Helicity decoupling
in the massless limit of massive tensor fields[*]

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Abstract

   Massive and massless potentials play an essential role in the perturbative formulation of particle interactions. Many difficulties arise due to the indefinite metric in gauge theoretic approaches, or the increase with the spin of the UV dimension of massive potentials. All these problems can be evaded in one stroke: modify the potentials by suitable terms that leave unchanged the field strengths, but are not polynomial in the momenta. This feature implies a weaker localization property: the potentials are "string-localized". In this setting, several old issues can be solved directly in the physical Hilbert space of the respective particles: We construct stress-energy tensors for massless fields of any helicity (thus evading the Weinberg-Witten theorem). We can control the separation of helicities in the massless limit of higher spin fields and conversely we recover massive potentials with $2s+1$ degrees of freedom by a smooth deformation of the massless potentials ("fattening"). We arrive at a simple understanding of the van Dam-Veltman-Zakharov discontinuity concerning, e.g., the distinction between a massless or a very light graviton. Finally, the use of string-localized fields opens new perspectives for interacting quantum field theories with, e.g., vector bosons or gravitons.

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[*]An abridged version of this paper, focussing on spin $s = 1$ and $s = 2$, is [MRS17].

1 Overview

Hilbert space positivity is indispensible for the interpretation of every quantum theory. However, for massless potentials of spin $s \geq 1$ there is a conflict between positivity, covariance, and local commutation relations (well-known for the Maxwell potential [W95]).

The massless potentials are, however, required to formulate interactions through minimal coupling, as in QED. In covariant frameworks, one uses auxiliary potentials with indefinite 2-point functions, hence defined on indefinite state spaces (Krein spaces). The free field strength lives on the Hilbert space defined by its own positive-definite 2-point function, obtained by taking the curls of the indefinite 2-point function of the potential.

However, in the interacting theory, one has to work in the Krein space, and proceed to a Hilbert space for the gauge invariant quantities by the Gupta-Bleuler or the more modern BRST cohomological methods.

On the other hand, massive potentials suffer from their bad UV behaviour which jeopardizes perturbative interactions by the power counting argument, or one uses the Feynman gauge which brings back the problems of indefinite metric.

The purpose of this contribution is to formulate and investigate a unified setting for potentials describing both massless and massive vector and tensor bosons, that live in Hilbert space and have the good UV behaviour needed for renormalizable perturbative interactions. The Hilbert space is that of the field strengths, which have no positivity problems. Our focus is here on the free fields, and in particular on the limit $m \to 0$ that is smooth in this setting. We comment on the issues concerning interactions in appropriate places, and otherwise refer to the literature.

Classical aspects of higher spin fields were treated in various papers, in particular in [F78, FV87, V00, V04]. For low spin, Lagrangean quantization respects quantum positivity (although positivity has no classical counterpart) but for higher spin there are problems. By constructing our free fields in the unitary Wigner representations, we can fully explore the interplay between positivity and causal localization and solve various problems that cannot be addressed in the indefinite gauge theoretical formalism.

Before we turn to our new results, we briefly recall the well-known general problematics for free fields.

We shall write 2-point functions throughout as

$$(\Omega, X(x)Y(y)\Omega) = \int d\mu_m(p) \cdot e^{-ip(x-y)} \cdot {}_m M^{X,Y}(p),$$

where $d\mu_m(p) = \frac{d^4 p}{(2\pi)^3} \delta(p^2 - m^2)\theta(p^0)$. Our sign convention is $\eta_{00} = +1$.

Example 1.1 Massless fields. *For $s = 1$, the massless (= Maxwell) field strength has the 2-point function*

$${}_0 M^{F_{\mu\nu}, F_{\kappa\lambda}} = -p_\mu p_\kappa \eta_{\nu\lambda} + p_\nu p_\kappa \eta_{\mu\lambda} + p_\mu p_\lambda \eta_{\nu\kappa} - p_\nu p_\lambda \eta_{\mu\kappa}.$$

The field strength is constructed on the Fock space over the unitary Wigner represen-tations of the Poincaré group with helicity $h = \pm 1$, as exposed in standard textbooks [W95]. To find potentials such that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, one has several options: E.g., the Coulomb gauge potential with $A_0^{\rm C} = 0$ and 2-point function

$$_0 M^{A_i^{\rm C}, A_j^{\rm C}} = \delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2} \tag{1.1}$$

can be defined on the same Hilbert space, but it fails to transform as a vector field and is non-local [W95]. The Krein potential with 2-point function

$$_0 M^{A_\mu^{\rm K}, A_\kappa^{\rm K}} = -\eta_{\mu\kappa}$$

obviously violates positivity, and cannot be defined on the Hilbert space of the field strength. When the potentials are required for the formulation of interaction, one has to compromise between positivity or Lorentz invariance, and preference is usually given to covariance.

The situation is similar with $s = 2$. We only display for later reference the indefinite 2-point function of the Krein potential which reproduces the positive[1] 2-point function of the field strength[2] $F^{(2)}_{[\mu_1\nu_1][\mu_2\nu_2]}$ by taking the curl in all indices:

$$_0 M^{A_{\mu\nu}^{\rm K}, A_{\kappa\lambda}^{\rm K}} = \frac{1}{2}\left[\eta_{\mu\kappa}\eta_{\nu\lambda} + \eta_{\mu\nu}\eta_{\kappa\lambda}\right] - \frac{1}{2}\eta_{\mu\nu}\eta_{\kappa\lambda}. \tag{1.2}$$

In the massive case with integer spin, the problems are of a different nature. By field equations like

$$\partial^\mu F_{\mu\nu} = -m^2 A_\nu^{\rm P} \tag{1.3}$$

for the Proca field, the potential can be defined from its field strength. When the latter is constructed on the Fock space over the (m, s) Wigner representation [W95], the resulting positive 2-point function of the potential is a polynomial in the projection matrix orthogonal to the momentum

$$\pi_{\mu\nu}(p) = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}, \tag{1.4}$$

see the Example 1.2 below.

Due to the momenta in the numerator, massive potentials have strong short-distance fluctuations which give them a UV dimension $d_{UV} = s + 1$ and, by power counting, jeopardize the renormalizability of minimal couplings to higher spin currents.

Only their field strengths have a massless limit because the curls "kill" the mass denominators along with the momenta in the numerators. Therefore, one may choose the Feynman gauge by replacing $\pi_{\mu\nu}$ by $\eta_{\mu\nu}$, which gives the same field strengths and avoids the bad UV behaviour, but brings back negative norm states.

The case $s = 2$ illustrates that, even if the massless limit of the field strength exists, it does not coincide with the massive field strength.

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[1]By "positive", it is actually understood "positive-semidefinite", accounting for the null states due to equations of motion.

[2]For $s \geq 2$ there are "partial" field strengths by taking the curl in $1 \leq n \leq s$ indices [F39]. Unless stated otherwise, we always understand the highest field strength (curl in all indices).

Example 1.2 Massive fields. *For $s = 1$, we have the Proca field Eq. (1.3) satisfying $\partial^\mu A_\mu^{\mathrm{P}} = 0$. Its positive 2-point function is*

$$_m M^{A_\mu^{\mathrm{P}}, A_\nu^{\mathrm{P}}} = -\pi_{\mu\nu}(p). \tag{1.5}$$

For $s = 2$, the symmetric, traceless and conserved Proca field[3] is given by the field equations $\partial^\mu \partial^\nu F_{[\mu\kappa][\nu\lambda]} = \partial^\mu(-m^2 F_{[\mu\kappa]\lambda}^{\mathrm{P}}) = (-m^2)^2 A_{\kappa\lambda}^{\mathrm{P}}$. Its 2-point function is

$$_m M^{A_{\mu\nu}^{\mathrm{P}}, A_{\kappa\lambda}^{\mathrm{P}}} = \frac{1}{2}\big[\pi_{\mu\kappa}\pi_{\nu\lambda} + \pi_{\mu\lambda}\pi_{\kappa\nu}\big] - \frac{1}{3}\pi_{\mu\nu}\pi_{\kappa\lambda}. \tag{1.6}$$

We emphasize the coefficient $-\frac{1}{3}$ of the third term in the 2-point function Eq. (1.6), that ensures the vanishing of the trace, as opposed to $-\frac{1}{2}$ in the massless case Eq. (1.2), which ensures that there are precisely two helicity states. In particular, the massless field strength is not the limit of the massive field strength as $m \to 0$.

The discrepancy of coefficients is the origin of the DVZ observation due to van Dam and Veltman [vDV70] and Zakharov [Z70], that in interacting models with $s \geq 2$, scattering amplitudes are discontinuous in the mass at $m = 0$, i.e., the scattering on massless gravitons (say) is significantly different from the scattering on gravitons of a very small mass. The DVZ discontinuity has been used to argue that, by measuring the deflection of light in a gravitational field, gravitons must be exactly massless.

A second famous result about the higher-spin massless case is the Weinberg-Witten theorem [WW80, L84, L08]. It states that for $s \geq 2$, no point-localized stress-energy tensor exists such that the Poincaré generators are moments of its zero-components:

$$P_\sigma \overset{!}{=} \int_{x_0=t} d^3\vec{x}\, T_{0\sigma}, \qquad M_{\sigma\tau} \overset{!}{=} \int_{x_0=t} d^3\vec{x}\, (x_\sigma T_{0\tau} - x_\tau T_{0\sigma}). \tag{1.7}$$

The absence of a stress-energy tensor also obstructs the semiclassical coupling of massless higher spin matter to gravity.

We are going to shed new light on these results, for arbitrary integer spin s.

The free massive Proca potential of spin s

$$A_{\mu_1\dots\mu_s}^{\mathrm{P}}(x)$$

is a completely symmetric traceless and conserved tensor fields of rank s. Its positive 2-point function is a polynomial in the tensors $\pi_{\mu\nu}(p)$, with coefficients dictated by symmetry and tracelessness, cf. Sect. 2. It is defined on the same Fock space as its field strength. This potential has no massless limit, and its UV dimension is $s + 1$.

We are going to define on the Fock space of the Proca potential new fields $A_{\mu_1\dots\mu_r}^{(r)}$ of rank $0 \leq r \leq s$ with the following properties.

1. All $A^{(r)}$ have UV dimension $d_{UV} = 1$ and are regular in the massless limit.
2. The potential A^{P} can be decomposed into the fields $A^{(r)}$ ($r \leq s$) in such a way that all contributions of UV dimension > 1 and all terms that are singular as $m \to 0$ are isolated as explicit inverse powers of m times derivatives of $A^{(r)}$ with lower rank $r < s$ (called "escort fields").

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[3] Albeit historically incorrect, we adopt the name "Proca field" also for higher spin.

3. The massive symmetric tensor fields $A^{(r)}$ are neither traceless nor conserved, and they are coupled among each other (with mass-dependent coefficients) through their traces and divergences. In the massless limit, they become traceless and conserved, and their field equations and 2-point functions decouple.

4. At $m = 0$, the escort $A^{(0)}$ is the canonical massless scalar field φ. The tensors $A^{(r>0)}$ are potentials for the field strengths of helicity $h = \pm r$ [W95].

5. Conversely, the given massless field $A^{(s)}$ of any helicity $h = \pm s$ can be made massive ("fattening") by simply changing the dispersion relation $p^0 = \omega_m(\vec{p})$ without loosing positivity. The fattened field brings along with it all lower rank fields $A^{(r)}$ by virtue of the coupling through the divergence, and the Proca field A^{P} can be restored from the massive field $A^{(s)}$. We give a surprisingly simple formula involving only derivatives (Prop. 3.12).

6. We construct a stress-energy tensor, that is regular in the massless limit, such that Eq. (1.7) are the generators of the Poincaré group. In the limit, we obtain a stress-energy tensor that decouples into a direct sum of mutually commuting stress-energy tensors $T^{(r)}$ for the helicity fields $A^{(r)}$.

By Item 6, the massless limit describes the exact decoupling of the lower helicities, along with the splitting of the (m, s) Wigner representation into massless helicity representations with $h = \pm r$ $(r = 1, \ldots, s)$ and $h = 0$:

$$P_\sigma = \bigoplus_{r=0}^{s} P_\sigma^{(r)}, \qquad M_{\sigma\tau} = \bigoplus_{r=0}^{s} M_{\sigma\tau}^{(r)} \tag{1.8}$$

In particular, the number $2s + 1$ of one-particle states at any fixed momentum is preserved in the limit. In contrast, the "fattening" of the massless helicity s field can increase the number of one-particle states, because its 2-point function is a semi-definite quadratic form of rank 2 that becomes rank $2s + 1$ under the deformation. Having less null states, it describes more physical states.

These facts (for $s = 2$) yield an obvious explanation of the DVZ discontinuity [vDV70, Z70]: The potential that continuously connects to the massless helicity $h = \pm 2$ potential is $A^{(2)}$. Coupling through the singular Proca potential or its analog in the indefinite Feynman gauge ([vDV70, Eq. (28)]), one has contributions from all $r \leq 2$ at each positve mass. Rejecting at $m = 0$ the helicities $|h| < 2$ causes the discontinuity. A coupling through $A^{(2)}$ at every mass would instead smoothly decrease the contributions of the lower helicities.

The apparent "jump" of the coefficient from $-\frac{1}{3}$ in Eq. (1.6) (massive) to $-\frac{1}{2}$ in Eq. (1.2) (massless) is by itself *not* a discontinuity, but a consequence of the reorganization of the fields before the limit is taken. The coefficient $-\frac{1}{3}$ pertains to the potential $A^{\mathrm{P}}_{\mu\kappa}$, while the coefficient $-\frac{1}{2}$ pertains to $A^{(2)}_{\mu\kappa}$.

The stated properties 1–6 of the massless potentials and stress-energy tensors are clearly at variance with many No-Go theorems, including the Weinberg-Witten theorem. The explanation is that the new potentials $A^{(r)}$ have a weaker localization property than assumed in [WW80]. Their 2-point functions involve a suitable tensor $E_{\mu\nu}(p)$ whose substitution for the singular (as $m \to 0$) tensor $\pi_{\mu\nu}(p)$ or indefinite

tensor $\eta_{\mu\nu}$ (i) preserves positivity, (ii) does not affect the field strengths, and (iii) has a regular limit $m \to 0$.

The usual No-Go theorems can be traced back to the fact that a tensor $E_{\mu\nu}(p)$ with the stated properties does not exist, if it is allowed to be a function of the momentum only. Instead,

$$E(e, e')_{\mu\nu}(p) := \eta_{\mu\nu} - \frac{p_\mu e_\nu}{(pe)_+} - \frac{e'_\mu p_\nu}{(pe')_+} + \frac{(ee')p_\mu p_\nu}{(pe)_+(pe')_+} \qquad (1.9)$$

are functions (actually distributions: $1/(pe)_+ = 1/((pe + i0))$ stands for the distribution in p and e defined by the integral $\int_{\mathbb{R}_+} du\, e^{ip(x+ue)} = \frac{i}{p \cdot e + i0} e^{ipx}$) of two four-vectors e, e'. Thus, the fields whose 2-point functions are polynomials in $E_{\mu\nu}$ depend on e, and fail to be local because the dependence on p is not polynomial.

Every denominator $(pe)_+$ can be achieved by integrating e^{ipx} along a ray $x + \mathbb{R}_+ e$ (henceforth called "string"): In momentum space, the integration

$$X(x, e) \equiv (I_e X)(x) := \int_{\mathbb{R}_+} du\, X(x + ue) \qquad (1.10)$$

produces precisely the denominators $i((pe) + i0)^{-1}$ in the creation part and $-i((pe) - i0)^{-1}$ in the annihilation part of a field X.[4] Thus, the potentials with the properties 1–6 are "string-localized", cf. the comments below. In fact, they are (iterated) string integrals Eq. (1.10) over the point-localized field strengths.

It follows from Eq. (1.10) (and later generalizations involving iterations of the integral operation I_e) that the Poincaré transformations of string-localized fields are

$$U(a, \Lambda) A_{\mu_1 \ldots \mu_r}(x, e) U(a, \Lambda)^* = \Lambda^{\nu_1}{}_{\mu_1} \cdots \Lambda^{\nu_r}{}_{\mu_r} A_{\nu_1 \ldots \nu_r}(a + \Lambda x, \Lambda e), \qquad (1.11)$$

i.e., the direction of the string is transformed along with its apex x and the tensor components of the field tensor. Unlike a fixed string direction, a transforming direction does not violate covariance.

String-localization requires some comments. First, it is not a feature of the associated particles, which are always the same massive spin s particles, but of the fields used to couple them to other particles. (The only exception would be particles in the infinite-spin representations [MSY06, LMR16], that are beyond the scope of the present analysis.) As we have discussed before, we understand string-localized potentials mainly as a device to set up renormalizable interaction terms that are equivalent to but better behaved (see Example 1.3 below) than their non-renormalizable point-localized counterparts.

It was already discussed in [L84] that the Weinberg-Witten theorem does not exclude non-local densities. The string-localized stress-energy tensors realize this possibility. More importantly, they may be used to couple higher spin matter to gravity.

Admitting string-localized potentials is not in conflict with the principle of causality, which is as imperative in relativistic quantum field theory as Hilbert space positivity.

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[4]Therefore, a 2-point function like Eq. (1.18) will rather involve $E(-e, e')$; we shall adjust this in Sect. 3 by considering correlations between like ${}_m M^{A(-e), A(e')}$.

While their field strengths are point-localized, the string-localized potentials satisfy causal commutation relations according to their localization: two such fields commute whenever every point on the string $x + \mathbb{R}_+ e$ is spacelike to every point on the other string $x' + \mathbb{R}_+ e'$. If the strings are chosen spacelike ($e^2 = -1$), then such pairs of spacelike separated strings are abundant. Even with lightlike directions, there are sufficiently many strings to construct scattering states in Haag-Ruelle theory, provided asymptotic cluster properties can be established. Lightlike strings may have advantages for some purposes [GMV17]. For the purpose of this work, we do not need to specify e further.

The need to use quantities localized in spacelike cones, in order to connect massive particle states with the vacuum in certain classes of interacting theories, was already established by Buchholz and Fredenhagen [BF82] by an analysis of the localization properties of charges within the general theory of local observables.

Example 1.3 *[S16] The merit of using string-localized fields in perturbation theory may be illustrated with QED: The coupling of the current to the indefinite massless potential A^{K} is replaced by a coupling $j^\mu A^{\mathrm{P}}_\mu$ to the massive Proca potential A^{P}. This avoids introducing negative-norm states, but the interaction of UV dimension 5 is non-renormalizable because of the UV dimension 2 of the Proca field. Now, the decomposition (see Sect. 1.1) $A^{\mathrm{P}}_\mu(x) = A_\mu(x, e) - m^{-1}\partial_\mu a(x, e)$ into a string-localized potential and its escort comes to bear: $A_\mu(e)$ has UV dimension 1 and is regular at $m = 0$. The UV-divergent part of the interaction is "carried away" by the escort field: $-m^{-1} j^\mu \partial_\mu a(e) = -\partial_\mu(m^{-1} j^\mu a(e))$ is a total derivative and may be discarded from the interaction Lagrangean. The remaining string-localized (but equivalent to the point-localized) interaction $j^\mu A_\mu(e)$ has UV dimension 4, and can be taken at $m = 0$. Thus, the role of the escort fields is the controlled disposal of the UV divergence from the outset, with the benefit of avoiding Krein space and compensating ghost fields.*

The same strategy applies whenever the string-dependence of an interaction term is a total derivative. The ongoing analysis of perturbation theory with string-localized interactions [S16, MS17, M17] gives strong evidence that the resulting theory is order-by-order renormalizable, and in the case of QED equivalent to the "usual" QED.

The scattering matrix can be made independent of the string direction e, provided a suitable renormalization condition is satisfied. This condition may require the presence of further "induced" interaction terms and/or fields of lower spin. E.g., in scalar massive QED, the cubic part of the string-local minimal coupling induces also the quartic part [S16]. Massive vector bosons (like W and Z bosons) can be coupled without spontaneous symmetry breaking. The string-independence of their self-interaction can only be achieved with the help of a boson with properties like the Higgs, including a quartic self-interaction [S16, MS17].

These observations are very analogous to the analysis by Scharf et al. [DS99, DS00, S01, DGSV10] (in a gauge-theoretic point-localized setting) where the presence of a Higgs boson is required by BRST invariance of causal perturbation theory with self-interacting massive vector bosons. Indeed, the condition of e-independence can also be formulated in a cohomological manner. Yet, the precise relation between gauge invariance and string-independence remains to be explored. But beyond this analogy,

it becomes clear that the role of the Higgs boson is not the generation of the mass, but the preservation of the renormalizability and locality [S16, MS17].

A new renormalizable interaction in the string-localized setting could be the coupling of matter to (massive) gravity through the string-localized potentials $A^{(2)}$, mentioned above in the context of the DVZ discontinuity.

We list a few more features pertaining to the string-localized fields $A^{(r)}$.

7. In the massless limit, the escort $A^{(0)}(x, e) \to \varphi(x)$ becomes independent of e.

8. The string-localized potentials $A^{(r)}$ ($1 \leq r \leq s$) remain string-localized in the massless limit, but their highest field strengths $F^{(r)}$ become point-localized, and coincide with the field strengths obtained from indefinite massless potentials. The potentials $A^{(r)}$ were previously constructed in [PY12] directly in the massless Fock space without an approximation from $m > 0$.

9. Concerning Item 6 above, we proceed in two steps: we first separate from a suitable point-localized stress-energy tensor "irrelevant terms" that do not contribute to the Poincaré generators Eq. (1.7). Instead, they "carry away" all the singularities when $m \to 0$:

$$T_{\rho\sigma}(x) = T_{\rho\sigma}^{\mathrm{reg}}(x, e) + \text{irrelevant terms},$$

The string-localized stress-energy tensor T^{reg} is regular in the limit $m \to 0$. At $m = 0$ it decouples as

$$T_{\rho\sigma}^{\mathrm{reg}}(x, e)\big|_{m=0} = \sum_{r \leq s} T_{\rho\sigma}^{(r)}(x, e) + \text{irrelevant terms}, \tag{1.12}$$

where $T^{(r)}$ are quadratic in $A^{(r)}$ and mutually commute with each other. Their moments generate the Poincaré transformations of the massless fields $A^{(r)}$ in the helicity $h = \pm r$ Wigner representations.

The massless string-localized stress-energy tensors $T^{(r)}$ are displayed in Prop. 4.6.

It is well known that "the" stress-energy tensor is by no means unique. The requirements are that it is conserved so that the momenta P_σ in Eq. (1.7) are independent of the time t; that it is symmetric so that also the Lorentz generators $M_{\sigma\tau}$ in Eq. (1.7) are independent of t; and that the commutators of the generators with the fields implement the given infinitesimal Poincaré transformations, where the commutation relations of the fields are fixed by their 2-point functions.

For $s \geq 2$ one can easily add local terms whose densities are spatial derivatives, and hence do not change the generators. Thus, the local densities are ambiguous while only the total quantities are meaningful – a familiar phenomenon in general relativity.

We give a brief discussion of earlier proposals in Sect. 4, before we present various equivalent (always differing by irrelevant terms) stress-energy tensors with different merits.

1.1 Examples: $s = 1$ vs. $s \geq 2$

The case $s = 1$ is very simple. The 2-point function Eq. (1.5) implies

$$_m M^{mA_\mu^{\mathrm{P}}, mA_\nu^{\mathrm{P}}} = p_\mu p_\nu - m^2 \eta_{\mu\nu}, \quad _m M^{F_{\mu\nu}, mA_\kappa^{\mathrm{P}}} = im(p_\mu \eta_{\nu\kappa} - p_\nu \eta_{\mu\kappa}). \tag{1.13}$$

Hence, in the massless limit mA_μ^{P} decouples from the field strength and becomes the derivative of the scalar free field φ with $_0 M^{\partial_\mu \varphi, \partial_\nu \varphi} = p_\mu p_\nu$.

The string-localized setting produces the massless scalar without derivative. More importantly, it yields the decomposition Item 2 (underlying Example 1.3)

$$A_\mu^{\mathrm{P}}(x) = A_\mu(x, e) - m^{-1} \partial_\mu a(x, e), \tag{1.14}$$

where

$$A_\mu(x, e) \quad := \quad I_e(F_{\mu\nu})(x)e^\nu \equiv \int_{\mathbb{R}_+} du\, F_{\mu\nu}(x + ue)e^\nu, \tag{1.15}$$

$$a(x, e) \quad := \quad m \cdot I_e(A_\nu^{\mathrm{P}})(x)e^\nu, \tag{1.16}$$

are string-localized fields, regular in the massless limit. Both A_μ and a satisfy the Klein-Gordon equation with mass m and are coupled by

$$\partial^\mu A_\mu(x, e) = -ma(x, e). \tag{1.17}$$

One may as well use Eq. (1.17) as the definition of $a(x, e)$ and derive Eq. (1.16) as a consequence.

In the massless limit, the field equation Eq. (1.17) decouples a from A_μ. The former converges to the massless scalar, and the latter converges to a string-localized massless potential of the Maxwell field strength $F_{\mu\nu}$. In terms of 2-point functions, these claims are explicitly seen by computing[5]

$$_m M^{A_\mu(-e), A_\nu(e')} = -\eta_{\mu\nu} + \frac{p_\mu e_\nu}{(pe)_+} + \frac{e'_\mu p_\nu}{(pe')_+} - \frac{(ee')p_\mu p_\nu}{(pe)_+(pe')_+} \equiv -E(e, e')_{\mu\nu}(p). \tag{1.18}$$

Now, Eq. (1.17) gives

$$_m M^{a(-e), A_\nu(e')} \quad = \quad im\Big(\frac{e'_\nu}{(pe')_+} - \frac{(ee')p_\nu}{(pe)_+(pe')_+}\Big), \tag{1.19}$$

$$_m M^{a(-e), a(e')} \quad = \quad 1 - m^2 \frac{(ee')}{(pe)_+(pe')_+}. \tag{1.20}$$

At $m = 0$, the 2-point function of a is independent of e and e', hence $\varphi = a(x, e)|_{m=0}$ is independent of e. Its one-particle state is the remnant of the massive particle state with transverse angular momentum.

The taming of the UV behaviour, underlying the application in Example 1.3, is seen from Eq. (1.18): it occurs because the momentum factors in the denominator balance those in the numerator [MO16]. The same persists at every spin s.

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[5]Because the fields are distributions in x (or p) and in e, we have to admit different string directions for the two field entries, and the choice "$-e$" is mainly a matter of convenience (to become useful for higher spin).

The decoupling in the case $s = 2$ is more delicate, in that it requires a second step. We define the symmetric string-localized potential as the two-fold string integral over the field strength

$$A_{\mu\nu}(x, e) := (I_e^2 F_{[\mu\kappa][\nu\lambda]})(x)e^\kappa e^\lambda, \tag{1.21}$$

and its escort fields by

$$
\begin{aligned}
a_\mu^{(1)}(x, e) &:= -m^{-1}\partial^\nu A_{\mu\nu}(x, e), \\
a^{(0)}(x, e) &:= -m^{-1}\partial^\mu a_\mu^{(1)}(x, e).
\end{aligned} \tag{1.22}
$$

They are regular at $m = 0$ because $\partial^\nu F_{[\mu\kappa][\nu\lambda]} = -m^2 F_{[\mu\kappa]\lambda}^{\mathrm{P}}$ (the partial field strength), and $\partial^\mu F_{[\mu\kappa]\lambda}^{\mathrm{P}} = -m^2 A_{\kappa\lambda}^{\mathrm{P}}$. From the 2-point function

$$_mM^{A_{\mu\nu}(-e),A_{\kappa\lambda}(e')} = \frac{1}{2}\Big[E(e, e')_{\mu\kappa}E(e, e')_{\nu\lambda} + (\kappa \leftrightarrow \lambda)\Big] - \frac{1}{3}E(e, e)_{\mu\nu}E(e', e')_{\kappa\lambda}, \tag{1.23}$$

all other 2-point functions can be computed by descending with Eq. (1.22).

It turns out that the mixed correlations $_mM^{A,a^{(1)}}$ and $_mM^{a^{(1)},a^{(0)}}$ are $O(m)$, hence the even and odd rank fields decouple in the massless limit. But the correlation among the even rank fields are $O(1)$ and become at $m = 0$

$$
\begin{aligned}
_0M^{a^{(0)}(-e),A_{\mu\nu}(e')} &= -\frac{1}{3}E(e', e')_{\mu\nu}(p) \tag{1.24} \\
_0M^{a^{(0)}(-e),a^{(0)}(e')} &= \frac{2}{3}. \tag{1.25}
\end{aligned}
$$

In particular, $A_{\mu\nu}(x, e)$ and $a^{(0)}(x, e)$ do not decouple in the massless limit. They are decoupled by defining

$$A_{\mu\nu}^{(2)}(e) := A_{\mu\nu}(e) + \frac{1}{2}E_{\mu\nu}(e, e)a^{(0)}(e), \tag{1.26}$$

where the operator

$$E(e, e)_{\mu\nu} = \eta_{\mu\nu} + (e_\nu\partial_\mu + e_\mu\partial_\nu)I_e + e^2\partial_\mu\partial_\nu I_e^2 \tag{1.27}$$

acts by multiplication with $E(e, e)_{\mu\nu}(p)$ and $E(e, e)_{\mu\nu}(-p) = E(-e, -e)_{\mu\nu}(p)$ on the creation and annihilation parts, respectively. With this redefinition, the decoupling is exact at $m = 0$.

It turns out that $_mM^{A_{\mu\nu}^{(2)}(-e),A_{\kappa\lambda}^{(2)}(e')}$ is the same as Eq. (1.23) but with the coefficient $-\frac{1}{2}$ rather than $-\frac{1}{3}$ for the third term. In particular, because the derivative terms $O(p/(pe))$ do not contribute to the field strength, $A^{(2)}$ becomes at $m = 0$ a string-localized potential for the massless field strength $F^{(2)}$ of helicity $h = \pm 2$. Unlike the point-localized Krein potential A^{K} with 2-point function Eq. (1.2), $A^{(2)}$ is positive, traceless, and conserved, and it satisfies the axial gauge condition $e^\mu A_{\mu\nu}^{(2)} = 0$.

The decomposition, referred to in Item 2 above, is

$$A_{\mu\nu}^{\mathrm{P}}(x) = A_{\mu\nu}(x, e) - m^{-1}\big(\partial_\mu a_\nu^{(1)} + \partial_\nu a_\mu^{(1)}\big)(x, e) + m^{-2}\partial_\mu\partial_\nu a^{(0)}(x, e), \tag{1.28}$$

where in turn, $A_{\mu\nu}$ is expressed in terms of $A^{(2)}_{\mu\nu}$ and $a^{(0)}$, and $a^{(1)} =: \sqrt{1/2} \cdot A^{(1)}$ and $a^{(0)} =: \sqrt{2/3} \cdot A^{(0)}$. The normalizations are such that in the massless limit $A^{(0)}(x,e)$ becomes the massless scalar field, and $A^{(1)}_\nu$ becomes the string-localized potential for the Maxwell field, as above.

The pattern persists for higher spin: the decoupling fields arise by linear combinations with powers of $E(e,e)$ acting on the lower escort fields. We systematize these results for general integer spin in Sect. 3.

It may be interesting to notice that one can average the $s = 1$ potential $A(x, e)$ in e over the spacelike sphere with $e^0 = 0$. The resulting field is, at $m = 0$, the well-known Maxwell potential in the Coulomb gauge ($A^C_0 = 0$) Eq. (1.1).

The same average of the string-localized potential $A^{(2)}_{\mu\nu}(x, e)$ involves divergent quadratic moments of $e/(pe)_+$. However, at $m = 0$, only mixed correlations of orthogonal components contribute, which formally average to zero, and one gets the Coulomb gauge potential ($A^C_{0\mu} = 0$) also for $s = 2$.

2 Preliminaries on point-localized fields

2.1 Massive case

The massive Proca field $A^P_{\mu_1 \dots \mu_s}$ of spin s is a completely symmetric traceless and conserved tensor field satisfying the Klein-Gordon equation:

$$\eta^{\mu_i \mu_j} A^P_{\mu_1 \dots \mu_s} = 0, \qquad \partial^{\mu_j} A^P_{\mu_1 \dots \mu_s} = 0, \qquad (\Box + m^2) A^P_{\mu_1 \dots \mu_s} = 0. \tag{2.1}$$

with 2-point function

$$_m M^{A^P_{\mu_1 \dots \mu_s}, A^P_{\nu_1 \dots \nu_s}} = \sum_{2n \leq s} \widetilde{b^s_n} (\pi_{\mu\mu})^n (\pi_{\nu\nu})^n (\pi_{\mu\nu})^{s-2n}. \tag{2.2}$$

The sum extends over all inequivalent attributions of the available indices of the given form, namely either $\pi_{\mu_i \mu_j}$, $\pi_{\nu_i \nu_j}$, or $\pi_{\mu_i \nu_j}$, to the schematically displayed factors.

The coefficients $\widetilde{b^s_n} = (-1)^s \frac{n!}{s!(\frac{1}{2}-s)_n}$ [6] ensure the vanishing of the traces, while the fact that $\pi_{\mu\nu}(p) p^\nu = 0$ ensures the vanishing of the divergences.

The same formula can also be derived from the (m, s) Wigner representation:

$$A^P_{\mu_1 \dots \mu_s} = \int d\mu_m(p) \left[e^{ipx} v^a_{\mu_1 \dots \mu_s}(p) a^*_a(p) + \text{h.c.} \right] \tag{2.3}$$

where the tensors $v^a_{\mu_1 \dots \mu_s}(p)$ intertwine the (m, s) Wigner representation of the Lorentz group with the symmetric traceless tensor representation [W95], and the spin indices a are summed over. The coefficients $\widetilde{b^s_n}$ in Eq. (2.2) are due to the projection operator (involved in the intertwiners $v^a_{\mu_1 \dots \mu_s}$) onto the spin s representation in the s-fold tensor product of vector representations of the little group SO(3) (= traceless symmetric tensors in $(\mathbb{C}^3)^{\otimes s}$ [G78, Eq. (1.13)]).

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[6] The alternating overall sign is due to our sign convention of the metric.

To keep track of the combinatorics for general s, it will be advantageous to trade the indices for a "polarization vector" $f \in \mathbb{R}^4$ and write

$$X(f) \equiv X_{\mu_1\ldots\mu_r} f^{\mu_1} \ldots f^{\mu_r},$$

when X is a symmetric rank r tensor. Then the divergence $(\partial X)_{\mu_2\ldots\mu_r} := \partial^{\mu_1} X_{\mu_1\ldots\mu_r}$ and the trace $(\text{Tr}\,X)_{\mu_3\ldots\mu_r} := \eta^{\mu_1\mu_2} X_{\mu_1\ldots\mu_r}$ are given by

$$r \cdot (\partial X)(f) = (\partial_x \cdot \partial_f X(f)), \qquad r(r-1) \cdot (\text{Tr}\,X)(f) = \Box_f X(f).$$

In this notation, the Proca correlation in momentum space is

$$_m M^{A^{\text{P}}(f), A^{\text{P}}(f')} = \sum\nolimits_{2n \leq s} b_n^s \, (f^t \pi f)^n (f'^t \pi f')^n (f^t \pi f')^{s-2n}$$

whose coefficients differ from $\widetilde{b}_n^s$ by a counting factor of equivalent terms:

$$b_n^s = \left[\binom{s}{2n} (2n-1)!! \right]^2 (s-2n)! \cdot \widetilde{b}_n^s = (-1)^s \frac{1}{4^n n!} \frac{s!}{(s-2n)!} \frac{1}{(\frac{1}{2} - s)_n}.$$

In D dimensions (where $\text{Tr}\,(\pi) = D - 1$, little group $\text{SO}(D-1)$), the Pochhammer symbol $(\frac{1}{2} - s)_n$, arising from the projection onto traceless symmetric tensors in $(\mathbb{C}^{D-3})^{\otimes s}$, would be replaced by $(\frac{5-D}{2} - s)_n$.

2.2 Massless case

For the massless case, point-localized covariant potentials with a positive 2-point function do not exist. From the pair of Wigner representations $(m = 0, h = \pm s)$ one can construct point-localized covariant field strengths $F^{(s)}_{[\mu_1\nu_1]\ldots[\mu_s\nu_s]}$ whose 2-point function are the curls of the indefinite 2-point function

$$_0 M^{A^{\text{K}}_{\mu_1\ldots\mu_s}, A^{\text{K}}_{\nu_1\ldots\nu_s}} = \sum\nolimits_{2n \leq s} \widetilde{c}_n^s \, (\eta_{\mu\mu})^n (\eta_{\nu\nu})^n (\eta_{\mu\nu})^{s-2n}, \tag{2.4}$$

(notation as in Eq. (2.2)) or equivalently,

$$_0 M^{A^{\text{K}}(f), A^{\text{K}}(f')} = \sum\nolimits_{2n \leq s} c_n^s \, (f^t \eta f)^n (f'^t \eta f')^n (f^t \eta f')^{s-2n}.$$

The coefficients are

$$c_n^s = \left[\binom{s}{2n} (2n-1)!! \right]^2 (s-2n)! \cdot \widetilde{c}_n^s = (-1)^s \frac{1}{4^n n!} \frac{s!}{(s-2n)!} \frac{1}{(1-s)_n}.$$

In D dimensions (little group $\text{E}(D-2) = \text{SO}(D-2) \ltimes \mathbb{R}^{D-2}$ with $\mathbb{R}^{D-2}$ represented trivially), the Pochhammer symbol $(1-s)_n$, arising from the projection onto traceless symmetric tensors in $(\mathbb{C}^{D-2})^{\otimes s}$, would be $(\frac{6-D}{2} - s)_n$.

3 String-localized fields: general integer spin s

Throughout this section the spin s is fixed, and does not always appear explicitly in the notation; i.e., fields like $a^{(r)}$ or numbers like $b_{nn'}^{rr'}$ will depend also on s.

3.1   Escort fields

Let $A^{\mathrm{P}}_{\mu_1\dots\mu_s}$ and $F_{[\mu_1\nu_1]\dots[\mu_s\nu_s]}$ be the Proca potential and its field strength. Let e be a (spacelike) unit vector. We introduce the symmetric string-localized potential

$$a^{(s)}_{\mu_1\dots\mu_s}(x,e) := (I^s_e F_{[\mu_1\nu_1]\dots[\mu_s\nu_s]})(x)e^{\nu_1}\dots e^{\nu_s} \tag{3.1}$$

and its escort fields for $0 \le r < s$

$$a^{(r)}_{\mu_1\dots\mu_r}(x,e) := -m^{-1} \cdot \partial^\mu a^{(r+1)}_{\mu_1\dots\mu_r\mu}(x,e). \tag{3.2}$$

Remark 3.1 $a^{(s)}_{\mu_1\dots\mu_s}$ *coincides with the string-localized field denoted* $A_{\mu_1\dots\mu_s}$ *in [S15, S16, MO16].* $a^{(r)}$ *($r < s$) are related to the escort fields* $\phi^{(r)}$ *introduced there by derivatives of lower* $\phi^{(q)}$ *($q < r$) and an overall power of the mass:*

$$a^{(r)}_{\mu_1\dots\mu_s}(x,e) = m^{s-r} \sum_{q \le r} \partial_\mu \dots \partial_\mu \phi^{(q)}_{\mu\dots\mu}(x,e)$$

where for each $q \le r$ *the sum extends over all* $\binom{r}{q}$ *inequivalent permutations of the indices. This can be seen from [MO16, Eq. (4)] by taking divergences and using the Klein-Gordon equation. Lemma 3.3 below justifies our departure from the previous definition.*

The definition Eq. (3.1) involves the operations curl, contraction with e and string integration on each Lorentz index of $A^{\mathrm{P}}_{\mu_1\dots\mu_s}$. This means that $a^{(s)}_{\mu_1\dots\mu_s}$ arises from Eq. (2.3) by multiplication of the intertwiner $v^a_{\nu_1\dots\nu_s}$ with the matrix

$$J_\mu{}^\nu(p,e) = \delta^\nu_\mu - \frac{p_\mu e^\nu}{(pe)_+}, \tag{3.3}$$

in each Lorentz index. It is obviously

$$J_\mu{}^\nu(p,e)p_\nu = 0, \qquad e^\mu J_\mu{}^\nu(p,e) = 0. \tag{3.4}$$

Corollary 3.2 *The "axial gauge" condition [PY12]* $e^\mu a^{(r)}_{\mu\mu_2\dots\mu_r}(e) = 0$ *holds.*

Proof: Obvious from the second of Eq. (3.4) and the definition Eq. (3.2).   □

The conservation of A^{P} means that $p^{\mu_i}v^a_{\mu_1\dots\mu_s}(p) = 0$ in Eq. (2.3). Because $ip^\mu J_\mu{}^\nu = ip^\nu - im^2\frac{e^\nu}{(pe)_+}$, it follows for $r \le s$

$$a^{(r)}_{\mu_1\dots\mu_r}(x,e) = \int d\mu_m(p)\Big[e^{ipx}\prod_{k=1}^r J^{\nu_k}_{\mu_k}(p,e)\prod_{k=r+1}^s \frac{ime^{\nu_k}}{(pe)_+}v^a_{\nu_1\dots\nu_s}(p)a^*_a(p) + \text{h.c.}\Big] \tag{3.5}$$

Lemma 3.3 *The fields* $a^{(r)}_{\mu_1\dots\mu_r}(x,e)$ *($0 \le r \le s$) are regular in the limit* $m \to 0$.

Proof: The 2-point functions of $a^{(r)}_{\mu_1...\mu_r}$ arise from Eq. (2.2) by multiplying with the matrices J and contracting with $im\frac{e}{(pe)_+}$ according to Eq. (3.5). By the first of Eq. (3.4), every matrix J kills one singular factor $p_\mu p_\nu/m^2$ of Eq. (2.2), and the powers of m coming with the contractions balance the remaining singularity. $\qquad\square$

We have the (preliminary) decomposition of the point-localized field $A^{\mathrm{P}}_{\mu_1...\mu_s}$ into $a^{(s)}$ and its escort fields:

Proposition 3.4 *The massive point-localized potential of spin s can be written as*

$$A^{\mathrm{P}}_{\mu_1...\mu_s}(x) = \prod_{k=1}^{s} \left(\delta^{\nu_k}_{\mu_k} + m^{-2}\partial_{\mu_k}\partial^{\nu_k}\right)a^{(s)}_{\nu_1...\nu_s}(x,e). \tag{3.6}$$

It decomposes into regular string-localized escort fields with inverse mass coefficients

$$A^{\mathrm{P}}_{\mu_1...\mu_s}(x) = \sum_{r\leq s}(-m^{-1})^{s-r}\partial_\mu...\partial_\mu a^{(r)}_{\mu...\mu}(x,e) \tag{3.7}$$

where for each $r \leq s$ the sum extends over all $\binom{s}{r}$ inequivalent permutations of the indices.

Proof: In momentum space, the differential operator in Eq. (3.6) is $\pi^{\otimes s}$. The identity follows from Eq. (3.5) (with $r = s$), because $\pi J = \pi$ and $\pi^{\otimes s}v^a = v^a$ since A^{P} is conserved. The derivatives $m^{-2}\partial_\mu\partial^\nu$ involved in Eq. (3.6) turn $a^{(r)}$ into $-m^{-1}\partial_\mu a^{(r-1)}$ by Eq. (3.2). This gives Eq. (3.7). $\qquad\square$

The string-localized fields $a^{(r)}$ are dynamically coupled among each other. We have

Proposition 3.5 *The regular escort fields $a^{(r)}_{\mu_1...\mu_r}$ are coupled through the field equations*

$$\partial^{\mu_1}a^{(r)}_{\mu_1...\mu_r} = -m\,a^{(r-1)}_{\mu_2...\mu_r}, \qquad \eta^{\mu_1\mu_2}a^{(r)}_{\mu_1...\mu_r} = -a^{(r-2)}_{\mu_3...\mu_r}. \tag{3.8}$$

By the first equation, every escort $a^{(r)}$ still "contains" all the lower escorts $a^{(r')}$ $(r' < r)$. The divergence will decouple in the massless limit from the lower escorts, while the trace doesn't. Subtracting the traces would instead bring back the coupling through the divergences. This is the reason why the decomposition in Prop. 3.4 is only preliminary.

Proof: The first equation is just the definition Eq. (3.2). The second follows from

$$(J^t\eta J)^{\nu_1\nu_2} = \eta^{\nu_1\nu_2} - \frac{p^{\nu_1}e^{\nu_2} + e^{\nu_1}p^{\nu_2}}{(pe)_+} + m^2\frac{e^{\nu_1}e^{\nu_2}}{(pe)^2_+}$$

together with the fact that A^{P} is traceless and conserved, hence p^ν and $\eta^{\nu_1\nu_2}$ act trivially in Eq. (3.5). $\qquad\square$

We display the 2-point function $_mM^{a^{(s)}(-e),a^{(s)}(e')}$. Every factor π in Eq. (2.2) is hit by two of the matrices $J(p,e')$ or $\overline{J(p,-e)} = J(p,e)$. We therefore define

$$E_{\mu\nu}(e_1,e_2)(p) := (J(p,e_1)\pi(p)J(p,e_2)^t)_{\mu\nu} = (J(p,e_1)\eta J(p,e_2)^t)_{\mu\nu} \tag{3.9}$$

which is precisely the distribution defined in Eq. (1.9), and abbreviate (for $f, f' \in \mathbb{R}^4$)

$$E_{ff} \equiv f^t E(e,e) f, \quad E_{ff'} \equiv f^t E(e,e') f', \quad E_{f'f'} \equiv f'^t E(e',e') f'.$$

Then we have from Eq. (2.2) and Eq. (3.5) with $r = s$:

$$_m M^{a^{(s)}(-e)(f), a^{(s)}(e')(f')} = \sum_n b_n^s \, (E_{ff})^n (E_{f'f'})^n (E_{ff'})^{s-2n}. \qquad (3.10)$$

3.2   Decoupling in the massless limit

The massless results of this section are equivalent to results obtained recently by Plaschke and Yngvason [PY12, Sect. 4A]. While these authors consider Wigner intertwiners directly at $m = 0$, we exhibit smooth families of fields $A^{(r)}|_{m \geq 0}$.

We turn to the task of a complete decoupling at $m = 0$. We do this by a study of the 2-point functions. In a positive metric, decoupling the 2-point functions implies the decoupling of the field equations.

The 2-point functions of the massive escort fields $a^{(r)}$ do not decouple. In order to compute them efficiently, we cast Eq. (3.5) into the form of a "generating functional":

$$\sum_{r \leq s} \binom{s}{r} a^{(r)}(e)(f) = Z(f,e) := A^{\mathrm{P}}(J_e^t f + m e I_e).$$

Here I_e is the string integration, understood in this formula as an operation acting on the field, and J_e acts by multiplication with $J_\mu{}^\nu(p,e)$ and its complex conjugate on the creation resp. annihilation part of the field. Then

$$_m M^{Z(f,-e), Z(f',e')} = \sum_{r,r'} \binom{s}{r} \binom{s}{r'} \cdot {}_m M^{a^{(r)}(-e)(f) a^{(r')}(e')(f')}.$$

Given the l.h.s. as a function of f and f', the correlations between $a^{(r)}$ and $a^{(r')}$ can be read off by selecting the terms of the appropriate homogeneities in f and in f'.

In order to compute the l.h.s., we have to contract each factor $\pi_{\mu\mu}$ in Eq. (2.2) twice with $(J(p,e)^t f - ime/(pe)_+)^\mu$, each factor $\pi_{\nu\nu}$ twice with $(J(p,e')^t f + ime'/(pe')_+)^\nu$, and each factor $\pi_{\mu\nu}$ with both vectors. Because of the first of Eq. (3.4) and Eq. (3.9), and because $(me/(pe)_+)^t \pi (me/(pe)_+) = -1 + O(m^2)$, all these contractions are of the form $E + 1 + O(m)$ resp. $E - 1 + O(m)$, and one arrives at

$$_m M^{Z(f,-e), Z(f',e')} = \sum_{n \leq s} b_n^s \, (E_{ff} + 1)^n (E_{f'f'} + 1)^n (E_{ff'} - 1)^{s-2n} + O(m).$$

We get the massless 2-point functions:

Proposition 3.6 *At $m = 0$, one has*

$$_0 M^{a^{(r)}(-e)(f), a^{(r')}(e')(f')} = \sum_{r-2n=r'-2n'} b_{nn'}^{rr'} (E_{ff})^n (E_{f'f'})^{n'} (E_{ff'})^{r-2n} \qquad (3.11)$$

with

$$\binom{s}{r} \binom{s}{r'} \cdot b_{nn'}^{rr'} = \sum_m \binom{m}{n} \binom{m}{n'} \binom{s-2m}{r-2n} \cdot b_m^s. \qquad (3.12)$$

In particular, $_0 M^{a^{(r)}(-e), a^{(r')}(e')} = 0$ if $r - r'$ is odd.

Proof: Eq. (3.12) are the coefficients of the respective terms of homogeneity r in f and r' in f'. $\qquad\Box$

One could also have computed Eq. (3.11) by descending from Eq. (3.10) with Eq. (3.8) at $m > 0$, and then taking $m \to 0$.

We now set out to "diagonalize" the mixed 2-point functions Eq. (3.11) with the help of the operator $E(e, e)_{\mu\nu}$ given in Eq. (1.27). We write $E_{ff} \equiv f^t E(e, e) f$.

Proposition 3.7 *The combinations*

$$A^{(r)}(f) = \sum_{2k \le r} \alpha_k^r \cdot (E_{ff})^k a^{(r-2k)}(f) \tag{3.13}$$

are traceless at $m = 0$ if and only if

$$\alpha_k^r = \frac{(r - 2k + 2)(r - 2k + 1)}{4k(r - k)} \alpha_{k-1}^r \qquad \Leftrightarrow \qquad \frac{\alpha_k^r}{\alpha_0^r} = (-1)^{r-k} c_k^r$$

with c_k^r given in Sect. 3.2.

Proof: The recursion follows by applying $\Box_f$ to Eq. (3.13) and noticing that $\mathrm{Tr}\,(E) = 2 + O(m^2)$, $\mathrm{Tr}\,(a^{(r)}) = -a^{(r-2)}$, and $E_\nu{}^\mu a^{(r)}_{\mu\mu_2\ldots\mu_r} = a^{(r)}_{\nu\mu_2\ldots\mu_r} + O(m)$ because $\partial a^{(r)} = O(m)$ and $ea^{(r)} = 0$ (Eq. (3.8) and Cor. 3.2). The recursion is solved by $\alpha_k^r = (-1)^{r-k} c_k^r \cdot \alpha_0^r$, where α_0^r will be used later for normalization. $\qquad\Box$

Because the definition Eq. (3.13) is upper triangular in r, the inverse formula is of the same form. We did, however, not succeed to compute its coefficients in closed form.

The operators E_{ff} and $E_{f'f'}$ involved in the field definitions produce the factors denoted with the same symbols (cf. Eq. (3.10)) in the 2-point functions. Therefore, the correlations among $A^{(r)}(f)|_{m=0}$ are of the same general form as Eq. (3.11) with different coefficients. Because $A^{(r)}$ are traceless, the same must be true for their correlations. This implies their decoupling:

Proposition 3.8

$$_0 M^{A^{(r)}(e)(f), A^{(r')}(e')(f')} = \delta_{rr'} N_r \cdot \sum_{2n \le r} c_n^r \, (E_{ff})^n (E_{f'f'})^n (E_{ff'})^{r-2n} \tag{3.14}$$

with the same coefficients $c_n^r = (-1)^r \frac{1}{4^n n!} \frac{r!}{(r-2n)!} \frac{1}{(1-r)_n}$ as in Sect. 3.2. The proper normalization $N_r = 1$ can be achieved by adjusting α_0^r.

Proof: We make a general ansatz with coefficients $c_{nn'}^{rr'}$ with $r - 2n = r' - 2n'$. The vanishing of $\Box_f$ and of $\Box_{f'}$ gives conflicting recursions for $c_{nn'}^{rr'}$ unless $r = r'$. If $r = r'$, the recursion implies the displayed coefficients. $\qquad\Box$

While Eq. (3.13) are defined for $m \ge 0$, the decoupling is exact only at $m = 0$.

Corollary 3.9 *The massless symmetric tensor potentials $A^{(r)}(x,e)$ are traceless (by construction) and conserved. They satisfy in addition the axial gauge condition*

$$e^\mu A^{(r)}_{\mu\mu_2\ldots\mu_r}(x,e) = 0.$$

They are string-localized potentials for the field strengths associated with the Wigner representations of helicity $h = \pm r$ [W95]. They coincide with the potentials given in [PY12, Sect. 4A].

Proof: When the divergence is taken, the derivative may be contracted with an index of E or with an index of $a^{(r-2k)}$. The former contributions are $Ep = O(m^2)$, the latter are $O(m)$ by Eq. (3.8), hence the divergence vanishes at $m = 0$. The axial gauge is a consequence of Cor. 3.2 and the fact that $e^\mu E(e,e)_{\mu\nu} = 0$. The last statements are immediate because $E_{\mu\nu}$ differs from $\eta_{\mu\nu}$ by derivative terms that do not contribute to the field strengths; and the coefficients are the same as in Eq. (2.4). □

It remains to relate the normalization N_r in Eq. (3.14) (which should be $= 1$ in the standard normalization Sect. 3.2) to $(\alpha_0^r)^2$ from Eq. (3.13). Because it is the coefficient of the purely mixed term $(E_{ff'})^r$ in Eq. (3.11), it is easy to see from Eq. (3.13) and Eq. (3.14) that $N_r c_0^r = (\alpha_0^r)^2 b_{00}^{rr}$, with $b_{00}^{rr} = \binom{s}{r}^{-1} \sum_{2m \le s-r} \frac{1}{4^m m!} \frac{(r-s)_{2m}}{(\frac{1}{2}-s)_m}$ given by Eq. (3.12). So the proper normalization is fixed by

$$(\alpha_0^r)^2 = (-1)^r (b_{00}^{rr})^{-1} = \binom{s}{r} \frac{\Gamma(\frac{1}{2}+s)\Gamma(1+r)}{\Gamma(\frac{1}{2}+\frac{r+s}{2})\Gamma(1+\frac{r+s}{2})}. \tag{3.15}$$

Remark 3.10 *(i) The decoupled massless fields $A^{(r)}$ are independent of the spin $s \ge r$ of the massive field in whose decomposition they emerge in the massless limit.*

(ii) The axial gauge condition in Cor. 3.9 ensures the reduction of the degrees of freedom as compared to the massive representation of spin r (relevant little group $\mathrm{SO}(D-2) = \mathrm{E}(D-2)/\mathbb{R}^{D-2}$ vs. $\mathrm{SO}(D-1)$ in D dimensions).

(iii) For the 2-point functions of the components $A^{(r)}_{\mu_1\ldots\mu_r}$, the factors E_{ff}, $E_{ff'}$ etc. in Eq. (3.14) have to be replaced by corresponding components of the tensors $E(e,e)(p)$:

$${}_0 M^{A^{(r)}_{\mu_1\ldots\mu_r}(e),A^{(r)}_{\nu_1\ldots\nu_r}(e')} = \sum \widetilde{c_n^r} \, (E(e,e)_{\mu\mu})^n (E(e',e')_{\nu\nu})^n (E(e,e')_{\mu\nu})^{r-2n}$$

(notation as in Eq. (2.2)).

(iv) Taking the total curl, kills all factors p_μ in all E tensors. Therefore the 2-point functions of the highest field strengths $F^{(r)}_{[\mu_1\nu_1]\ldots[\mu_r\nu_r]}$ are the same as if they were derived from point-localized potentials $A^{\mathrm{K}(r)}$ with indefinite 2-point functions Eq. (2.4). The Krein potentials are neither traceless nor conserved.

(v) In particular, the field strengths are independent of e, hence they are point-localized fields, and (by the same argument as the one leading to Eq. (3.10))

$$A^{(r)}_{\mu_1\ldots\mu_r}(x,e) = (I_e^r F^{(r)}_{[\mu_1\nu_1]\ldots[\mu_r\nu_r]})(x) e^{\nu_1} \ldots e^{\nu_r}. \tag{3.16}$$

3.3   "Fattening"

The 2-point function Eq. (3.14) with $r = s$ is exact also for $m > 0$.[7]

Thus, if one takes the massless string-localized potential $A^{(s)}|_{m=0}$ with 2-point function Eq. (3.14) (with $r = s$) as the starting point, one can get the mass by simply taking the arguments of the functions $E_{\mu\nu}(p)$ on the mass-shell. The previous analysis, where we have derived this massive 2-point function from a positive theory, shows that this deformation preserves positivity. But it decreases the number of null states of the 2-point function, viewed as a quadratic form, because the massive potential is not conserved, and hence it increases the number of particle states.

Remark 3.11 *The fattening allows to continuously "turn on the mass" in interactions with vector or tensor bosons without appealing to the Higgs mechanism and the "eating of the Goldstone boson". See the comments in Sect. 1.*

One can also get back the Proca potential $A^{\mathrm{P}}(x)$ as derivatives of the fattened potential $A^{(s)}(x, e)$:

Proposition 3.12 *The point-localized Proca potential can be restored from the string-localized massive helicity $h = \pm s$ field $A^{(s)}|_{m>0}$ by "applying the Proca 2-point function Eq. (2.2)", regarded as a differential operator ($\pi_{\mu\nu} = \eta_{\mu\nu} + m^{-2}\partial_\mu\partial_\nu$):*

$$A^{\mathrm{P}}_{\mu_1\ldots\mu_s}(x) = (-1)^s \cdot {}_m M^{A^{\mathrm{P}}_{\mu_1\ldots\mu_s},A^{\mathrm{P}\nu_1\ldots\nu_s}} \cdot A^{(s)}_{\nu_1\ldots\nu_s}|_m(x,e)$$

Proof: We multiply the 2-point function in the form Eq. (2.2) on $A^{(s)}$ in the form Eq. (3.13). In the first step, we notice that every factor $E_{\nu\nu}$ contained the field[8] annihilates the 2-point function because the latter is conserved and traceless. Thus, we may replace $A^{(s)}$ by its leading term $a^{(s)}$ ($k = 0$ in Eq. (3.13), $\alpha_0^s = 1$). In the second step, we notice that every factor $\pi^{\nu\nu}$ in the 2-point function annihilates $a^{(s)}$ by virtue of Eq. (3.8). Thus we may replace the 2-point function by its leading term $n = 0$ in Eq. (2.2), which is $(-\pi)^{\otimes s}$. The claim then follows from Eq. (3.6).   □

Proposition 3.13 *Conversely, we have the formulae*

$$a^{(s)}_{\mu_1\ldots\mu_s}(x,e) = (-1)^s \cdot {}_m M^{a^{(s)}_{\mu_1\ldots\mu_s}(-e),a^{(s)\nu_1\ldots\nu_s}(e)} \cdot A^{\mathrm{P}}_{\nu_1\ldots\nu_s}(x) \qquad (3.17)$$

for $m > 0$, and (after taking the limit $m \to 0$ of the regular field $a^{(s)}$)

$$A^{(s)}_{\mu_1\ldots\mu_s}(x,e) = (-1)^s \cdot {}_0 M^{A^{(s)}_{\mu_1\ldots\mu_s}(e),A^{(s)\nu_1\ldots\nu_s}(e)} \cdot a^{(s)}_{\nu_1\ldots\nu_s}(x,e) \qquad (3.18)$$

for $m = 0$, to restore the massless helicity field $A^{(s)}$ from the Proca field. In position space, the 2-point functions Eq. (3.10), Eq. (3.14) are understood as integro-differential operators, cf. Eq. (1.27).

---

[7]This is not true for the massive fields $A^{(r)}$ with $r < s$. Due to their coupling to fields with $r' > r$, their 2-point functions are not just polynomials in $E_{\mu\nu}(p)$, cf. Eq. (3.11).

[8]We suppress sub-indices like $E_{\nu_i\nu_j}$ in this and all similar arguments to follow.

Proof: For Eq. (3.17), we notice that every factor $E^{\nu\nu}$ annihilates A^{P} (traceless and conserved), hence only $n=0$ in the 2-point function contributes, and the factors $E_\mu{}^\nu$ act on A^{P} like $\delta_\mu^\nu - \frac{p_\mu e^\nu}{(pe)_+} = J_\mu{}^\nu$. This gives $a^{(s)}$ by Eq. (3.5). For Eq. (3.18), we notice that at $m=0$, $E_\mu{}^\nu$ acts on $a^{(s)}$ like δ_μ^ν by the first of Eq. (3.8) and Cor. 3.2, and $E^{\nu\nu}$ acts like $\eta^{\nu\nu}$. Thus, the second of Eq. (3.8) implies the claim. $\qquad\square$

4 Stress-energy tensor

4.1 The point-localized stress-energy tensor for $m > 0$

We refer to App. A for some comments on stress-energy tensors and Lagrangeans for free fields of higher spin.

For our purposes here, it suffices to "read back" a suitable stress-energy tensor for the Proca field $A^{\mathrm{P}}_{\mu_1\ldots\mu_s}$ from a simple form of the Poincaré generators.

Proposition 4.1 *The generators of the Poincaré transformations of the Proca field can be written as*

$$P_\sigma = (-1)^s \int d^3\vec{x}\Big[-\frac{1}{4} A^{\mathrm{P}}_{\mu_1\ldots\mu_s} \overset{\leftrightarrow}{\partial}_\sigma \overset{\leftrightarrow}{\partial}_0 A^{\mathrm{P}\mu_1\ldots\mu_s}\Big], \tag{4.1}$$

$$M_{\sigma\tau} = (-1)^s \int d^3\vec{x}\Big[-\frac{1}{4}\Big(x_\sigma \cdot A^{\mathrm{P}}_{\mu\times} \overset{\leftrightarrow}{\partial}_0\overset{\leftrightarrow}{\partial}_\tau A^{\mathrm{P}\mu\times} - (\sigma\leftrightarrow\tau)\Big) - s A^{\mathrm{P}}_{\sigma\times} \overset{\leftrightarrow}{\partial}_0 A^{\mathrm{P}\times}_\tau\Big], \tag{4.2}$$

where $X_\times Y^\times$ stands for the contraction in $s-1$ indices $\mu_2\ldots\mu_s$.

Here and everywhere below, normal ordering is understood.

Before we give the proof, we state the corollary:

Corollary 4.2 *The generators Eq. (4.1) and Eq. (4.2) can be obtained from the "reduced stress-energy tensor"*

$$T^{\mathrm{red}}_{\rho\sigma} := (-1)^s\Big[-\frac{1}{4} A^{\mathrm{P}}_{\mu\times} \overset{\leftrightarrow}{\partial}_\rho\overset{\leftrightarrow}{\partial}_\sigma A^{\mathrm{P}\mu\times} - \frac{s}{2} \partial^\mu\Big(A^{\mathrm{P}}_{\rho\times} \overset{\leftrightarrow}{\partial}_\sigma A^{\mathrm{P}\times}_\mu + (\rho\leftrightarrow\sigma)\Big)\Big]. \tag{4.3}$$

See App. A for how T^{red} relates to more familiar stress-energy tensors.

Eq. (4.1) and the first term in Eq. (4.3) already appear in [F39]. The second term in Eq. (4.3) does not contribute to the momenta, but it produces the last term in Eq. (4.2), which is necessary in order to get the correct infinitesimal boosts. This will become apparent in the proof of Prop. 4.1. The first term in Eq. (4.3) and the two parts of the derivative term are separately conserved w.r.t. both indices ρ and σ by virtue of Lemma B.1(i) resp. (ii).

Proof of Cor. 4.2: We have to do the integrals Eq. (1.7) at fixed $x^0 = t$. The first part of Eq. (4.3) obviously gives Eq. (4.1) and the first terms of Eq. (4.2). The two pieces of the second part do not contribute to P_σ, and they give rise to the last term of Eq. (4.2) by Lemma B.1(i) and (ii), respectively. $\qquad\square$

Proof of Prop. 4.1: The argument for P_σ can essentially be found in [F39], except that the commutator Eq. (4.4) has been guessed not quite correct [F39, Eq. (4.2)]. We display the argument here because we shall use many variants of it below. See also footnote 8.

The 2-point function Eq. (2.2) fixes the commutation relation

$$[A^{\mathrm{P}}_{\mu_1\dots\mu_s}(x), A^{\mathrm{P}}_{\nu_1\dots\nu_s}(y)] = (-1)^s D_{\mu_1\dots\mu_s,\nu_1\dots\nu_s} \Delta_m(x-y) \tag{4.4}$$

where $(-1)^s D_{\mu_1\dots\mu_s,\nu_1\dots\nu_s} = {}_m M^{A^{\mathrm{P}}_{\mu_1\dots\mu_s}, A^{\mathrm{P}}_{\nu_1\dots\nu_s}}$ is the 2-point function regarded as a differential operator ($\pi_{\mu\nu} = \eta_{\mu\nu} + m^{-2}\partial_\mu\partial_\nu$) acting on the commutator function $\Delta_m(x-y)$ of the scalar free field. The commutator of P_σ with $A^{\mathrm{P}}_{\nu_1\dots\nu_s}$ is

$$[P_\sigma, A^{\mathrm{P}}_{\nu_1\dots\nu_s}(y)] = -\frac{1}{2}\int d^3\vec{x}\, D_{\mu_1\dots\mu_s,\nu_1\dots\nu_s} \Delta_m(x-y) \overset{\leftrightarrow}{\partial_0}\overset{\leftrightarrow}{\partial_\sigma} A^{\mathrm{P}\,\mu_1\dots\mu_s}(x).$$

The derivatives ∂_μ appearing in pieces of the differential operator D can be partially integrated using Lemma B.1(i), with $\Theta_{\rho\sigma}$ of the form $\partial_\mu(D'_{\dots}\Delta_m \overset{\leftrightarrow}{\partial_\rho}\overset{\leftrightarrow}{\partial_\sigma} A^{\mathrm{P}\,\mu\dots})$, suppressing further indices. After partial integration, the derivatives act on the field $A^{\mathrm{P}\,\mu\dots}(x)$ where they vanish. Thus, one may replace all operators of the form $\pi_{\mu\nu}$ and $\pi_{\mu\mu}$ in D by $\eta_{\mu\nu}$ and $\eta_{\mu\mu}$. Because the latter also kill the field $A^{\mathrm{P}\,\mu\dots\mu}(x)$, only the contribution $n=0$ of the 2-point function Eq. (2.2) (that specifies the operator D) survives, and D may be replaced by the "identity operator" $(\eta_{\mu\nu})^{\otimes s}$. At this point, the integral can be immediately performed: because Eq. (4.1) integrated at $x^0 = t$ is independent of t, one may choose $x^0 = y^0$, and use the equal-time properties of the scalar commutator function: $\Delta_m(x)|_{x^0=0} = 0$ and $\partial_0\Delta_m(x)|_{x^0=0} = -i\delta(\vec{x})$. We get the desired result $[P_\sigma, A^{\mathrm{P}}_{\nu\dots\nu}(y)] = -i\partial_\sigma A^{\mathrm{P}}_{\nu\dots\nu}(y)$.

The argument for the Lorentz generators is more involved. The commutator of the first terms in Eq. (4.2) with $A^{\mathrm{P}}_{\nu_1\dots\nu_s}$ is

$$-\frac{1}{2}\int d^3\vec{x}\, x_\sigma D_{\mu_1\dots\mu_s,\nu_1\dots\nu_s} \Delta_m(x-y) \overset{\leftrightarrow}{\partial_0}\overset{\leftrightarrow}{\partial_\tau} A^{\mathrm{P}\,\mu_1\dots\mu_s}(x) - (\sigma \leftrightarrow \tau).$$

All terms involving $\partial_\mu\partial_\mu$, either from $\pi_{\mu\mu}$ or from $\pi_{\mu\nu}\pi_{\mu\nu}$ within D, vanish because $\int d^3\vec{x}\, \partial_\mu[D''_{\dots}\Delta_m \overset{\leftrightarrow}{\partial_0}\overset{\leftrightarrow}{\partial_\sigma} A^{\mathrm{P}\,\mu\dots}] = 0$ (using Lemma B.1(i) twice). Thus, the only contributions are due to $(\eta_{\mu\nu})^{\otimes s}$ and s terms $(\eta_{\mu\nu})^{\otimes s-1}m^{-2}\partial_\mu\partial_\nu$. The former give rise, if evaluated at $x^0 = y^0$, to the infinitesimal transformation of the point x:

$$-\frac{1}{2}\int d^3\vec{x}\, x_\sigma \Delta_m(x-y) \overset{\leftrightarrow}{\partial_0}\overset{\leftrightarrow}{\partial_\tau} A^{\mathrm{P}}_{\nu_1\dots\nu_s} = -i(x_\sigma\partial_\tau - x_\tau\partial_\sigma)A^{\mathrm{P}}_{\nu_1\dots\nu_s}.$$

The latter give rise, again by Lemma B.1(i), to the undesired term

$$-\frac{1}{2m^2}\sum_{i=1}^s \int d^3\vec{x}\, \partial_{\nu_i}\Delta_m \overset{\leftrightarrow}{\partial_0}\overset{\leftrightarrow}{\partial_\sigma} A^{\mathrm{P}}_{\tau\nu_1\dots\widehat{\nu_i}\dots\nu_s} - (\sigma \leftrightarrow \tau) = \frac{i}{m^2}\sum_{i=1}^s \partial_{\nu_i} F^{\mathrm{P}}_{[\sigma\tau]\nu_1\dots\widehat{\nu_i}\dots\nu_s}.$$

On the other hand, the commutator of the last term in Eq. (4.2) with $A^{\mathrm{P}}_{\nu_1\dots\nu_s}$ is

$$-s\int d^3\vec{x}\, D_{\sigma\mu_2\dots\mu_s,\nu_1\dots\nu_s} \Delta_m(x-y) \overset{\leftrightarrow}{\partial_0} A^{\mathrm{P}\,\mu_2\dots\mu_s}_\tau(x) - (\sigma \leftrightarrow \tau).$$

Again, all terms involving ∂_μ vanish by Lemma B.1(i), and terms involving $\eta_{\mu\mu}$ vanish because A^{P} is traceless. Thus, only the terms $\pi_{\sigma\nu}(\eta_{\mu\nu})^{\otimes s-1}$ survive:

$$= -\sum_{i=1}^s \int d^3\vec{x} \left(\eta_{\sigma\nu_i} + m^{-2}\partial_\sigma\partial_{\nu_i}\right) \Delta_m(x-y) \overset{\leftrightarrow}{\partial_0} A^{\mathrm{P}}{}_{\tau\nu_1\ldots\widehat{\nu_i}\ldots\nu_s}(x) - (\sigma \leftrightarrow \tau).$$

The contribution from $\eta_{\sigma\nu_i}$ gives the infinitesimal transformation of the tensor indices

$$-i \sum_{i=1}^s \left(\eta_{\sigma\nu_i} \overset{\leftrightarrow}{\partial_0} A^{\mathrm{P}}{}_{\tau\nu_1\ldots\widehat{\nu_i}\ldots\nu_s} - \eta_{\tau\nu_i} \overset{\leftrightarrow}{\partial_0} A^{\mathrm{P}}{}_{\sigma\nu_1\ldots\widehat{\nu_i}\ldots\nu_s}\right).$$

The remaining contribution from $m^{-2}\partial_\sigma\partial_{\nu_i}$ is

$$-\frac{1}{m^2} \sum_{i=1}^s \int d^3\vec{x}\, \partial_\sigma\partial_{\nu_i}\Delta_m(x-y) \overset{\leftrightarrow}{\partial_0} A^{\mathrm{P}}{}_{\tau\nu_1\ldots\widehat{\nu_i}\ldots\nu_s}(x) - (\sigma \leftrightarrow \tau)$$

and cancels with the previous undesired term thanks to the identity

$$\int d^3\vec{x} \left[X \overset{\leftrightarrow}{\partial_0}\overset{\leftrightarrow}{\partial_\sigma} Y + 2\partial_\sigma X \overset{\leftrightarrow}{\partial_0} Y\right] = \int d^3\vec{x}\, \partial_\sigma\left[X \overset{\leftrightarrow}{\partial_0} Y\right] = 0 \tag{4.5}$$

(once more by Lemma B.1(i), writing $\partial_\sigma = \partial^\mu\eta_{\mu\sigma}$). $\qquad\qquad\square$

4.2 The string-localized stress-energy tensors for $m = 0$

We are going to separate "irrelevant contributions" from the reduced stress-energy tensor, that do not contribute to the generators. It is, however, more practical, to perform the corresponding partial integrations inside the generators Eq. (4.1), Eq. (4.2), and read back a resulting stress-energy tensor, as we have done before. In the first step, the partial integrations remove all terms that are singular in the massless limit.

We insert the preliminary decomposition Eq. (3.7) of the point-localized potential A^{P} in terms of derivatives of string-localized fields $a^{(r)}$ into the Poincaré generators Eq. (4.1) and Eq. (4.2), and partially integrate all the derivatives of the decomposition. The result is

Proposition 4.3 *Expressed in terms of string-localized fields $a^{(r)}$ ($r \le s$), the Poincaré generators are*

$$P_\sigma = \sum_{r=0}^s \binom{s}{r}(-1)^r \int d^3\vec{x} \left[-\frac{1}{4}a^{(r)}_{\mu_1\ldots\mu_r}(x,e) \overset{\leftrightarrow}{\partial_0}\overset{\leftrightarrow}{\partial_\sigma} a^{(r)\mu_1\ldots\mu_r}(x,e')\right],$$

$$M_{\sigma\tau} = \sum_{r=0}^s \binom{s}{r}(-1)^r \int d^3\vec{x}$$
$$\left[-\frac{1}{4}x_\sigma a^{(r)}_{\mu\times}(x,e) \overset{\leftrightarrow}{\partial_0}\overset{\leftrightarrow}{\partial_\tau} a^{(r)\mu\times}(x,e') - \frac{r}{2}a^{(r)}_{\sigma\times}(x,e) \overset{\leftrightarrow}{\partial_0} a^{(r)\times}_\tau(x,e')\right] - (\sigma \leftrightarrow \tau)$$

for any pair e, e', and at all values of the mass m.

Remark 4.4 *The integrands must be understood as distributions in e and e' separately, i.e., they should be averaged with test functions $h(e, e')$ (of total weight one, in order to preserve the generators, which are independent of e, e'). The reason is as follows.*

All quadratic expressions are understood as Wick products. Wick products of string-localized fields at coinciding x and at coinciding e are well-defined as distributions in x and e [M17]. For point-localized fields, the "time $t = 0$ integral" of $:\phi(x)^2:$ (i.e., the extension of this distribution to the singular "test function" $\delta(x^0)1_{\mathbb{R}^3}$) is not defined as an operator: only its matrix elements in a dense set of states are defined. For string-localized fields with spacelike e, the situation is worse: not even matrix elements of the integral over $:\phi(x, e)^2:$ can be defined, because they produce conflicting denominators $1/(pe)_-(pe)_+$. Therefore, even if e, e' are arbitrary, one must not put $e' = e$ and regard the integrand as a distribution in e.

With lightlike e, the fields are functions of e rather than distributions [MO16, footnote 3], and the problem is absent.

Proof of Prop. 4.3: We insert the expansion Eq. (3.7) of $A^{\mathrm{P}}(x)$ in terms of derivatives of $a^{(r)}(e)$ resp. $a^{(r')}(e')$ into Eq. (4.1).

It is routine work to partially integrate all the derivatives coming from Eq. (3.7), using Lemma B.1(i) again and again. The field equations Eq. (3.8) produce positive powers of the mass m, that cancel all inverse powers of the expansion: Partially integrating $\partial_\mu a^{(r)}_{...}(e)$ against $a^{(r')\mu\cdots}(e')$, one gets $m a^{(r)}_{...}(e) \cdots a^{(r'-1)\cdots}(e')$ by Eq. (3.8), and vice versa. Partially integrating $\partial_\mu a^{(r)}_{...}(e)$ against $\partial^\mu a^{(r')\cdots}(e')$, one gets $m^2 a^{(r)}_{...}(e) \cdots a^{(r')\cdots}(e')$ by the Klein-Gordon equation. In the expansion of the momenta Eq. (4.1), the number of terms with a contractions between derivatives, b contractions between $a^{(r)}(e)$ and a derivative, b' contractions between a derivative and $a^{(r')}(e')$, and c contractions between $a^{(r)}(e)$ and $a^{(r')}(e')$, such that $r = b + c$, $r' = b' + c$ and $a + b + b' + c = s$, is $\frac{s!}{a!b!b'!c!}$. Each such term after partial integration becomes (schematically) $(-1)^{b+b'} a^{(c)} \cdots a^{(c)}$ times the same operator quadratic in $a^{(c)}$. Therefore the combinatorics is done by observing that $\sum_{a+b+b'=s-c}(-1)^{b+b'}\frac{s!}{a!b!b'!c!} = (-1)^{s-c}\binom{s}{c}$.

The expansion of the Lorentz generators Eq. (4.2) is likewise just a counting issue, where special care has to be taken with the tensor indices σ, τ in the second contribution to $M_{\sigma\tau}$. When they are attached to derivatives, they cancel against the results of partial integrations according to Lemma B.1(i) in the first term. $\square$

The formulae in Prop. 4.3 have the merit that they do not contain any singular fields, and one may read back a conserved and symmetric massive string-localized stress-energy tensor $T^{\mathrm{reg}}_{\sigma\rho}(e, e')$ that is regular at $m = 0$, in exactly the same way as was done in Cor. 4.2 from Prop. 4.1. The limit $m \to 0$ can be taken directly by putting $m = 0$. But these steps are of little use, because the intermediate escort fields $a^{(r)}$ do not decouple. We must in turn express $a^{(r)}$ in Prop. 4.3 in terms of the decoupling string-localized fields $A^{(r-2k)}$. The following result holds only at $m = 0$, where the decoupling of 2-point functions is exact.

Proposition 4.5 *At $m = 0$, one has Eq. (1.8):*

$$P_\sigma = \bigoplus_{r=0}^{s} P_\sigma^{(r)}, \qquad M_{\sigma\tau} = \bigoplus_{r=0}^{s} M_{\sigma\tau}^{(r)} \tag{4.6}$$

where for any e, e' (with the same caveat as in Remark 4.4)

$$P_\sigma^{(r)} = (-1)^r \int d^3\vec{x} \left[-\frac{1}{4} A_{\mu_1\ldots\mu_r}^{(r)}(x, e) \overset{\leftrightarrow}{\partial_0}\overset{\leftrightarrow}{\partial_\sigma} A^{(r)\mu_1\ldots\mu_r}(x, e') \right], \tag{4.7}$$

$$M_{\sigma\tau}^{(r)} = (-1)^r \int d^3\vec{x} \tag{4.8}$$

$$\left[-\frac{1}{4} x_\sigma A_{\mu\times}^{(r)}(x, e) \overset{\leftrightarrow}{\partial_0}\overset{\leftrightarrow}{\partial_\tau} A^{(r)\mu\times}(x, e') - \frac{r}{2} A_{\sigma\times}^{(r)}(x, e) \overset{\leftrightarrow}{\partial_0} A_\tau^{(r)\times}(x, e') \right] - (\sigma \leftrightarrow \tau).$$

The notation in Eq. (4.6) asserts that the generators $P_\sigma^{(r)}$ and $M_{\sigma\tau}^{(r)}$ commute with $A^{(r')}$ and consequently with $P_\sigma^{(r')}$ and $M_{\sigma\tau}^{(r')}$ $(r' \neq r)$, and hence generate the infinitesimal Poincaré transformations of $A^{(r)}$ according to Eq. (1.11).

Proof: We insert the expansion Eq. (3.13) in terms of $E_{\mu\mu}(e, e)^k a_{\mu\ldots\mu}^{(r-2k)}(e)$ into $A^{(r)}(e)$ in Eq. (4.7). We partially integrate the derivatives contained in the factors $E(e, e)$ (cf. Eq. (1.27)). When they hit $A^{(r)}(e')$, they vanish because $A^{(r)}$ are conserved at $m = 0$. The remaining contribution $\eta_{\mu\mu}$ of $E_{\mu\mu}$ is directly contracted with $A^{(r)}(e')$, and vanishes because $A^{(r)}$ are traceless at $m = 0$. Thus, only the leading term $A^{(r)}(e) = \alpha_0^r a^{(r)}(e) + \ldots$ contributes. Now, we expand $A^{(r)}(e')$ and partially integrate the derivatives contained in $E^{\mu\mu}(e', e')$ onto $a^{(r)}(e)$, where they vanish because $a^{(r)}$ are conserved at $m = 0$. But $a^{(r)}$ are not traceless, and $E(e', e')^k$ acts like $\eta^k a^{(r)}(e) = (-1)^k a^{(r-2k)}(e)$ by Eq. (3.8). It remains to add up the coefficients

$$\sum_{2k \leq s-r} (\alpha_0^{r+2k})^2 c_k^{r+2k} = (-1)^r \binom{s}{r}.$$

(We were not able to establish this identity for finite sums of rational numbers in closed form, but have verified them numerically until $s = 100$.)

Again, the case of the Lorentz generators requires a more involved combinatorics. Let us consider the first step: the partial integration of derivatives $\partial_\mu a'(e)$ contained in $E(e, e)^k a^{(r-2k)}(e)$ against $A^{(r)}(e')$. By Lemma B.1(i), the partial integrations within the first term in Eq. (4.8) give undesired non-vanishing contributions of the form

$$-\frac{1}{4} \cdot 2 \cdot \frac{r(r-1)}{2} \int d^3\vec{x} \left[a'(x, e) \overset{\leftrightarrow}{\partial_0}\overset{\leftrightarrow}{\partial_\sigma} A_\tau^{(r)\cdots}(x, e') \right] - (\sigma \leftrightarrow \tau),$$

where the factor $2 \cdot \frac{r(r-1)}{2}$ counts the assignments of the other contracted indices. On the other hand, when the index σ is attached to a factor E in the second term of Eq. (4.8), it gives the undesired term

$$-\frac{r}{2} \cdot (r-1) \int d^3\vec{x} \left[\partial_\sigma a'(x, e) \overset{\leftrightarrow}{\partial_0} A_\tau^{(r)\cdots}(x, e') \right] - (\sigma \leftrightarrow \tau)$$

with another counting factor. These terms cancel each other by virtue of Eq. (4.5). In the second step: the partial integration of derivatives $\partial^\mu a''(e')$ contained in $E(e', e')$

within $A^{(r)}(e')$ against $a^{(r)}(e)$, the cancellations occur with the same pattern. This shows the equality of the generators in Prop. 4.5 and Prop. 4.3.

The final statements are immediate: $A^{(r)}$ mutually commute, because their mixed 2-point functions vanish. Hence the "r" generators commute with the "r'" fields and generators. Then the "r" generators act on the "r" fields like the full generators P_σ and $M_{\sigma\tau}$, hence they implement the correct Poincaré transformations. $\qquad\square$

One can now read back conserved and symmetric massless string-localized stress-energy tensors $T^{(r)}_{\sigma\rho}(r)$ from Eq. (4.7), Eq. (4.8).

Proposition 4.6 *The generators Eq. (4.7) and Eq. (4.8) can be obtained from the string-localized massless stress-energy tensors for every $r \geq 1$:*

$$
T^{(r)}_{\rho\sigma}(x, e, e') \;\; := \;\; (-1)^r \Big[- \frac{1}{4} A^{(r)}_{\mu\times}(x, e) \overset{\leftrightarrow}{\partial_\rho} \overset{\leftrightarrow}{\partial_\sigma} A^{(r)\mu\times}(x, e') \tag{4.9}
$$
$$
- \frac{r}{4} \partial^\mu \Big(A^{(r)}_{\rho\times}(x, e) \overset{\leftrightarrow}{\partial_\sigma} A^{(r)\times}_\mu(x, e') \;\; \begin{matrix} +(e \leftrightarrow e') \\ +(\rho \leftrightarrow \sigma) \end{matrix} \Big) \Big].
$$

Proof: The argument is the same as with Cor. 4.2. $\qquad\square$

The stress-energy tensors $T^{(r)}$ do not depend on the spin $s \geq r$ of the reduced stress-energy tensor Eq. (4.3) from which they were extracted at $m = 0$. By Eq. (3.16), they can also be expressed in terms of the corresponding field strengths $F^{(r)}$, that are directly obtained from the massless helicity $h = \pm r$ Wigner representations [W95].

As compared to other proposals [FV87, V00, V04, L08, BBS12] evoking an interplay of infinitely many spins, M-theory, and non-commutative geometry, the string-localized stress-energy tensors of Prop. 4.6 for every pair of helicities $h = \pm s$ are perhaps the most conservative way around the Weinberg-Witten theorem. They are even "less non-local" than the examples with unpaired helicities proposed in [L84]. We are presently investigating how they may be used to (semiclassically) couple massless higher spin matter to gravity.

Remark 4.7 *For charge operators like*

$$
Q = (-1)^s i \int_{x^0 = t} d^3\vec{x}\, A^{\mathrm{P}*}_{\mu_1\ldots\mu_s}(x) \overset{\leftrightarrow}{\partial_0} A^{\mathrm{P}\mu_1\ldots\mu_s}(x)
$$

for complex potentials, one can proceed in complete analogy as with the momentum operators, and obtains string-localized massless conserved currents

$$
J^{(r)}_\rho(x, e, e') = \frac{i}{2} A^{(r)*}_{\mu_1\ldots\mu_r}(x, e) \overset{\leftrightarrow}{\partial_\rho} A^{(r)\mu_1\ldots\mu_r}(x, e'). \tag{4.10}
$$

The string-localized densities $T^{(r)}_{\rho\sigma}$ and $J^{(r)}_\rho$ may be averaged over the directions of their strings (cf. Remark 4.4) with test functions of arbitrarily small support. Hence, they can be localized in arbitrarily narrow spacelike cones.

5 Conclusion

We have introduced string-localized potentials for massive particles of integer spin s, that admit a smooth massless limit to potentials with individual helicities $h = \pm r$, $r \leq s$. We have elobarated several remarkable properties of the massless limit, including an inverse prescription how to pass from the massless to the massive potentials via a manifestly positive deformation of the 2-point function.

As a byproduct, we could construct string-localized currents and stress-energy tensors for massless fields of any helicity, that evade the Weinberg-Witten theorem in a very conservative way.

Our results also allow to approximate string-localized fields in the massless infinite-spin Wigner representations [MSY06] by the massive scalar escort fields $A^{(0)}$ of spin $s \to \infty$, $m^2 s(s+1) = \kappa^2 = \mathrm{const.}$ (Work in progress [MRS].)

The feature of string-localization arises just by multiplication operators in momentum space (of a special form), acting on the intertwiner functions that define covariant fields in terms of creation and annihilation operators of the (m, s) Wigner representations.

In particular, string-localization of the fields does not change the nature of the particles that they describe, nor does it relax any of the fundamental principles of relativistic quantum field theory. We emphasize that we regard fields (associated with a given particle) mainly as a device to formulate interaction Lagrangeans. String-localized interactions are admissible whenever their string-dependence is a total derivative. In that case, string-localized fields have the primary benefit of a better UV behaviour than point-localized fields associated with the same particles. They therefore admit the formulation of interactions that are otherwise only possible at the expense of introducing states of negative norm and compensating ghost fields.

The renormalized perturbation theory of interactions mediated by string-localized fields is presently investigated. It bears formal analogies with BRST renormalization, but is more economic (by avoiding auxiliary unphysical degrees of freedom), and much closer to the fundamental principles of relativistic quantum field theory.

The necessity of using string-localized quantities to connect the vacuum state with scattering states in theories with short-range interactions was exhibited much earlier by Buchholz and Fredenhagen [BF82] who investigated, in the framework of algebraic quantum field theory, the localization properties of particle states in charged sectors relative to the vacuum. Their conclusion was that, depending on the given model, the best possible localization is in an arbitrarily narrow spacelike cone, and that in the presence of a mass gap it cannot be worse in general.

The emerging renormalized perturbation theory using string-localized fields [S15, S16, M17, MS17, GMV17] is the practical realization of this insight.

A Stress-energy tensors for higher spin fields

[FR04] and [GM92] give excellent discussions of how to properly define stress-energy tensors. We focus only on a few facts.

It is well-known from the example of the free Maxwell field, that the canonical definition

$$T_{\rho\sigma} = \sum \frac{\partial L}{\partial \partial^\rho \phi} \partial_\sigma \phi - \eta_{\rho\sigma} L[\phi],$$

where the sum extends over all independent fields, may not give rise to a symmetric stress-energy tensor. Consequently its Lorentz generators defined by Eq. (1.7) are not time-independent, even if L is Lorentz invariant. In the Maxwell case, the canonical stress-energy tensor is also not gauge invariant, and both defects can be cured "in one stroke" by adding a trivially conserved term $\partial_\kappa(F^{\mu\kappa} A_\nu)$. There are many prescriptions (e.g., [B39, R40]) to obtain symmetric stress-energy tensors in the general case.

The modern approach uses the Hilbert stress-energy tensor that is defined by varying a generally covariant version of the action [R40, HE73, FR04, GM92] with respect to the metric, and then putting $g_{\mu\nu} = \eta_{\mu\nu}$:

$$T_{\rho\sigma}(x) := 2 \frac{\delta S}{\delta g^{\rho\sigma}(x)}\Big|_{g=\eta}.$$

The Hilbert tensor is always symmetric and conserved. In both approaches, one first needs a Lagrangean whose Euler-Lagrange equations are the equation of motion.

This question has been addressed by Fierz and Pauli [FP39] and Fronsdal [F78] for free massive spin s fields; they used auxiliary fields to ensure the vanishing of the divergence. When varying with respect to the metric, one may omit terms involving the divergence and the auxiliary fields that vanish by virtue of the equations of motion. For $s = 2$, this gives

$$L = \frac{1}{4} F^{\mathrm{P}}_{[\mu\nu]\kappa} F^{\mathrm{P}[\mu\nu]\kappa} - \frac{m^2}{2} A^{\mathrm{P}}_{\nu\kappa} A^{\mathrm{P}\nu\kappa}. \tag{A.1}$$

The generally covariant action is

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{4} g^{\mu\mu'} g^{\nu\nu'} F^{\mathrm{P}}_{[\mu\nu]\kappa} F^{\mathrm{P}}_{[\mu'\nu']\kappa'} - \frac{m^2}{2} g^{\nu\nu'} A^{\mathrm{P}}_{\nu\kappa} A^{\mathrm{P}}_{\nu'\kappa'} \right) g^{\kappa\kappa'}$$

where $F^{\mathrm{P}}_{[\mu\nu]\kappa} := D_\mu A^{\mathrm{P}}_{\nu\kappa} - D_\nu A^{\mathrm{P}}_{\mu\kappa} = \partial_\mu A^{\mathrm{P}}_{\nu\kappa} - \partial_\nu A^{\mathrm{P}}_{\mu\kappa} - (\Gamma^\lambda_{\mu\kappa} A^{\mathrm{P}}_{\nu\lambda} - \Gamma^\lambda_{\nu\kappa} A^{\mathrm{P}}_{\mu\lambda})$. The variation of $g^{\mu\mu'}$ and $g^{\nu\nu'}$ and the factor $\sqrt{-g}$ in S give the stress-energy tensor

$$T^{(\mathrm{Fierz})}_{\rho\sigma} = \eta^{\lambda\lambda'} F^{\mathrm{P}}_{[\rho\lambda]}{}^\mu F^{\mathrm{P}}_{[\sigma\lambda']\mu} - m^2 A^{\mathrm{P}}_\rho{}^\mu A^{\mathrm{P}}_{\sigma\mu} - \eta_{\rho\sigma} L. \tag{A.2}$$

This tensor was first considered by Fierz [F39]. However, unlike the case of antisymmetrized indices, the Christoffel symbols for the indices κ, κ' do not drop out; and the contraction by $g^{\kappa\kappa'}$ carries another dependence on the metric, so that we have

Proposition A.1 *The Hilbert stress-energy tensor is* $T_{\rho\sigma} = T^{(\mathrm{Fierz})}_{\rho\sigma} + \Delta T_{\rho\sigma}$ *with*

$$\Delta T_{\rho\sigma} = -\frac{1}{2} \partial^\mu \left[A^{\mathrm{P}\lambda}_\rho F^{\mathrm{P}}_{[\sigma\lambda]\mu} + A^{\mathrm{P}\lambda}_\sigma F^{\mathrm{P}}_{[\rho\lambda]\mu} + A^{\mathrm{P}\lambda}_\mu \left(F^{\mathrm{P}}_{[\lambda\rho]\sigma} + F^{\mathrm{P}}_{[\lambda\sigma]\rho} \right) \right]. \tag{A.3}$$

Fierz [F39] has shown that $T^{(\text{Fierz})}$ produces the Hamiltonian

$$P_0 = -\frac{1}{4} \int d^3\vec{x}\, A^{\text{P}\,\mu\nu}\, \overset{\leftrightarrow}{\partial_0}\overset{\leftrightarrow}{\partial_0}\, A^{\text{P}}_{\mu\nu},$$

and one easily verifies that the commutator is $i[P_0, A^{\text{P}}_{\mu\nu}] = \partial_0 A^{\text{P}}_{\mu\nu}.$[9] The same is true for all P_σ. Fierz has actually given a hierarchy of s linearly independent stress-energy tensors $T^{(q)}$ for the free massive spin s field. They involve an increasing number $q = 1, \ldots, s$ of derivatives of the potential, and overall factors $(-2m^2)^{-(q-1)}$. They all produce the same generators P_σ that implement the correct infinitesimal translations $i[P_\sigma, A^{\text{P}}_{\mu_1\ldots\mu_s}] = \partial_\sigma A^{\text{P}}_{\mu_1\ldots\mu_s}$.

The Fierz stress-energy tensors also all produce the same generators $M^{(\text{Fierz})}_{\sigma\tau}$, but the latter do not implement the correct infinitesimal Lorentz transformations! E.g., for $s = 2$, one finds $i[M^{(\text{Fierz})}_{0i}, A^{\text{P}}_{00}] = (x_0\partial_i - x_i\partial_0)A^{\text{P}}_{00} + A_{i0} - m^{-2}\partial_0 F^{\text{P}}_{[0i]0}$ rather than the correct $i[M_{0i}, A^{\text{P}}_{00}] = (x_0\partial_i - x_i\partial_0)A^{\text{P}}_{00} + 2A_{i0}$.

This defect is precisely cured by the correction $\Delta T_{\rho\sigma}$ in the Hilbert stress-energy tensor, given in Eq. (A.3). For general spin s, one computes the Hilbert stress-energy tensor $T = T^{(\text{Fierz})} + \Delta T$ where $T^{(\text{Fierz})}$ is exactly as in Eq. (A.2) with an overall sign $(-1)^s$ (due to our sign convention of the metric) and the contracted index μ replaced by $s - 1$ contracted indices $\mu_2\ldots\mu_s$, and ΔT is exactly as in Eq. (A.3) with the same overall sign factor $(-1)^s$, with an additional factor $s - 1$ (due to its origin from $s - 1$ additional contractions and Christoffel symbols for $s - 1$ indices), and every field equipped with $s - 2$ additional contracted indices $\mu_3\ldots\mu_s$. This finally gives the Lorentz generators Eq. (4.1) and Eq. (4.2), from which we have read back the reduced stress-energy tensor Eq. (4.3). (The two parts in Eq. (4.3) do not separately correspond to $T^{(\text{Fierz})}$ and ΔT.)

(We are not aware of a general argument that the Hilbert tensor always, also in the presence of constraints, yields the correct generators. This issue is not explicitly mentioned in the literature, including the reviews [GM92, FR04].)

Because T^{red} has the same generators as $T = T^{(\text{Fierz})} + \Delta T$, their densities differ by spatial derivatives ("irrelevant terms"). One can verify this by hand (but we spare the reader this cumbersome exercise), by first rewriting $T^{(\text{Fierz})}$ with the help of the identities

$$\eta^{\lambda\lambda'} F^{\text{P}}_{[\rho\lambda]\times} F^{\text{P}}_{[\sigma\lambda']}{}^{\times} - m^2 A^{\text{P}}_{\rho\times} A^{\text{P}}_{\sigma} = F^{\text{P}}_{[\rho\mu]\times}\partial_\sigma A^{\text{P}\mu\times} - \partial^\mu\big(F^{\text{P}}_{[\rho\mu]\times} A^{\text{P}\times}_\sigma\big)$$

and

$$-\frac{1}{4}F^{\text{P}[\mu\nu]\times} F^{\text{P}}_{[\mu\nu]\times} + \frac{m^2}{2}A^{\text{P}\nu\times} A^{\text{P}}_{\nu\times} = -\frac{1}{2}\,\partial^\mu\Big[A^{\text{P}\nu\times} F^{\text{P}}_{[\mu\nu]\times}\Big],$$

then adding ΔT, and finally showing that the difference from Eq. (4.3) does not contribute to the generators according to Lemma B.1(i) and (ii) (where Θ are various contributions to the stress-energy tensor).

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[9] see Prop. 4.1.

B   A useful lemma

The following (rather trivial, but very useful) lemma deals with a covariant form of partial integration of four-derivatives in spatial (fixed-time) integrals.

Lemma B.1 *With a tensor $\Theta_{\rho\sigma}$ we associate the "charges" (not necessarily independent of t) $\Pi_\sigma := \int_{x_0=t} d^3\vec{x}\,\Theta_{0\sigma}$ and $\Omega_{\sigma\tau} := \int_{x_0=t} d^3\vec{x}\,(x_\sigma\,\Theta_{0\tau} - x_\tau\,\Theta_{0\sigma})$. We assume all fields or functions to have sufficiently rapid decay in spatial directions, so that boundary terms do not matter.*

(i) If $\Theta_{\rho\sigma}$ is of the form

$$\Theta_{\rho\sigma} = \partial^\mu\big(Y_\mu\,\overset{\leftrightarrow}{\partial}_\rho\,Z_\sigma\big)$$

(or a sum of terms[10] of the same structure), where Y and Z are solutions to the Klein-Gordon equation, then $\partial^\rho\Theta_{\rho\sigma} = 0$ trivially. The charges $\Pi_\sigma := \int_{x_0=t} d^3\vec{x}\,\Theta_{0\sigma} = 0$ vanish, and the charges $\Omega_{\sigma\tau}$ are

$$\Omega_{\sigma\tau} = \int_{x_0=t} d^3\vec{x}\,\big(Y_\tau\,\overset{\leftrightarrow}{\partial}_0\,Z_\sigma - Y_0\,\overset{\leftrightarrow}{\partial}_\tau\,Z_\sigma\big) - (\sigma \leftrightarrow \tau).$$

(ii) The same is true with

$$\Theta_{\rho\sigma} = \partial^\mu X_{[\mu\rho]\sigma},$$

where $[\mu\rho]$ stands for an anti-symmetric index pair, and

$$\Omega_{\sigma\tau} = \int_{x_0=t} d^3\vec{x}\,\big(X_{[\tau 0]\sigma} - X_{[\sigma 0]\tau}\big).$$

(iii) In order for $\Omega_{\sigma\tau}$ to vanish, the respective integrands have to be spatial derivatives.

Proof: $\Theta_{0\sigma} = \partial^\mu X_{[\mu 0]\sigma}$ in (ii) is a spatial derivative, because the term $\mu = 0$ is absent by anti-symmetry. The claim follows by partial integration. (i) is a special case of (ii) by writing $\Theta_{\rho\sigma} = \partial^\mu\big(Y_\mu\,\overset{\leftrightarrow}{\partial}_\rho\,Z_\sigma - Y_\rho\,\overset{\leftrightarrow}{\partial}_\mu\,Z_\sigma\big)$. The statement (iii) is trivial. $\square$

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[10]A term $Y\,\overset{\leftrightarrow}{\partial}_\rho\,Z_{\mu\sigma} = Y\delta_\mu^\lambda\,\overset{\leftrightarrow}{\partial}_\rho\,Z_{\lambda\sigma}$ can be written as a sum over terms of this form.

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