \Omega_{+1}] = \Omega_0 = 2it\Omega_{+1}, \qquad [z'_- \mathbb{I}_2, \Omega_0] = \Omega_{-1} = -2t^2\Omega_{+1}. \tag{4.40}$$

One should note that the commutator $[z'_- \mathbb{I}_2, \Omega_{-1}] = 0$ is vanishing, so that no further operator is generated.

The operators $\Omega_{\pm1}, \Omega_0$ close an $sl(2)$ algebra. We have, indeed,

$$[\Omega_0, \Omega_{\pm1}] = \pm 2\Omega_{\pm1}, \qquad [\Omega_{+1}, \Omega_{-1}] = -2\Omega_0. \tag{4.41}$$

The connection between the pure 2×2 matrix Calogero system obtained from (2.19) at $\epsilon = 0$ and the 2×2 matrix Calogero-DFF system obtained from (2.19) at $\epsilon = 1$ can now be made explicit, by adapting to the present case the construction discussed in [18]. Under redefinition of the time coordinate and a similarity transformation, the suitably normalized Cartan operator $N\Omega_0$ (N is a constant) is mapped into the DFF $\Omega_{\epsilon=1}$ operator. As a corollary, the symmetry operators of the pure Calogero system ($\epsilon = 0$) are mapped into the symmetry operators of the Calogero system with oscillatorial damping ($\epsilon = 1$).

The new time variable τ is introduced from the equation

$$t \quad = \quad Ce^{2i\tau}, \tag{4.42}$$

(C is here a suitable constant), such that the time-derivative operator entering $N\Omega_0$ can be expressed as $2iNt\partial_t = i\partial_\tau$. This is accomplished by choosing $N = i$, $C = -\frac{i}{2}$.

The further similarity transformation which maps the symmetry generators of $\Omega_{\epsilon=0}$ into the symmetry generators of $\Omega_{\epsilon=1}$ is given by

$$\mathfrak{g} \quad \mapsto \check{\mathfrak{g}} = e^{S_2}e^{S_1}\mathfrak{g}e^{-S_1}e^{-S_2}, \tag{4.43}$$

where

$$S_1 = i\tau(x\partial_x + \frac{1}{2}), \qquad S_2 = \frac{1}{2}x^2. \tag{4.44}$$

It is rather straightforward to check that, in particular, we get

$$i\Omega_0 \quad \mapsto i\check{\Omega}_0 = \mathbb{I}_2(i\partial_\tau + \tfrac{1}{2}\partial_x^2 - \tfrac{1}{8}\tfrac{\nu^2}{x^2} - \tfrac{1}{2}x^2) + K\tfrac{1}{4}\tfrac{\nu}{x^2}. \tag{4.45}$$

The similarity transformation (4.43) and the time redefinition (4.42) apply to all 2×2 matrix symmetry generators. We have, in particular,

$$\begin{array}{lcl} \Sigma_1 & \mapsto & \Xi_2 = e_{12}e^{-i\tau}(\partial_x + \dfrac{\nu}{2x} - x), \\ \Sigma_3 & \mapsto & \Xi_4, \\ \Sigma_2 & \mapsto & -\dfrac{i}{2}\Xi_1, \\ \Sigma_4 & \mapsto & -\dfrac{i}{2}\Xi_3. \end{array} \tag{4.46}$$

The operators in the r.h.s. coincide, for the time variable τ, with the corresponding operators given in (2.22). It follows, in particular, that $osp(2|2)$ is a symmetry superalgebra of the matrix system with oscillatorial damping. Since the $osp(2|2)$ generators are connected by similarity transformations their (anti)commutation relations are preserved.

It is convenient to define the hermitian generators $\widehat{Q}_1, \widehat{Q}_2, \widehat{Q}'_1, \widehat{Q}'_2$ through the positions

$$\begin{aligned} \widehat{Q}_1 &= \frac{i}{\sqrt{2}}(\Xi_1+\Xi_4), \\ \widehat{Q}_2 &= \frac{1}{\sqrt{2}}(\Xi_4-\Xi_1) = iK\widehat{Q}_1, \\ \widehat{Q}'_1 &= \frac{i}{\sqrt{2}}(\Xi_2+\Xi_3), \\ \widehat{Q}'_2 &= \frac{1}{\sqrt{2}}(\Xi_3-\Xi_2) = iK\widehat{Q}'_1. \end{aligned} \tag{4.47}$$

In the presence of the oscillatorial damping we obtain two $\mathcal{N}=2$ supersymmetric quantum mechanics, since

$$\begin{aligned} \{\widehat{Q}_i, \widehat{Q}_j\} &= 2\delta_{ij}\widehat{H}, \\ \{\widehat{Q}'_i, \widehat{Q}'_j\} &= 2\delta_{ij}\widehat{H}'. \end{aligned} \tag{4.48}$$

The supersymmetric Hamiltonians are respectively given by

$$\begin{aligned} \widehat{H} &= \mathbb{I}_2(-\frac{1}{2}\partial_x^2+\frac{1}{2}x^2+\frac{1}{8}\frac{\nu^2}{x^2}+\frac{\nu}{2})+\frac{1}{2}K(1-\frac{\nu}{2x^2}), \\ \widehat{H}' &= \mathbb{I}_2(-\frac{1}{2}\partial_x^2+\frac{1}{2}x^2+\frac{1}{8}\frac{\nu^2}{x^2}-\frac{\nu}{2})-\frac{1}{2}K(1+\frac{\nu}{2x^2}). \end{aligned} \tag{4.49}$$

The supersymmetric Hamiltonians correspond to a shift of the original H_{osc} Hamiltonian with oscillatorial damping, whose expression is

$$H_{osc} = \mathbb{I}_2(-\frac{1}{2}\partial_x^2+\frac{1}{2}x^2+\frac{1}{8}\frac{\nu^2}{x^2})-\frac{1}{4}K\frac{\nu}{x^2}. \tag{4.50}$$

Indeed, we have

$$\widehat{H} = H_{osc}+\frac{1}{2}(K+\mathbb{I}_2\nu), \qquad \widehat{H}' = H_{osc}-\frac{1}{2}(K+\mathbb{I}_2\nu). \tag{4.51}$$

The shift operators $\pm\frac{1}{2}(K+\mathbb{I}_2\nu)$ commute with H_{osc}; we have $[H_{osc},(K+\mathbb{I})_2\nu]=0$.

An $osp(1|2)$ symmetry superalgebra can be expressed in terms of the operators A, $A^\dagger$ and their anticommutators, where

$$A = \frac{i}{\sqrt{2}}(\Xi_1+\Xi_3), \qquad A^\dagger = \frac{i}{\sqrt{2}}(\Xi_2+\Xi_4). \tag{4.52}$$

The Hamiltonian H_{osc} is recovered from the anticommutator

$$\{A, A^\dagger\} = 2H_{osc}. \tag{4.53}$$

The remaining two operators ($E_\pm$) that, together with $A, A^\dagger, H_{osc}$, close the $osp(1|2)$ superalgebra are given by

$$\begin{aligned} \{A, A\} &:= E_+ = e^{2it}[\mathbb{I}_2(-\partial_x^2-2x\partial_x-x^2+\frac{\nu^2}{4x^2}-1)-K\frac{\nu}{2x^2}] \\ \{A^\dagger, A^\dagger\} &:= E_- = e^{-2it}[\mathbb{I}_2(-\partial_x^2+2x\partial_x-x^2+\frac{\nu^2}{4x^2}+1)-K\frac{\nu}{2x^2}]. \end{aligned} \tag{4.54}$$

The operators $A, A^\dagger$ induce the deformed Heisenberg algebra since their commutator is given by

$$[A, A^\dagger] \;=\; \mathbb{I}_2 + \nu K. \tag{4.55}$$

5 Deformed Schrödinger and higher spin (super)algebras

We are now in the position to discuss the arising of infinitely generated deformed Schrödinger algebra and higher-spin superalgebra as dynamical symmetries of the pure and DFF Matrix Calogero PDEs. Pure and DFF Matrix Calogero dynamical symmetries are isomorphic, since they are related by the time redefinition (4.42) and the similarity transformation (4.43). It is therefore sufficient to present the results for the pure Matrix Calogero case.

The introduction of the deformed Schrödinger algebra requires to couple an $sl(2)\oplus u(1)$ invariant subalgebra to a pair of deformed Heisenberg oscillators. Several invariant deformed Schrödinger algebras can be recovered as a consequence. Indeed we can pick, e.g., as $sl(2)\oplus u(1)$ subalgebra, either the first-order differential operators $g\cdot\mathbb{I}_2$, with g denoting one of the four operators entering (2.3), or the second-order differential operators entering the even sector of the $osp(2|2)$ superalgebra and recovered from (3.30) and (3.32-3.34). Similarly, up to normalization, the deformed Heisenberg operators can be given by several possible pairs. We have, e.g., $Q_1, \widetilde{Q}_1$, $Q_2, \widetilde{Q}_2$ or $\overline{A}, \overline{A}^\dagger$, whose respective deformed Heisenberg commutators are given in formulas (3.36) and (3.38). The choice $\mathcal{Q}_1 = \sqrt{2}\widetilde{Q}_1$, $\mathcal{Q}_2 = -\sqrt{2}Q_1$, so that

$$\begin{aligned}
\mathcal{Q}_1 &= (e_{12}+e_{21})(it\partial_x + x) + (e_{12}-e_{21})it\frac{\nu}{2x},\\
\mathcal{Q}_2 &= -(e_{12}+e_{21})i\partial_x - (e_{12}-e_{21})i\frac{\nu}{2x},
\end{aligned} \tag{5.56}$$

with

$$[\mathcal{Q}_1, \mathcal{Q}_2] \;=\; i(\mathbb{I}_2 + \nu K), \quad (K = e_{11} - e_{22}), \tag{5.57}$$

is particularly convenient. The limit $\nu \to 0$ exists. At $\nu = 0$ (in the absence of the Calogero potential) they coincide, respectively, with the ordinary Galileo boost and space translation multiplied by $e_{12}+e_{21}$ (namely, the operator exchanging the fields entering a 2-component multiplet). The closure, at $\nu = 0$, of the ordinary $1+1$-dimensional Schrödinger algebra with the extra $sl(2)\oplus u(1)$ first-order differential operators $g\cdot\mathbb{I}_2$ introduced above, immediately follows.

At $\nu \neq 0$ the appearance in the r.h.s. of (5.57) of the Klein operator K implies that (an infinite tower of) new generators must be introduced to close a Lie algebra. Indeed, for $\alpha = 1, 2$, we have $[\mathcal{Q}_\alpha, K] = 2\mathcal{Q}_\alpha K$, so that $[\mathcal{Q}_\alpha, \mathcal{Q}_\alpha K] = 2\mathcal{Q}_\alpha^2 K$, $[\mathcal{Q}_\alpha, \mathcal{Q}_\alpha^2 K] = 2\mathcal{Q}_\alpha^3 K$ and so on.

We can now introduce the associative algebra of Weyl ordered (hermitian) monomials

$$\mathcal{Q}_{\alpha(n)} = \mathcal{Q}_{\alpha_1\alpha_2\cdots\alpha_n} := \sum_{\alpha=1,2;\sigma} \frac{1}{n!}\, \mathcal{Q}_{\sigma(\alpha_1)}\mathcal{Q}_{\sigma(\alpha_2)}\cdots\mathcal{Q}_{\sigma(\alpha_n)}\,, \quad n = 0, 1, \ldots,$$

(the σ's denote the members of the permutation group acting on n elements), together with their product with the Klein operator K. We have, at a fixed value ν,

$$Aq(2;\nu) := \{(-iK)^b\, \mathcal{Q}_{\alpha(n)}\,, \quad n = 0, 1, \ldots, \quad b = 1, 2\}. \tag{5.58}$$

The associative algebra $Aq(2;\nu)$ defines the Universal Enveloping Algebra of the deformed Heisenberg algebra (5.57). This algebra was first introduced in [6] in the context of the quantization of observables on a hyperboloid.

As a vector space $Aq(2;\nu)$ can be endowed with two different types of brackets:
i) either ordinary brackets realized by commutators or
ii) $\mathbb{Z}_2$-graded brackets realized by (anti)commutators.

In the first case $Aq(2;\nu)$ is an infinite-dimensional Lie algebra (the Jacobi identities being satisfied by construction) that we will denote as $q(2;\nu)_{[,]}$:

$$q(2;\nu)_{[,]} := \{Aq(2;\nu)\,|\,[\mathfrak{a},\mathfrak{b}]\,\in\,Aq(2;\nu)\,,\,\forall\,\mathfrak{a},\,\mathfrak{b}\}. \tag{5.59}$$

The infinite-dimensional Lie algebra $q(2;\nu)_{[,]}$ is another realization of the deformed Schrödinger algebra. Since the operators $\mathcal{Q}_\alpha$ commute with the (2.19) 2×2 matrix operator $\Omega_{\epsilon=0}$, by construction $q(2;\nu)_{[,]}$ is a dynamical symmetry of the Pure Calogero Matrix PDE; indeed we have

$$[(-iK)^b\mathcal{Q}_{\alpha_1\alpha_2\cdots\alpha_n},\Omega_{\epsilon=0}] \;=\; 0.$$

In the second case $Aq(2;\nu)$ is an infinite-dimensional Lie superalgebra satisfying the graded Jacobi identities and whose consistent $\mathbb{Z}_2$-graded brackets, the (anti)commutators, are introduced through the position

$$[\mathfrak{a},\mathfrak{b}\} := \mathfrak{a}\mathfrak{b}-(-1)^{\epsilon_\mathfrak{a}\epsilon_\mathfrak{b}}\mathfrak{b}\mathfrak{a}. \tag{5.60}$$

The $\mathbb{Z}_2$-grading of the generators, $\epsilon_\mathfrak{a}=0,1$, is determined by, respectively, the even or odd power of the $\mathcal{Q}_\alpha$'s operators entering any given monomial (the presence of the Klein operator K does not affect the $\mathbb{Z}_2$-grading). We denote the superalgebra, with the given $\mathbb{Z}_2$-graded brackets, as

$$q(2;\nu)_{[,\}} := \{Aq(2;\nu)\,|\,[\mathfrak{a},\mathfrak{b}\}\,\in\,Aq(2;\nu)\,,\,\forall\,\mathfrak{a},\,\mathfrak{b}\}. \tag{5.61}$$

This ν-dependent class of superalgebras was introduced in [6] and, together with a Lorentz invariant supertrace formula, used in the construction of higher spin Chern-Simons supergravity (see also [19]). In [6] these superalgebras were simply denoted as "$q(2;\nu)$". We introduced here the "$[,\}$" label to distinguish them from the "$[,]$"-labeled deformed Schrödinger algebras. We keep, however, the simplified notation when no confusion arises.

The infinite-dimensional, ν-dependent, $q(2;\nu)$ superalgebra is a higher-spin superalgebra satisfying the correct spin-statistics connection. In order to understand its relation with higher spin, two main features have to be recalled. The first one consists in the fact that $q(2;\nu)$ contains the finite dimensional Lie superalgebra $osp(2|2)$ as subalgebra, so that $osp(2|2)\subset q(2;\nu)$; the second feature is due to the fact that $osp(2|2)$ admits a covariant description in terms of three types of indices. They are the vector indices $\mu,\lambda,\ldots=0,1,2$ labeling a three-dimensional Minkowski space, the (Majorana) spinorial indices $\alpha,\beta,\ldots=1,2$ labeling the associated real 2-component spinors and the scalar indices $A,B,\ldots=1,2$ labeling an internal space.

In the covariant notation the $osp(2|2)$ fermions are expressed as $\mathcal{Q}_\alpha^A$, where one can set

$$\mathcal{Q}_\alpha^1 := \mathcal{Q}_\alpha, \qquad \mathcal{Q}_\alpha^2 := iK\mathcal{Q}_\alpha. \tag{5.62}$$

We note that the $\mathcal{Q}_\alpha^A$'s operators are Hermitian.

The even sector is expressed by the 2×2 diagonal operators J^μ and R ("R" stands for the R-symmetry $u(1)$-generator) which can be recovered from the saturated (due to the presence of both spinorial α,β and internal A,B indices) generalized supersymmetry (see [20])

$$\{\mathcal{Q}_\alpha^A,\mathcal{Q}_\beta^B\} \;=\; \delta^{AB}(C\gamma_\mu)_{\alpha\beta}J^\mu+\epsilon^{AB}C_{\alpha\beta}R. \tag{5.63}$$

The Charge Conjugation matrix $C_{\alpha\beta}$ ($C=e_{12}-e_{21}$) and its inverse are used to raise/lower the spinorial indices; the diagonal metric $\eta_{\mu\nu}$ ($\eta_{\mu\nu}=diag(-1,+1,+1)$) and its inverse raise/lower the vector indices; the three gamma matrices ${(\gamma_\mu)_\alpha}^\beta$ ($\gamma_0 = e_{12}-e_{21}$, $\gamma_1 = e_{12}+e_{21}$, $\gamma_2 = e_{11}-e_{22}$) satisfy the composition law of the split-quaternions given by $\gamma_\mu\gamma_\nu = \eta_{\mu\nu}\mathbb{I}_2 + \epsilon_{\mu\nu\lambda}\gamma^\lambda$, with the totally antisymmetric Levi-Civita pseudo-tensor $\epsilon_{\mu\nu\lambda}$ normalized so that $\epsilon_{012} = 1$; by definition $(C\gamma_\mu)_{\alpha\beta} = C_{\alpha\alpha'}{(\gamma_\mu)_\beta}^{\alpha'}$. It is worth pointing out that the Charge Conjugation matrix and the gamma matrices act on the spinorial indices of the algebra and not on the 2-component wave function of the original PDE. The normalization of the antisymmetric tensor ϵ^{AB} is $\epsilon^{12} = 1$.

Straightforward computations show that the 2×2 matrix differential operators J^μ, R are connected with the $\mathbf{H}, \mathbf{D}, \mathbf{K}, \mathbf{J}$ operators of formulas (3.30) and (3.32-3.34) through the positions

$$J^0 = -2(\mathbf{K}+\mathbf{H}), \quad J^1 = 2(\mathbf{K}-\mathbf{H}), \quad J^2 = 2\mathbf{D}, \quad R = 2\mathbf{J}. \tag{5.64}$$

The closure of the $osp(2|2)$ superalgebra is guaranteed by the non-vanishing commutators

$$\begin{aligned}
[J_\mu, J_\nu] &= 4i\epsilon_{\mu\nu\lambda}J^\lambda, \\
[J_\mu, \mathcal{Q}^A_\alpha] &= 2i{(\gamma_\mu)_\alpha}^\beta\mathcal{Q}^A_\beta, \\
[R, \mathcal{Q}^A_\alpha] &= -2i{S^A}_B\mathcal{Q}^B_\alpha.
\end{aligned} \tag{5.65}$$

In the last equation the matrix ${S^A}_B$ is given by $S = e_{12} - e_{21}$.

Due to the presence of the covariant $osp(2|2)$ subalgebra, the remaining generators of $q(2;\nu)$ are accommodated into higher spin representations.

As mentioned before an associative algebra, called $\widehat{Aq}(2;\nu)$, of invariant operators for the DFF Matrix Calogero PDE is recovered from $Aq(2;\nu)$ by applying the time redefinition (4.42) and the similarity transformation (4.43). From $\widehat{Aq}(2;\nu)$ one recovers the deformed Schrödinger algebra $\widehat{q}_{[,]}(2;\nu)$ and the higher spin superalgebra $\widehat{q}_{[,\}}(2;\nu)$. They are isomorphic realizations of $q_{[,]}(2;\nu)$ and $\widehat{q}_{[,\}}(2;\nu)$, respectively.

6 Conclusions

As we have seen the Matrix Calogero models under consideration possess, in particular, a deformed Schrödinger algebra and the higher spin superalgebra $q_{[,\}}(2;\nu)$ as dynamical symmetries. The deformed Schrödinger algebra is obtained by computing, similarly to [14], the class of linear operators which leave invariant the space of solutions of the given PDEs. A whole universal enveloping algebra, $Aq(2;\nu)$, is generated by requiring the closure of the Lie algebra induced by the commutators. When the Calogero potential coupling parameter ν takes the 0 value, the infinite dimensional algebra can be truncated to the standard Schrödinger algebra.

We further observed that the matrix Calogero models and (at $\nu = 0$) the free particle multiplet, exhibit (relativistic) higher spin symmetries. These results suggest that, extending the conjecture of [13], the non-relativistic Calogero system studied so far may be dual, in a certain regime and in a certain asymptotic non-relativistic geometry (see [21]), to the vacuum of the (bulk) Vasiliev's higher spin gravity. As a matter of fact, the ν-parameter is related to the vacuum expectation value of the scalar field in the Vasiliev-Prokuskin higher spin gravity [22] in $2+1$ dimensions.

We should mention that another class of $(1+1)$-dimensional non-relativistic PDEs, unlike the Matrix Calogero models here discussed, inherently contain higher spin operators [23, 18]. These PDEs are invariant under the centrally extended ℓ-Conformal Galilei algebras, with $\ell = \frac{1}{2} + \mathbb{N}_0 \geq \frac{3}{2}$. In these models the higher-spin generators span a spin-ℓ representation of an $sl(2)$ subalgebra constructed from $\ell + \frac{1}{2}$ Heisenberg algebras ($\ell = \frac{1}{2}$ for the free particle in one dimension). An open problem consists in understanding whether, for these systems, a deformation of the centrally extended ℓ-Conformal Galilei algebra, induced by matrix-Calogero potentials, is allowed.

We expect, more generally, that systems possessing conformal Galileo invariance could be related to higher spin gravities. This type of non-relativistic dualities, if they indeed exist, would complement the dualities conjectured in [24, 25] (see also[26]).

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