From worldline to quantum superconformal mechanics with/without oscillatorial terms: $D(2,1;\alpha)$ and $sl(2|1)$ models

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Abstract

In this paper we quantize superconformal σ-models defined by worldline supermultiplets. Two types of superconformal mechanics, with and without a DFF term, are considered. Without a DFF term (Calogero potential only) the supersymmetry is unbroken.

The models with a DFF term correspond to deformed (if the Calogero potential is present) or undeformed oscillators. For these (un)deformed oscillators the classical invariant superconformal algebra acts as a spectrum-generating algebra of the quantum theory.

Besides the $osp(1|2)$ examples, we explicitly quantize the superconformally-invariant worldline σ-models defined by the $\mathcal{N}=4$ $(1,4,3)$ supermultiplet (with $D(2,1;\alpha)$ invariance, for $\alpha \neq 0,1$) and by the $\mathcal{N}=2$ $(2,2,0)$ supermultiplet (with two-dimensional target and $sl(2|1)$ invariance). The parameter α is the scaling dimension of the $(1,4,3)$ supermultiplet and, in the DFF case, has a direct interpretation as a vacuum energy. In the DFF case, for the $sl(2|1)$ models, the scaling dimension λ is quantized (either $\lambda = \frac{1}{2} + \mathbb{Z}$ or $\lambda = \mathbb{Z}$). The ordinary two-dimensional oscillator is recovered from $\lambda = -\frac{1}{2}$. The spectrum of the theory is decomposed into an infinite set of lowest weight representations of $sl(2|1)$. Surprisingly, extra fermionic raising operators, not belonging to $sl(2|1)$, allow to construct the whole spectrum from a single (for $\lambda = \frac{1}{2} + \mathbb{Z}$) bosonic vacuum.

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1 Introduction

In this paper we quantize superconformal σ-models defined by worldline supermultiplets. We consider two types of superconformal mechanics, parabolic or trigonometric [1], namely in the absence or, respectively, in the presence of an oscillatorial DFF term [2].

In the absence of a DFF term the systems under consideration possess only a Calogero potential [3]; they are supersymmetric and with a continuous spectrum. In the presence of a DFF term they correspond to deformed (if the Calogero potential is present) or undeformed oscillators with a discrete, bounded from below, spectrum. For these (un)deformed oscillators the classical invariant superconformal algebra acts as a spectrum-generating algebra of the quantum theory.

We illustrate at first our method with two $osp(1|2)$-invariant examples, the ordinary one-dimensional harmonic oscillator being recovered in the trigonometric case. Later we explicitly quantize the superconformally-invariant worldine σ-models defined by
i) the $\mathcal{N}=4$ $(1,4,3)$ supermultiplet with scaling dimension $\alpha \neq 0, -1$ (these models are classically invariant under the exceptional $D(2,1;\alpha)$ Lie superalgebra) and
ii) the $\mathcal{N}=2$ $(2,2,0)$ supermultiplet of scaling dimension λ (these models present a two-dimensional target and classical $sl(2|1)$-invariance).

For the $(1,4,3)$ supermultiplet, at the special $\alpha = -\frac{1}{2}$ value, the Calogero potential terms are vanishing. For this value the invariant superalgebra is $D(2,1;-\frac{1}{2}) = D(2,1) \approx osp(4|2)$. An interesting result, in the $(1,4,3)$ trigonometric case, consists in the direct and simple interpretation of α as a vacuum energy (if α is regarded as an external control parameter, it determines the Casimir energy of the system).

For the $sl(2|1)$ models the scaling dimension λ is quantized (either $\lambda = \frac{1}{2} + \mathbb{Z}$ or $\lambda = \mathbb{Z}$). In the trigonometric case the ordinary two-dimensional oscillator (without Calogero potential terms) is recovered from the special $\lambda = -\frac{1}{2}$ value after a superselection of the spectrum, defined by a projection operator, is imposed. The spectrum of the theory turns out to be decomposed into an infinite set of lowest weight representations of $sl(2|1)$. By construction, the role of $sl(2|1)$ as a spectrum-generating algebra is expected. What is quite unexpected and surprising is the further result that extra fermionic raising operators, not belonging to the $sl(2|1)$ superalgebra, allow to construct the whole spectrum from the single $\lambda = \frac{1}{2} + \mathbb{Z}$ bosonic vacuum (in Appendix **A** this action is visualized in diagrams).

Models of superconformal mechanics have been investigated in [4]–[12] (see, e.g., the review [13] and references therein). For superconformal actions with oscillator potentials see [14, 15, 1]. (Super)conformal mechanics is currently a very active area of research; among the motivations for this interest one can mention the AdS_2/CFT_1 correspondence [16, 17], or the possibility to apply it to test particles moving in the proximity of the horizon of certain black holes, see [11].

$\mathcal{N}=4$ superconformal models based on the exceptional (see [18]) Lie superalgebra $D(2,1;\alpha)$ were investigated in [19]–[26]. The models considered in those works, mostly classical, are supersymmetric; for that reason they do not allow the presence of the oscillatorial DFF terms (in Appendix **C** we comment about the "soft" supersymmetry property of the oscillatorial models). The recognition in [28] that conformal mechanics could allow new potentials, permitted the introduction in [1] of the trigonometric (read, oscillatorial) classical $D(2,1;\alpha)$ models.

The scheme of the paper is the following.

Sections **2**, **3**, **4** are propaedeutic. In Section **2** we discuss the change of coordinates from linear to non-linear realizations of the superconformal algebras (the "constant kinetic basis") which allows us to present the worldline superconformal σ-models in the Hamiltonian framework. A detailed description of the passage from classical Lagrangians to Hamiltonians is given in Sec-

tion **3**. In Section **4** the quantization procedure and the construction of the Noether charges is explained for two examples, the parabolic and trigonometric $osp(1|2)$-invariant σ-models. Section **5** contains the main results for the quantization of the parabolic (i.e. both superconformal and supersymmetric) quantum models with $D(2,1;\alpha)$-invariance, based on the $\mathcal{N}=4$ worldline supermultiplet $(1,4,3)$, and $sl(2,1)$-invariance, based on the $\mathcal{N}=(2,2,0)$ worldline supermultiplet. In Section **6** the main results of their quantum trigonometric versions are derived. These systems contain DFF terms and are "softly supersymmetric". They correspond to (un)deformed oscillators. The main results are the derivation of the vacuum energy in terms of the α scaling dimension for the $(1,4,3)$ supermultiplet and the derivation of the spectrum-generating superalgebra for the (un)deformed two-dimensional oscillator with quantized scaling dimension λ. In Appendix **A** diagrams are presented illustrating the decomposition of the two-dimensional oscillators in terms of the $sl(2|1)$ lowest weight representations, interconnected by the puzzling extra fermionic raising and lowering operators introduced in Section **6**. For completeness in Appendix **B** the classical version of the trigonometric $\mathcal{N}=2$ $(2,2,0)$ superconformal σ-model is presented. Finally, in Appendix **C** we discuss the "soft supersymmetry" of the (un)deformed oscillators and the role, for these theories, of the spectrum generating superalgebras. In the Conclusions we present the open questions raised by our analysis.

2 Worldline (super)conformal σ-models in constant kinetic basis

A convenient approach, in constructing one-dimensional superconformal σ-models, consists in starting from a linear D-module representation of the superconformal algebra. Once such a representation is known, the Lagrangian defining the superconformally invariant action can be systematically constructed by applying fermionic generators to a prepotential function which depends only on the propagating bosons. The requirement of superconformal invariance, imposed as a constraint, determines the specific form of the prepotential. This method (and its applications) has been discussed in [1].

The kinetic term $\Phi(\vec{x})\frac{1}{2}\delta_{ij}(\dot{x}_i\dot{x}_j+\ldots)$ of the derived Lagrangian is an ordinary constant kinetic term multiplied by a conformal factor $\Phi(\vec{x})$ which is a function of the propagating bosons. In order to apply the standard methods of quantization we need to reabsorb the conformal factor. One way to do this consists in introducing a new set of fields. In the new basis of fields the kinetic term is expressed as a constant coefficient (hence the name "constant kinetic basis" given in [1]); the superalgebra, on the other hand, is realized non-linearly.

In [1] the procedure of changing the basis (from the "linear" to the "constant kinetic" basis) was sketched for certain D-module representations acting on supermultiplets consisting of a single propagating boson. We discuss it here in a more general framework.

Let us consider a D-module irrep of a $\mathcal{N}$-extended superconformal algebra (for our purposes $\mathcal{N}=1,2,4,8$) acting on a $(k,\mathcal{N},\mathcal{N}-k)$ supermultiplet [29, 30, 31, 32] (namely, k propagating bosons, $\mathcal{N}$ fermions and $\mathcal{N}-k$ bosonic auxiliary fields). In the linear basis the propagating bosons are labeled as x_1, ..., x_k, the fermions as ψ_1, ..., $\psi_{\mathcal{N}}$ and the auxiliary bosons as b_1, ..., $b_{\mathcal{N}-k}$. The kinetic term in the Lagrangian is given by

$$\frac{1}{2}r^{-\frac{1+2\lambda}{\lambda}}(\dot{x}_m\dot{x}_m+i\omega\psi_\beta\dot{\psi}_\beta-\omega^2 b_n b_n). \tag{1}$$

In the above equation the convention over repeated indices is used. The constant ω is dimensionless (and can be set equal to unity) in the parabolic case, while it is dimensional, see [1], in the hyperbolic/trigonometric case. The function r is $r=(x_m x_m)^{\frac{1}{2}}$ and the parameter λ is the scaling dimension of the supermultiplet. At $\lambda=-\frac{1}{2}$ the kinetic term is constant. For the

remaining $\lambda \neq -\frac{1}{2}$ values a change to a constant kinetic basis is required in order to present a kinetic term with constant coefficients. Let us denote the propagating bosons in the constant kinetic basis as y_1, ..., y_k, the fermions as χ_1, ..., $\chi_{\mathcal{N}}$ and the auxiliary bosons as a_1, ..., $a_{\mathcal{N}-k}$. The transformations passing from the "linear" to the "constant kinetic" basis are given by:
i) for the $(1, \mathcal{N}, \mathcal{N}-1)$ supermultiplets we have

$$y = -2\lambda x^{-\frac{1}{2\lambda}}, \qquad \chi_\beta = x^{-\frac{1+2\lambda}{2\lambda}}\psi_\beta, \qquad a_n = x^{-\frac{1+2\lambda}{2\lambda}} b_n; \tag{2}$$

in terms of the new fields equation (1) is expressed as

$$\frac{1}{2}(\dot{y}\dot{y} + i\omega\chi_\beta\dot{\chi}_\beta - \omega^2 a_n a_n); \tag{3}$$

ii) when $\mathcal{N} \geq 2$, for the $(2, \mathcal{N}, \mathcal{N}-2)$ supermultiplets it is convenient to use a complex notation for the propagating bosons and set

$$\begin{aligned} y &= -2\lambda(x_1 + ix_2)^{-\frac{1}{2\lambda}}, & y^* &= -2\lambda(x_1 - ix_2)^{-\frac{1}{2\lambda}}, \\ \chi_\beta &= r^{-\frac{1+2\lambda}{2\lambda}}\psi_\beta, & a_n &= r^{-\frac{1+2\lambda}{2\lambda}} b_n, \end{aligned} \tag{4}$$

so that the kinetic term can be expressed as

$$\frac{1}{2}(\dot{y}\dot{y}^* + i\omega\chi_\beta\dot{\chi}_\beta - \omega^2 a_n a_n); \tag{5}$$

iii) when $\mathcal{N} = 4, 8$ it is possible to construct a constant kinetic basis for any $(k, \mathcal{N}, \mathcal{N}-k)$ supermultiplet at the specific $\lambda = 1/2$ value of the scaling dimension via the transformations

$$y_m = \frac{x_m}{r^2}, \qquad \chi_\beta = \frac{\psi_\beta}{r^2}, \qquad a_n = \frac{b_n}{r^2}, \tag{6}$$

leading to the kinetic term

$$\frac{1}{2}(\dot{y}_m\dot{y}_m + i\omega\chi_\beta\dot{\chi}_\beta - \omega^2 a_n a_n). \tag{7}$$

For $\mathcal{N} = 4$ and $k \neq 2$, irreps of the exceptional superalgebras $D(2,1;\alpha)$ are recovered, see [25, 26, 1], from the $(k, 4, 4-k)$ supermultiplets according to the relation

$$\alpha = (2-k)\lambda. \tag{8}$$

At the special $\lambda = \frac{1}{2}$ value the associated superalgebra is $A(1,1)$ for the $(4,4,0)$ supermultiplet and $D(2,1)$ for the $(3,4,1)$ supermultiplet.

For $\mathcal{N} = 8$ and $k \neq 4$, irreps of superconformal algebras are recovered for each supermultiplet $(k, 8, 8-k)$ at the critical values of the scaling dimension given by

$$\lambda_k = \frac{1}{k-4}. \tag{9}$$

The special value $\lambda = \frac{1}{2}$ yields an irrep of $A(3,1)$ acting on the supermultiplet $(6,8,2)$. The reader is referred to [25, 26] for a detailed discussions on the criticality of the scaling dimension of the $\mathcal{N} = 4, 8$ superconformal algebras.

3 From Lagrangians to classical Hamiltonians: an application to the $osp(1|2)$-invariant σ-models

The quantization of the 1D superconformal σ-models follows the canonical procedure formalized by Dirac and based on the classical Hamiltonian formalism. Since these σ-models have fermionic degrees of freedom, the passage from the Lagrangian to the classical Hamiltonian formalism requires the use of Dirac brackets (see, e.g., [33]). The need for Dirac brackets becomes clear after inspecting equations (3), (5) and (7); it is due to the fact that the linear dependence on the fermionic velocities $\dot{\chi}_\beta$ forces us to extend the phase space of the system and treat the fermionic canonical momenta as constraints in this extended phase space. In Dirac's language these constraints are both *primary* (they hold even without using the equations of motion) and *second class* (namely, a constraint that has a non-vanishing Poisson brackets with at least one of the constraints).

This procedure, used throughout the paper, will be illustrated in detail for the simplest possibility given by the $osp(1|2)$-invariant σ-models (their two variants, *parabolic* and *hyperbolic/trigonometric*, see [1]). In the parabolic case the Hamiltonian is identified with a bosonic root of the superconformal algebra, while in the hyperbolic/trigonometric case it is associated with a Cartan element. The parabolic D-module reps describe systems which are supersymmetric, while the hyperbolic/trigonometric reps furnish only a *weak* version of supersymmetry, see the discussion in the Introduction. The hyperbolic and trigonometric models are interrelated via a Wick rotation of the dimensional parameter ω. The trigonometric case is here emphasized with respect to the hyperbolic one because it yields a bounded from below Hamiltonian.

In the rest of this Section we discuss in detail the Hamiltonian formulation of both parabolic and trigonometric $osp(1|2)$-invariant σ-models. The method, notations and conventions here presented are later applied to models with larger superconformal symmetry.

3.1 The $osp(1|2)$-invariant parabolic σ-model

In the constant kinetic basis the generators of the $osp(1|2)$ parabolic D-module rep read as

$$H = \begin{pmatrix} \partial_t & 0 \\ 0 & \partial_t \end{pmatrix}, \quad D = \begin{pmatrix} t\partial_t - \frac{1}{2} & 0 \\ 0 & t\partial_t \end{pmatrix}, \quad K = \begin{pmatrix} t^2\partial_t - t & 0 \\ 0 & t^2\partial_t \end{pmatrix},$$
$$Q = \begin{pmatrix} 0 & 1 \\ i\partial_t & 0 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} 0 & t \\ it\partial_t - i & 0 \end{pmatrix}. \tag{10}$$

The above generators act on the column vector supermultiplet $(y, \chi)^T$ possessing the scaling dimension $\lambda = -\frac{1}{2}$.

The bosonic generators H, D, K span the $sl(2)$ Lie subalgebra, while the fermionic generators Q, $\bar{Q}$ span the odd sector of $osp(1|2)$.

The associated $osp(1|2)$-invariant action is simply

$$\mathcal{S} = \int dt\mathcal{L} = \int dt \tfrac{1}{2}(\dot{y}^2 + i\chi\dot{\chi}). \tag{11}$$

Unlike the $\mathcal{N} \geq 2$ superconformal algebras discussed in the following, for $osp(1|2)$ the same action is recovered by starting from a generic D-module rep with scaling dimension $\lambda \neq -\frac{1}{2}$ and applying the (2) change of basis.

For a theory possessing bosons and fermions a conserved Noether charge is expressed, for a symmetry generator O, as

$$C_O = (\delta_O \phi_I) \frac{\partial \mathcal{L}}{\partial \dot{\phi}_I} - J_O, \tag{12}$$

where J_O stems from the variation $\delta_O \mathcal{L} = \frac{dJ_O}{dt}$; the sum over the repeated index I labeling the fields is understood. The given ordering of the right hand side of (12) is essential in dealing with Grassmann variables and derivatives.

For the case at hand the classical Noether charges are

$$C_H = \frac{\dot{y}^2}{2}, \quad C_D = \frac{t\dot{y}^2}{2} - \frac{y\dot{y}}{2}, \quad C_K = \frac{t^2\dot{y}^2}{2} - ty\dot{y} + \frac{y^2}{2}, \quad C_Q = \dot{y}\chi, \quad C_{\bar{Q}} = t\dot{y}\chi + y\chi. \tag{13}$$

The Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \tag{14}$$

lead to the equations of motion

$$\ddot{y} = 0, \qquad \dot{\chi} = 0. \tag{15}$$

The Grassmann variable in the classical $osp(1|2)$ model is a constant and plays essentially no physical role besides ensuring the $osp(1|2)$ invariance.

To introduce the Hamiltonian formalism we have to compute the conjugate momenta given by

$$p = \frac{\partial \mathcal{L}}{\partial \dot{y}} = \dot{y}, \quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\chi}} = -\frac{i\chi}{2}. \tag{16}$$

In the Hamiltonian framework the classical charges (13) are rewritten as

$$C_H = \frac{p^2}{2}, \quad C_D = \frac{tp^2}{2} - \frac{yp}{2}, \quad C_K = \frac{t^2p^2}{2} - typ + \frac{y^2}{2}, \quad C_Q = p\chi, \quad C_{\bar{Q}} = tp\chi + y\chi. \tag{17}$$

The last step requires defining the Dirac brackets. The second equation in (16) makes clear why Dirac brackets need to be introduced. The conjugate momentum π to the Grassmann variable χ is not an invertible function of the velocity $\dot{\chi}$. The second equation in (16) should therefore be viewed as a second class constraint on the phase space,

$$u = \pi + \frac{i\chi}{2}. \tag{18}$$

The super-Poisson bracket involving even or odd f, g functions is given by

$$\{f, g\}_P = \sum_I (-1)^{deg(f)\cdot deg(g)} \frac{\partial f}{\partial \phi_I}\frac{\partial g}{\partial \pi_I} - \frac{\partial f}{\partial \pi_I}\frac{\partial g}{\partial \phi_I}, \tag{19}$$

where the degree function deg is 0 if evaluated on bosons and 1 on fermions.

Denoting with u_i the set of all second class contraints, the Dirac bracket reads as

$$\{f, g\}_D = \{f, g\}_P - \sum_{k,l} \{f, u_k\}_P U^{-1}_{kl} \{u_l, g\}_P, \tag{20}$$

where $U_{kl} = \{u_k, u_l\}_P$ is a matrix constructed from the super-Poisson brackets of all second class constraints.

u entering (18) is a second class constraint, since it satisfies

$$\{u, u\}_P = -i.$$

A straightforward computation gives the non-vanishing Dirac brackets

$$\{y, p\}_D = 1, \qquad \{\chi, \chi\}_D = -i. \tag{21}$$

We can derive, with the use of the Dirac brackets, the equations of motion in the Hamiltonian formalism and compute (recovering $osp(1|2)$) the superalgebra satisfied by the (17) conserved charges.

In terms of Dirac brackets the Hamilton's equations are

$$\dot{\phi} = \frac{\partial \phi}{\partial t} + \{\phi, C_H\}_D. \tag{22}$$

For the case at hand we get

$$\dot{p} = 0, \qquad \dot{\chi} = 0, \tag{23}$$

which, together with the $p = \dot{y}$ position, allow to recover (15).

3.2 The $osp(1|2)$-invariant trigonometric σ-model

In the trigonometric case the passage from the Lagrangian to the Hamiltonian formalism follows the same steps as before. We therefore skip unnecessary comments.

In the constant kinetic basis the generators of the $osp(1|2)$ trigonometric D-module rep are

$$H = e^{i\omega t}\begin{pmatrix} \frac{1}{\omega}\partial_t - \frac{i}{2}x & 0 \\ 0 & \frac{1}{\omega}\partial_t \end{pmatrix}, \quad D = \begin{pmatrix} \frac{1}{\omega}\partial_t & 0 \\ 0 & \frac{1}{\omega}\partial_t \end{pmatrix}, \quad K = e^{-i\omega t}\begin{pmatrix} \frac{1}{\omega}\partial_t + \frac{i}{2}x & 0 \\ 0 & \frac{1}{\omega}\partial_t \end{pmatrix},$$
$$Q = e^{\frac{i\omega t}{2}}\begin{pmatrix} 0 & 1 \\ \frac{i}{\omega}\partial_t + \frac{1}{2}x & 0 \end{pmatrix}, \qquad \bar{Q} = e^{-\frac{i\omega t}{2}}\begin{pmatrix} 0 & 1 \\ \frac{i}{\omega}\partial_t - \frac{1}{2}x & 0 \end{pmatrix}. \tag{24}$$

The $osp(1|2)$-invariant action is

$$\mathcal{S} = \int dt \mathcal{L} = \int dt \frac{1}{2}(\dot{y}^2 + i\omega\chi\dot{\chi}) - \frac{\omega^2}{8}y^2. \tag{25}$$

The derived conserved Noether charges are

$$C_H = e^{i\omega t}(\frac{1}{2\omega}\dot{y}^2 - \frac{i}{2}y\dot{y} - \frac{\omega}{8}y^2), \quad C_D = \frac{1}{2\omega}\dot{y}^2 + \frac{\omega}{8}y^2, \quad C_K = e^{-i\omega t}(\frac{1}{2\omega}\dot{y}^2 + \frac{i}{2}y\dot{y} - \frac{\omega}{8}y^2),$$
$$C_Q = e^{\frac{i\omega}{2}t}(\dot{y}\chi - \frac{i\omega}{2}y\chi), \quad C_{\bar{Q}} = e^{-\frac{i\omega}{2}t}(\dot{y}\chi + \frac{i\omega}{2}y\chi). \tag{26}$$

The Euler-Lagrange equations of motion are

$$\ddot{y} = -\frac{\omega^2 y}{4}, \qquad \dot{\chi} = 0. \tag{27}$$

The conjugate momenta are given by

$$p = \frac{\partial \mathcal{L}}{\partial \dot{y}} = \dot{y}, \quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\chi}} = -\frac{i\omega\chi}{2}. \tag{28}$$

In the Hamiltonian formulation, the (26) conserved charges are

$$C_H = e^{i\omega t}(\frac{1}{2\omega}p^2 - \frac{i}{2}yp - \frac{\omega}{8}y^2), \quad C_D = \frac{1}{2\omega}p^2 + \frac{\omega}{8}y^2, \quad C_K = e^{-i\omega t}(\frac{1}{2\omega}p^2 + \frac{i}{2}yp - \frac{\omega}{8}y^2),$$
$$C_Q = e^{\frac{i\omega}{2}t}(p\chi - \frac{i\omega}{2}y\chi), \quad C_{\bar{Q}} = e^{-\frac{i\omega}{2}t}(p\chi + \frac{i\omega}{2}y\chi). \tag{29}$$

The second equation in (28) gives the constraint in phase space

$$u = \pi + \frac{i\omega\chi}{2}, \tag{30}$$

which allows to compute the Dirac brackets as before. The non-vanishing Dirac brackets are

$$\{y, p\}_D = 1, \qquad \{\chi, \chi\}_D = -\frac{i}{\omega}. \tag{31}$$

The Hamilton's equations of motion are now written as

$$\dot{\phi} = \omega\{\phi, C_D\}_D + \frac{\partial\phi}{\partial t}. \tag{32}$$

One should note that, while in the parabolic σ-model the charge C_H is the physical Hamiltonian and the symmetry operator H is the generator of the time translations, in the trigonometric σ-model the physical hamiltonian is given by ωC_D, the Cartan generator ωD being the generator of the time translations. One can readily check that equation (32) leads to

$$\dot{p} = -\frac{\omega^2 y}{4}, \qquad \dot{\chi} = 0, \tag{33}$$

which reproduce (27) by taking into account that $p = \dot{y}$.

4 The quantization. Quantum versus classical Noether charges and the $osp(1|2)$ models

The canonical quantization of the models presented in Section **3** is realized by substituting the Dirac Brackets by the appropriate (based on the superalgebra structure) (anti)commutators, that we will denote with the "$[.,.\}$" symbol:

$$\{A, B\}_D \quad \rightarrow \quad \frac{1}{i\hbar}[A, B\}. \tag{34}$$

By applying (34) to (21) and (31) we get, respectively, the parabolic and trigonometric $osp(1|2)$-invariant quantum superconformal models.

We point out that, since the observables must be Hermitian operators, the parabolic and trigonometric quantum models correspond to different real forms (read, conjugations) of the invariant superalgebra. We illustrate in detail this feature, which is also valid for $\mathcal{N} \geq 2$ invariant theories.

4.1 The parabolic $osp(1|2)$-invariant quantum σ-model

The non-vanishing (anti)commutators recovered from (21) are

$$[\hat{y},\hat{p}]=i\hbar,\qquad \{\hat{\chi},\hat{\chi}\}=\hbar. \tag{35}$$

In the position-space representation the above operators are given by

$$\hat{y}=y,\qquad \hat{p}=-i\hbar\partial_y,\qquad \hat{\chi}=\sqrt{\tfrac{\hbar}{2}}. \tag{36}$$

The last equation is particularly important because it tells us that the fermionic field χ, classically represented by a Grassmann variable, becomes a Clifford variable $\hat{\chi}$ in the quantum version. The choice in (36) of representing $\hat{\chi}$ as a real number (that we can think of as the generator of the $Cl(1,0)$ Clifford algebra), is not unique. An alternative choice, which respects the $\mathbb{Z}_2$-graded structure of the super-vector space acted upon by the operators $\hat{y},\hat{p},\hat{\chi}$, consists in picking $\hat{\chi}$ as the 2×2 matrix element corresponding to the antidiagonal Clifford algebra generator $Cl(2,1)$ with positive square. In this $\mathbb{Z}_2$-graded representation, the operators $\hat{y},\hat{p},\hat{\chi}$ are

$$\hat{y}=\begin{pmatrix} y & 0\\ 0 & y\end{pmatrix},\quad \hat{p}=\begin{pmatrix} -i\hbar\partial_y & 0\\ 0 & -i\hbar\partial_y\end{pmatrix},\quad \hat{\chi}=\sqrt{\tfrac{\hbar}{2}}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad N_f=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}, \tag{37}$$

while N_f is the Fermion number operator.

The possibility, offered by the $\mathbb{Z}_2$-graded structure, of doubling the vector space, will be used in the following in constructing the trigonometric and $\mathcal{N}=2,4$ quantum models.

The parabolic quantum $osp(1|2)$ superalgebra obtained by the (34) quantization of the classical counterpart, leads to

$$\begin{aligned}
&[\hat{H},\hat{D}]=i\hbar\hat{H},\quad [\hat{H},\hat{K}]=2i\hbar\hat{D},\quad [\hat{K},\hat{D}]=-i\hbar\hat{K}\\
&[\hat{H},\hat{\tilde{Q}}]=i\hbar\hat{Q},\quad [\hat{K},\hat{Q}]=-i\hbar\hat{\tilde{Q}},\quad [\hat{Q},\hat{D}]=\frac{i\hbar}{2}\hat{Q},\quad [\hat{\tilde{Q}},\hat{D}]=-\frac{i\hbar}{2}\hat{\tilde{Q}},\\
&\{\hat{Q},\hat{Q}\}=2\hbar\hat{H},\quad \{\hat{Q},\hat{\tilde{Q}}\}=2\hbar D,\quad \{\hat{\tilde{Q}},\hat{\tilde{Q}}\}=2\hbar K.
\end{aligned} \tag{38}$$

The remaining (anti)commutators are vanishing.

The above superalgebra is realized by the quantum charges

$$\begin{aligned}
&\hat{H}=\frac{1}{2}\hat{p}^2,\quad \hat{D}=\frac{t}{2}\hat{p}^2-\frac{1}{4}(\hat{y}\hat{p}+\hat{p}\hat{y}),\quad \hat{K}=\frac{t^2}{2}\hat{p}^2-\frac{t}{2}(\hat{y}\hat{p}+\hat{p}\hat{y})+\frac{1}{2}\hat{y}^2,\\
&\hat{Q}=\hat{\chi}\hat{p},\qquad \hat{\tilde{Q}}=t\hat{\chi}\hat{p}-\hat{y}\hat{\chi}.
\end{aligned} \tag{39}$$

They are, up to symmetrization, identical to the classical charges. This is a unique feature of the $\mathcal{N}=1$ $osp(1|2)$-invariant models. From $\mathcal{N}\geq 2$ the models explicitly depend on the scaling dimension λ. As a result, the quantum versions of these theories require corrections which are traced backed to the mapping of the classical Grassmann variables into quantum Clifford generators.

The Hamiltonian $\hat{H}$ in (39) corresponds to the one-dimensional free particle. The operators $\hat{H}$, $\hat{D}$ $\hat{K}$ close the $sl(2)$ bosonic symmetry algebra of the system. $\hat{H}$ and $\hat{Q}$ gives the $\mathcal{N}=1$ algebra of the Supersymmetric Quantum Mechanics. In terms of the (36) realization ($\hat{\chi}$ is a real number) the parabolic $osp(1|2)$-invariant model admits no fermionic degrees of freedom. This is no longer the case (fermions are present) if the model is expressed via the (37) realization.

In the parabolic model all charges entering (39) are observables. The superalgebra (38) can be re-expressed in terms of the canonical $osp(1|2)$ Cartan-Weyl basis $H, F^{\pm}, E^{\pm}$ (such that all the structure constants are real), see [18], through the identifications

$$\hat{H} = -E^{-}, \quad \hat{D} = iH, \quad \hat{K} = -E^{+}, \quad \hat{Q} = 2F^{-}, \quad \hat{\bar{Q}} = 2iF^{+}. \tag{40}$$

The computation of the $osp(1|2)$ structure constants in the new basis is immediate.

The superalgebra conjugation corresponding to (39) reads, in the Cartan-Weyl basis, as

$$(E^{\pm})^{\dagger} = E^{\pm}, \quad H^{\dagger} = -H, \quad (F^{\pm})^{\dagger} = \mp(F^{\pm}). \tag{41}$$

Concerning the dimensional analysis of the model we can set, without loss of generality, $[\partial_t] = 1$. If we set the Planck constant $\hbar$ and the action $\mathcal{S}$ to be dimensionless, we therefore get $[\hat{y}] = -\frac{1}{2}$, $[\hat{p}] = \frac{1}{2}$, $[\hat{\chi}] = [\mathcal{S}] = 0$.

4.2 The trigonometric $osp(1|2)$-invariant quantum σ-model

The quantization of the trigonometric model follows the same lines of the parabolic one. Without loss of generality we can set $\omega = 1$, reproducing the non-vanishing (anti)commutators (35) and the (36) and (37) position-space representations for the operators $\hat{y}, \hat{p}, \hat{\chi}$.
The quantum trigonometric generators, identical to the classical ones up to symmetrization, are

$$\begin{aligned}
&\hat{H} = e^{it}(\frac{1}{2}\hat{p}^2 - \frac{i}{4}(\hat{y}\hat{p} + \hat{p}\hat{y}) - \frac{1}{8}\hat{y}^2), \quad \hat{K} = e^{-it}(\frac{1}{2}\hat{p}^2 + \frac{i}{4}(\hat{y}\hat{p} + \hat{p}\hat{y}) - \frac{}{8}\hat{y}^2), \\
&\hat{D} = \frac{1}{2}\hat{p}^2 + \frac{1}{8}\hat{y}^2, \qquad \hat{Q} = e^{\frac{it}{2}}(\hat{\chi}\hat{p} - \frac{i}{2}\hat{\chi}\hat{y}), \qquad \hat{\bar{Q}} = e^{-\frac{it}{2}}(\hat{\chi}\hat{p} + \frac{i}{2}\hat{\chi}\hat{y}).
\end{aligned} \tag{42}$$

In the (42) realization, the $osp(1|2)$ non-vanishing brackets reads as

$$\begin{aligned}
&[\hat{H}, \hat{D}] = \hbar\hat{H}, \quad [\hat{H}, \hat{K}] = 2\hbar\hat{D}, \quad [\hat{K}, \hat{D}] = -\hbar\hat{K}, \\
&[\hat{H}, \hat{\bar{Q}}] = \hbar\hat{Q}, \quad [\hat{K}, \hat{Q}] = -\hbar\hat{\bar{Q}}, \quad [\hat{Q}, \hat{D}] = \frac{\hbar}{2}\hat{Q}, \quad [\hat{\bar{Q}}, \hat{D}] = -\frac{\hbar}{2}\hat{\bar{Q}}, \\
&\{\hat{Q}, \hat{Q}\} = 2\hbar\hat{H}, \quad \{\hat{Q}, \hat{\bar{Q}}\} = 2\hbar D, \quad \{\hat{\bar{Q}}, \hat{\bar{Q}}\} = 2\hbar K.
\end{aligned} \tag{43}$$

The $osp(1|2)$ Cartan-Weyl basis is recovered, from the (42) trigonometric charges, via the identifications

$$\hat{H} = E^{-}, \quad \hat{D} = H, \qquad \hat{K} = -E^{+}, \quad \hat{Q} = 2iF^{-}, \quad \hat{\bar{Q}} = -2iF^{+}. \tag{44}$$

We obtain a different conjugation with respect to the parabolic case, given by

$$(E^{\pm})^{\dagger} = -E^{\mp}, \qquad H^{\dagger} = H, \qquad (F^{\pm})^{\dagger} = F^{\mp}. \tag{45}$$

In the trigonometric case the Hamiltonian is given by the $osp(1|2)$ Cartan generator $\omega\hat{D}$.

By taking into account the presence of the dimensional parameter ω that we set, for convenience, equal to 1 in the formulas above, the dimensional analysis of the trigonometric model gives us the dimensions $[t] = -1$, $[\hat{y}] = -\frac{1}{2}$, $[\hat{p}] = \frac{1}{2}$, $[\hat{\chi}] = -\frac{1}{2}$, $[\omega] = 1$, $[\mathcal{S}] = 0$.

5 Superconformal Quantum Mechanics with Calogero potentials: $1D$ $D(2,1;\alpha)$ and $2D$ $sl(2|1)$ models

In this Section we quantize the worldline superconformal σ-models recovered from the $\mathcal{N}=4$ $(1,4,3)$ (i.e., one-dimensional target) and $\mathcal{N}=2$ $(2,2,0)$ (i.e., two-dimensional target) parabolic supermultiplets. Unlike the $\mathcal{N}=1$ parabolic model analyzed in Section **4**, non-trivial potential terms and non-trivial quantum corrections to the classical Hamiltonians, appear.

The $\mathcal{N}=4$ $(1,4,3)$ parabolic model possesses a $D(2,1;\alpha)$ invariance, where $\alpha \neq 0,-1$ is identified with the scaling dimension of the supermultiplet. The Hamiltonian describes a particle moving on a line under an inverse square potential and includes spin-like degrees of freedom.

The $\mathcal{N}=2$ $(2,2,0)$ parabolic model possesses an $sl(2|1)$ invariance. Its Hamiltonian describes a particle moving on a plane under an inverse square potential and with a spin-orbit coupling.

5.1 The $\mathcal{N}=4$ $(1,4,3)$ parabolic model with $D(2,1;\alpha)$ invariance

A discussion of the classical $\mathcal{N}=4$ $(1,4,3)$ superconformal worldline model can be found, e.g., in [1]. We present here the quantization of this model repeating the same steps discussed in Section **4** for the $osp(1|2)$-invariant model.

The non-vanishing (anti)commutators obtained from quantizing the Dirac brackets are

$$[\hat{y},\hat{p}]=i, \qquad \{\hat{\chi}_\alpha,\hat{\chi}_\beta\}=\delta_{\alpha\beta}, \tag{46}$$

with $\alpha,\beta=0,...,3$. The above equations define the superalgebra $\mathfrak{h}_1\oplus C_4$, with the one-dimensional Heisenberg algebra $\mathfrak{h}_1$ in its even sector and the four $C\ell(4,0)$ Clifford algebra gamma-matrices in its odd sector. These gamma-matrices can be expressed as 4×4 complex matrices. We choose, to respect the $\mathbb{Z}_2$-graded structure of the superalgebra, block-antidiagonal gamma matrices, while representing the Heisenberg generators as block-diagonal operators.

The position-space representation of (46) is

$$\begin{aligned} &\hat{y}=y\mathbb{I}_4, \qquad \hat{p}=-i\partial_y\mathbb{I}_4, \\ &\hat{\chi}_0=\frac{1}{\sqrt{2}}\sigma_2\otimes\mathbb{I}_2, \quad \hat{\chi}_1=-\frac{1}{\sqrt{2}}\sigma_1\otimes\sigma_1, \quad \hat{\chi}_2=-\frac{1}{\sqrt{2}}\sigma_1\otimes\sigma_2, \quad \hat{\chi}_3=-\frac{1}{\sqrt{2}}\sigma_1\otimes\sigma_3, \end{aligned} \tag{47}$$

where $\mathbb{I}_n$ is the n x n identity matrix and the σ_i's $(i=1,2,3)$ are the Pauli matrices.

The quantum charges are given by

$$\begin{aligned}
\hat{H} &= (\frac{\hat{p}^2}{2}+\frac{(1+2\alpha)^2}{8\hat{y}^2})\mathbb{I}_4+\frac{1+2\alpha}{4\hat{y}^2}\mathcal{F}_4,\\
\hat{D} &= (\frac{t\hat{p}^2}{2}-\frac{1}{4}(\hat{y}\hat{p}+\hat{p}\hat{y})+\frac{t(1+2\alpha)^2}{8\hat{y}^2})\mathbb{I}_4+\frac{t(1+2\alpha)}{4\hat{y}^2}\mathcal{F}_4,\\
\hat{K} &= (\frac{t^2\hat{p}^2}{2}-\frac{t}{2}(\hat{y}\hat{p}+\hat{p}\hat{y})+\frac{\hat{y}^2}{2}+\frac{t^2(1+2\alpha)^2}{8\hat{y}^2})\mathbb{I}_4+\frac{t^2(1+2\alpha)}{4\hat{y}^2}\mathcal{F}_4,\\
\hat{Q}_0 &= \hat{\chi}_0\hat{p}+\frac{i(1+2\alpha)}{6}\epsilon_{ijk}\frac{\hat{\chi}_i\hat{\chi}_j\hat{\chi}_k}{\hat{y}},\\
\hat{Q}_i &= \hat{\chi}_i\hat{p}-\frac{i(1+2\alpha)}{2}\epsilon_{ijk}\frac{\hat{\chi}_0\hat{\chi}_j\hat{\chi}_k}{\hat{y}},\\
\hat{\bar{Q}}_0 &= t\hat{\chi}_0\hat{p}-\chi_0\hat{y}+\frac{it(1+2\alpha)}{6}\epsilon_{ijk}\frac{\hat{\chi}_i\hat{\chi}_j\hat{\chi}_k}{\hat{y}},\\
\hat{\bar{Q}}_i &= t\hat{\chi}_i\hat{p}-\chi_i\hat{y}-\frac{it(1+2\alpha)}{2}\epsilon_{ijk}\frac{\hat{\chi}_0\hat{\chi}_j\hat{\chi}_k}{\hat{y}},\\
\hat{J}_i &= -i(\frac{1}{2}\epsilon_{ijk}\hat{\chi}_j\hat{\chi}_k+\hat{\chi}_0\hat{\chi}_i),\\
\hat{L}_i &= -i(\frac{1}{2}\epsilon_{ijk}\hat{\chi}_j\hat{\chi}_k-\hat{\chi}_0\hat{\chi}_i).
\end{aligned}\tag{48}$$

In the above formulas we used the Fermi number operator $\mathcal{F}_4$, defined by $\mathcal{F}_{2n}=\begin{pmatrix}\mathbb{I}_n & 0\\ 0 & -\mathbb{I}_n\end{pmatrix}$. One should note that the quantum operators $\hat{H}$, $\hat{D}$, $\hat{K}$ contain an Ehrenfest quantum correction term, proportional to $\frac{\hbar^2(1+2\alpha)^2}{\hat{y}^2}\mathbb{I}_4$, which is not present in the classical charges. Its appearance can be traced to the change from classical Grassmann to quantum Clifford variables.

At a given value $\alpha\neq 0,-1$, the above operators close the exceptional superalgebra $D(2,1;\alpha)$. The R-symmetry generators $\hat{J}_i$ and $\hat{L}_i$, $i=1,2,3$, close two independent ($[\hat{J}_i,\hat{L}_j]=0$) $su(2)$ subalgebras.

In the Cartan-Weyl basis the non-vanishing $D(2,1;\alpha)$ brackets are given by

$$\begin{aligned}
&[H,E^{\pm}]=\pm E^{\pm},\quad [E^+,E^-]=2H,\quad [H,F^{\pm}_{\beta}]=\pm\frac{1}{2}F^{\pm}_{\beta},\quad [E^{\pm},F^{\mp}_{\beta}]=-F^{\pm}_{\beta},\\
&\{F^{\pm}_0,F^{\mp}_j\}=-\frac{i}{4}(\lambda J_j+(1+\lambda)L_j),\quad \{F^+_j,F^-_k\}=\epsilon_{jkl}(-\frac{i\lambda}{4}J_l+\frac{i(\lambda+1)}{4}L_l),\\
&\{F^{\pm}_{\beta},F^{\pm}_{\gamma}\}=\pm\frac{1}{2}\delta_{\beta\gamma}E^{\pm},\quad [J_j,F^{\pm}_0]=iF^{\pm}_j,\quad [J_j,F^{\pm}_k]=i(-\delta_{jk}F^{\pm}_0+\epsilon_{jkl}F^{\pm}_l),\\
&[L_j,F^{\pm}_0]=-iF^{\pm}_j,\qquad [L_j,F^{\pm}_k]=i(\delta_{jk}F^{\pm}_0+\epsilon_{jkl}F^{\pm}_l),\\
&[J_j,J_k]=2i\epsilon_{jkl}J_l,\qquad [L_j,L_k]=2i\epsilon_{jkl}L_l.
\end{aligned}\tag{49}$$

The above superalgebra is realized by the (48) quantum operators via the identifications

$$\hat{H}=-E^-,\quad \hat{D}=iH,\quad \hat{K}=-E^+,\quad \hat{Q}_{\beta}=2F^-_{\beta},\quad \hat{\bar{Q}}_{\beta}=2iF^+_{\beta},\quad \hat{J}_j=J_j,\quad \hat{K}_j=K_j.\tag{50}$$

The Hamiltonian operator $\hat{H}$, explicitly written in 4×4 supermatrix form, is given by

$$\hat{H}=\left(\begin{array}{c|c}(\frac{\hat{p}^2}{2}+\frac{4\alpha^2+8\alpha+3}{8\hat{y}^2})\mathbb{I}_2 & 0\\ \hline 0 & (\frac{\hat{p}^2}{2}+\frac{4\alpha^2-1}{8\hat{y}^2})\mathbb{I}_2\end{array}\right).\tag{51}$$

It is the Hamiltonian of the $\mathcal{N}=4$ super-Calogero model with $D(2,1;\alpha)$ invariance.

It contains a (purely bosonic) Calogero Hamiltonian in both its upper and lower diagonal blocks. We recall that the Calogero Hamiltonian $\mathcal{H}_C$ is given by

$$\mathcal{H}_C = \frac{1}{2}\hat{p}^2 + \frac{g^2}{\hat{y}^2}. \tag{52}$$

The self-adjointness of the Calogero Hamiltonian $\mathcal{H}_C$ depends on the value of the coupling parameter g. We refer to the [34, 35] papers for a thorough discussion of this subtle point.

For our purposes it is important to note here the relation between the coupling constant g and the scaling dimension parameter α. From [34] we know that $\mathcal{H}_C$ is self-adjoint, provided that the inequality $g^2 > -\frac{1}{8}$ is satisfied. Under this condition the boundary value problem

$$\mathcal{H}_C\phi_k = E_k\phi_k, \qquad \phi_k(0) = 0,$$

gives a continuous positive spectrum, $0 \leq E_k < \infty$, the eigenfunctions and eigenvalues being

$$\phi_k(y) = 2^{\mu-\frac{1}{2}}\Gamma(\mu+\frac{1}{2})(ky)^{-(\mu-\frac{1}{2})}J_{\mu-\frac{1}{2}}(ky)y^{\mu}, \qquad E_k = \frac{1}{2}k^2,$$

for

$$g^2 = \frac{1}{2}\mu(\mu-1). \tag{53}$$

Let us set

$$g_b^2 = \frac{4\alpha^2+8\alpha+3}{8}, \qquad g_f^2 = \frac{4\alpha^2-1}{8}, \tag{54}$$

for the Calogero parameters entering, respectively, upper and lower diagonal blocks of the blocks of the (51) Hamiltonian. It is quite rewarding that imposing, simultaneously, the $g_b^2, g_f^2 > -\frac{1}{8}$ condition, we end up with the $\alpha \neq 0, -1$ inequality for the scaling dimension. The class of exceptional $D(2,1;\alpha)$ superalgebras guarantee the existence of a well-defined Hamiltonian with a continuous positive spectrum bounded from below.

At the special $\alpha = -\frac{1}{2}$ value the Calogero potential terms (in both upper and lower blocks) vanish. Therefore, this special point corresponds to a free theory. At this given value, see [18], we have $D(2,1;-\frac{1}{2}) = D(2,1)$, so that the invariant superalgebra coincides with the classical $D(2,1) \approx osp(4|2)$ superalgebra.

We can express, from (53), g_b, g_f in term of their respective μ_b, μ_f parameters. From (54) μ_b, μ_f can be given in terms of α. The result is the linear relations

$$\mu_b = \tfrac{1}{2} \pm (\alpha+1), \quad \mu_f = \tfrac{1}{2} \pm \alpha. \tag{55}$$

In quantum mechanics the continuity conditions are also imposed imposed on the probability currents. Since the zero-energy wave functionis (up to a normalizing factor) $\phi_0(y) = y^{\mu}$, these conditions imply that both μ_b, μ_f must satisfy $\mu_b, \mu_f > \frac{1}{2}$ to ensure continuity at the origin. The (55) equations show that any $\alpha \neq 0, -1$ is suitable to fulfill these constraints.

As a final comment we point out that the energy levels of both bosonic (upper) and fermionic (lower) blocks are doubly degenerated. This degeneracy is removed by taking into account the hermitian operators $\hat{J}_3$, $\hat{L}_3$ which commute with $\hat{H}$. Indeed,

$$\hat{J}_3 = \left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & 0 \end{array}\right), \qquad \hat{L}_3 = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sigma_3 \end{array}\right), \tag{56}$$

are both diagonal and specify spin-like quantum numbers in the bosonic and fermionic sectors, respectively. We can say that the bosonic states have $\frac{1}{2}$ $\hat{J}$-spin and 0 $\hat{L}$-spin, while the fermionic states have 0 $\hat{J}$-spin and $\frac{1}{2}$ $\hat{L}$-spin .

5.2 The $\mathcal{N}=2$ $(2,2,0)$ parabolic model with $sl(2|1)$ invariance

The classical $sl(2|1)$-invariant action based on the parabolic D-module rep of the $(2,2,0)$ supermultiplet is presented in Appendix **B**. Its quantization is performed with the techniques previously outlined (introduction of the "constant kinetic basis", Dirac brackets, etc.). For this model it is convenient to express the two propagating bosons in terms of a complex field y.

We obtain the non-vanishing (anti)commutators

$$[y^*, p_{y^*}] = [y, p_y] = i\hbar, \quad \{\chi, \chi^\dagger\} = \tfrac{\hbar}{C}. \tag{57}$$

The fermions can be expressed as $\chi = \sqrt{\frac{\hbar}{C}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\chi^\dagger = \sqrt{\frac{\hbar}{C}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
Let us fix, for simplicity, $\hbar = 1$ and $C = \frac{1}{2}$. The quantum operators $\hat{Q}^1_-$, $\hat{Q}^2_-$ can be written as

$$\hat{Q}^1_- = i \begin{pmatrix} 0 & -A \\ A^\dagger & 0 \end{pmatrix}, \qquad \hat{Q}^2_- = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}, \tag{58}$$

where

$$A^\dagger = -\tfrac{i}{\sqrt{2}} e^{-i2\lambda\theta} \left(\partial_r + \tfrac{i}{r}\partial_\theta + \tfrac{2\lambda+1}{2r}\right), \quad A = -\tfrac{i}{\sqrt{2}} e^{i2\lambda\theta} \left(\partial_r - \tfrac{i}{r}\partial_\theta + \tfrac{2\lambda+1}{2r}\right), \tag{59}$$

are expressed in polar coordinates ($y = re^{i\theta}$, $y^* = re^{-i\theta}$).

The quantum hamiltonian is

$$\hat{H} = -\tfrac{1}{2}\left(\partial_r^2 + \tfrac{1}{r}\partial_r + \tfrac{1}{r^2}\partial_\theta^2\right) + i\tfrac{(2\lambda+1)}{2r^2}\sigma_z\partial_\theta + \tfrac{(2\lambda+1)^2}{8r^2}, \tag{60}$$

with σ_z the diagonal Pauli matrix. $\frac{(2\lambda+1)^2}{8r^2}$ is the Ehrenfest term resulting from quantization.

The remaining quantum Noether charges are

$$\begin{aligned} &\hat{L}_0 = t\hat{H} + \tfrac{i}{2}\left(r\partial_r + 1\right), \ \hat{L}_1 = t^2\hat{H} + it\left(r\partial_r + 1\right) + \tfrac{r^2}{2}, \ \hat{J} = -\tfrac{i}{2}\partial_\theta - \tfrac{2\lambda-1}{4}\sigma_z, \\ &\hat{Q}^1_+ = t\hat{Q}^1_- - i\tfrac{r}{\sqrt{2}} \begin{pmatrix} 0 & e^{i2\lambda\theta} \\ -e^{-i2\lambda\theta} & 0 \end{pmatrix}, \ \hat{Q}^2_+ = t\hat{Q}^2_- - \tfrac{r}{\sqrt{2}} \begin{pmatrix} 0 & e^{i2\lambda\theta} \\ e^{-i2\lambda\theta} & 0 \end{pmatrix}. \end{aligned} \tag{61}$$

The non-vanishing (anti)commutators, closing the $sl(2|1)$ superalgebra are ($m, n = 0, \pm 1$):

$$\begin{aligned} &[\hat{L}_n, \hat{L}_m] = i\left(m-n\right)\hat{L}_{m+n}, \quad [\hat{L}_0, \hat{Q}^a_\pm] = \frac{i}{2}\hat{Q}^a_\pm, \quad [\hat{L}_{\pm1}, \hat{Q}^a_\mp] = \mp i\hat{Q}^a_\pm, \\ &[\hat{J}, \hat{Q}^a_\pm] = \frac{i}{2}\epsilon_{ab}\hat{Q}^b_\pm, \quad \{\hat{Q}^a_\pm, \hat{Q}^b_\pm\} = 2\delta_{ab}\hat{L}_{\pm1}, \quad \{\hat{Q}^a_\pm, \hat{Q}^b_\mp\} = 2\delta_{ab}\hat{L}_0, \quad \{\hat{Q}^a_\pm, \hat{Q}^b_\mp\} = \pm 2\epsilon_{ab}\hat{J}, \end{aligned} \tag{62}$$

where $\hat{L}_{-1} = \hat{H}$, $a, b = 1, 2$ and $\epsilon_{12} = -\epsilon_{21} = 1$.

The eigenvalue equation $\hat{H}\psi_{E_{m\pm}} = E_{m\pm}\psi_{E_{m\pm}}$, for $E_{m\pm} > 0$, produces a continuum spectrum with eigenfunctions

$$\begin{aligned} \psi_{Em+}(r,\theta) &= J_{|\frac{2\lambda+1}{2}-m|}(\alpha r)e^{im\theta} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \psi_{Em-}(r,\theta) &= J_{|\frac{2\lambda+1}{2}+m|}(\alpha r)e^{im\theta} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{63}$$

where $J_{|\frac{2\lambda+1}{2}-m|}(\alpha r)$ and $J_{|\frac{2\lambda+1}{2}+m|}(\alpha r)$ are Bessel functions and $\alpha = \sqrt{2E}$.

To conclude the analysis of this model, we present it as a Supersymmetric Quantum Mechanics. Let us introduce

$$\hat{Q} = \frac{\hat{Q}^2_- + i\hat{Q}^1_-}{2} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad \hat{Q}^\dagger = \frac{\hat{Q}^2_- - i\hat{Q}^1_-}{2} = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}. \tag{64}$$

We get $\{\hat{Q}, \hat{Q}^\dagger\} = 2\hat{H}$ and $\hat{Q}^2 = (\hat{Q}^\dagger)^2 = 0$.

From the expressions (59), it follows that $\hat{Q}\psi_{E_{m-}} = \psi_{E_{(m+2\lambda)+}}$ and $\hat{Q}^\dagger\psi_{E_{m+}} = \psi_{E_{(m-2\lambda)-}}$. Since $m+2\lambda$ and $m-2\lambda$ need to be intege numbersr, $\hat{Q}\psi_{E_{m-}}$ and $\hat{Q}^\dagger\psi_{E_{m+}}$ belong to the Hilbert space only if 2λ is an integer number. A supersymmetric pair is therefore only encountered for the quantized values of the scaling dimension, either $\lambda \in \frac{1}{2} + \mathbb{Z}$ or $\lambda \in \mathbb{Z}$.

6 Superconformal Quantum Mechanics with DFF oscillator potential terms: $1D$ $D(2,1;\alpha)$ and $2D$ $sl(2|1)$ models

In this Section we quantize the worldline trigonometric σ-models obtained from the $\mathcal{N} = 4$ $(1,4,3)$ and $\mathcal{N} = 2$ $(2,2,0)$ supermultiplets (see Appendix **B**). They contain (besides a Calogero potential) an oscillatorial (DFF term) which furnishes a discrete, bounded from below, spectrum. The associated $D(2,1;\alpha)$ and, respectively, $sl(2|1)$superconformal algebras act as spectrum-generating algebras for these models.

The $D(2,1;\alpha)$ $(1,4,3)$ trigonometric σ-models shed some new light on the results of Calogero [3] and de Alfaro, Fubini and Furlan [2]. Indeed, their Casimir energy linearly depends (in two regions) on the scaling dimension parameter α (in contrast with the complicated dependence expressed in terms of the Calogero coupling constant, see [34]).

Interesting features are also presented by the $sl(2|1)$ $(2,2,0)$ trigonometric σ-models. The scaling dimension λ needs to be quantized (either $\lambda = \frac{1}{2} + \mathbb{Z}$ or $\lambda \in \mathbb{Z}$). At the special $\lambda = -\frac{1}{2}$ value the ordinary two-dimensional oscillator (since the Calogero potential vanishes at this special point) can be recovered. The Hilbert space of these class of models is decomposed into an infinite direct sum of $sl(2|1)$ lowest weight representations. An unexpected feature is the existence of fermionic raising operators (not entering the $sl(2|1)$ superalgebra) which allow, together with the $sl(2|1)$ raising operators, for $\lambda = \frac{1}{2} + \mathbb{Z}$ to recover the whole Hilbert space of the theory from the single bosonic vacuum. The existence of these extra fermionic operators is traced to the presence of a discrete symmetry.

6.1 The quantum $D(2,1;\alpha)$ trigonometric model from $\mathcal{N} = 4$ $(1,4,3)$

The quantization of this model follows the same steps as the quantization of the $osp(1|2)$-invariant trigonometric model described in Section **4**. We end up, just like its $\mathcal{N} = 4$ $(1,4,3)$ parabolic counterpart of Section **5**, with (anti)commutators defining the the $\mathfrak{h}_1 \oplus C_4$ superalgebra (46). We set, for convenience and without loss of generality, the dimensional parameter ω (its presence in the equations can be restored by means of dimensional analysis).

The quantum operators are ($\mathcal{F}_4$ is the Fermion Number operator introduced in (48))

$$\begin{aligned}
\hat{H} &= e^{it}(\frac{\hat{p}^2}{2} - \frac{i}{4}(\hat{y}\hat{p}+\hat{p}\hat{y}) - \frac{\hat{y}^2}{8} + \frac{(1+2\alpha)^2}{8\hat{y}^2})\mathbb{I}_4 + e^{it}\frac{1+2\alpha}{4\hat{y}^2}\mathcal{F}_4,\\
\hat{D} &= (\frac{\hat{p}^2}{2} + \frac{\hat{y}^2}{8} + \frac{(1+2\alpha)^2}{8\hat{y}^2})\mathbb{I}_4 + \frac{(1+2\alpha)}{4\hat{y}^2}\mathcal{F}_4,\\
\hat{K} &= e^{-it}(\frac{\hat{p}^2}{2} + \frac{i}{4}(\hat{y}\hat{p}+\hat{p}\hat{y}) - \frac{\hat{y}^2}{8} + \frac{(1+2\alpha)^2}{8\hat{y}^2})\mathbb{I}_4 + e^{-it}\frac{1+2\alpha}{4\hat{y}^2}\mathcal{F}_4,\\
\hat{Q}_0 &= e^{\frac{it}{2}}(\hat{\chi}_0\hat{p} - \frac{i}{2}\hat{\chi}_0\hat{y} + \frac{i(1+2\alpha)}{6}\epsilon_{ijk}\frac{\hat{\chi}_i\hat{\chi}_j\hat{\chi}_k}{\hat{y}}),\\
\hat{Q}_i &= e^{\frac{it}{2}}(\hat{\chi}_i\hat{p} - \frac{i}{2}\hat{\chi}_i\hat{y} - \frac{i(1+2\alpha)}{2}\epsilon_{ijk}\frac{\hat{\chi}_0\hat{\chi}_j\hat{\chi}_k}{\hat{y}}),\\
\hat{\bar{Q}}_0 &= e^{-\frac{it}{2}}(\hat{\chi}_0\hat{p} + \frac{i}{2}\hat{\chi}_0\hat{y} + \frac{i(1+2\alpha)}{6}\epsilon_{ijk}\frac{\hat{\chi}_i\hat{\chi}_j\hat{\chi}_k}{\hat{y}}),\\
\hat{\bar{Q}}_i &= e^{-\frac{it}{2}}(\hat{\chi}_i\hat{p} + \frac{i}{2}\hat{\chi}_i\hat{y} - \frac{i(1+2\alpha)}{2}\epsilon_{ijk}\frac{\hat{\chi}_0\hat{\chi}_j\hat{\chi}_k}{\hat{y}}),\\
\hat{J}_i &= -i(\frac{1}{2}\epsilon_{ijk}\hat{\chi}_j\hat{\chi}_k + \hat{\chi}_0\hat{\chi}_i),\\
\hat{L}_i &= -i(\frac{1}{2}\epsilon_{ijk}\hat{\chi}_j\hat{\chi}_k - \hat{\chi}_0\hat{\chi}_i).
\end{aligned} \tag{65}$$

The above operators realize the $D(2,1;\alpha)$ superalgebra (49) with the identifications

$$\hat{H} = E^-, \quad \hat{D} = H, \quad \hat{K} = -E^+, \quad \hat{Q}_\beta = 2iF^-_\beta, \quad \hat{\bar{Q}}_\beta = -2iF^+_\beta, \quad \hat{J}_j = J_j, \quad \hat{K}_j = K_j. \tag{66}$$

The quantum Hamiltonian $\hat{\mathcal{H}} \equiv \hat{D}$ is, explicitly,

$$\hat{D} = \left(\begin{array}{c|c} (\frac{\hat{p}^2}{2} + \frac{4\alpha^2+8\alpha+3}{8\hat{y}^2} + \frac{\hat{y}^2}{8})\mathbb{I}_2 & 0 \\ \hline 0 & (\frac{\hat{p}^2}{2} + \frac{4\alpha^2-1}{8\hat{y}^2} + \frac{\hat{y}^2}{8})\mathbb{I}_2 \end{array}\right). \tag{67}$$

Both upper (bosonic) and lower (fermionic) diagonal blocks of $\hat{D}$ contain a Calogero Hamiltonian with the DFF oscillatorial potential,

$$\hat{\mathcal{H}}_{DFF} = \frac{1}{2}\hat{p}^2 + \frac{g^2}{\hat{y}^2} + \frac{\hat{y}^2}{8}. \tag{68}$$

A detailed analysis of this Hamiltonian can be found in [3, 34]. Just like the parabolic case, the inequality $g^2 > -\frac{1}{8}$ guarantees the existence of physically acceptable solutions. The boundary value problem

$$\hat{\mathcal{H}}_{DFF}\phi_n = E_n\phi_n, \quad \phi_n(0) = 0, \qquad n = 0,1,2,\ldots, \tag{69}$$

implies the discrete spectrum

$$E_n = \frac{1}{2}(n+\nu+1), \tag{70}$$

with eigenfunctions given (up to normalizion) by

$$\phi_n(y) = y^{\nu+\frac{1}{2}}\exp(-\frac{y^2}{4})L^\nu_n(\frac{1}{2}y^2). \tag{71}$$

In the right hand side L_n^ν stands for the modified Laguerre polynomials. The parameter ν entering the Casimir energy $\frac{1}{2}(\nu+1)$ is

$$\nu = \frac{1}{2}(1+8g^2)^{\frac{1}{2}}. \tag{72}$$

Comparing equations (67) and (68) we see that g_b, g_f are again given by equations (54), so that $\alpha \neq 0, -1$ to ensure that both g_b^2 and g_f^2 are greater than $-\frac{1}{8}$.

Since the Hamiltonian is a Cartan generator of the (65) superalgebra, the whole spectrum can be recovered from a lowest weight representation of $D(2,1;\alpha)$, where the Q_β's are the lowering and the $\bar{Q}_\beta$'s are the raising operators. The vacuum $|\Lambda\rangle$ is introduced from

$$Q_\beta |\Lambda\rangle = 0, \qquad\qquad \beta = 0,1,2,3. \tag{73}$$

From the definition of the Q_β's in (65) the four differential equations (73) can be recasted into the single differential equation

$$(\hat{p} - \frac{i}{2}\hat{y} - \frac{i(1+2\alpha)}{2\hat{y}}\mathcal{F}_4)\,|\Lambda\rangle = 0. \tag{74}$$

In position-space representation, (74) splits into two separate equations for the bosonic (+) and respectively fermionic (−) subspaces,

$$\frac{d\phi_{0,\sigma}}{dy} = -\frac{1}{2}(y \pm \frac{1+2\alpha}{y})\phi_{0,\sigma}. \tag{75}$$

The label σ accounts, just as in the parabolic case, for the $\hat{J}$,$\hat{L}$-spin degrees of freedom.

Integrating the above equation we get, up to normalization, the vacuum solutions

$$\phi_{0,\sigma} \;\; = \;\; y^{\mp(\frac{1+2\alpha}{2})}\exp(-\frac{y^2}{4}). \tag{76}$$

This result is in agreement with (71) provided that we set

$$\nu_b = -(1+\alpha), \qquad \nu_f = \alpha. \tag{77}$$

This analysis forces us to conclude that two degenerate lowest energy vacua exist for $\alpha \neq -\frac{1}{2}$. They are bosonic for $\alpha < -\frac{1}{2}$ and fermionic for $\alpha > -\frac{1}{2}$. This is implied by equation (71) which tells us that any bosonic (fermionic) vacuum should be such that $\nu_b + \frac{1}{2} > 0$ ($\nu_f + \frac{1}{2} > 0$).

At the special $\alpha = -\frac{1}{2}$ value we have that $D(2,1;-\frac{1}{2}) \equiv D(2,1) \approx osp(4|2)$. The Calogero potential terms vanish both in the upper and lower diagonal blocks. At $\alpha = -\frac{1}{2}$ we recover four undeformed harmonic oscillator equations. All the states of the theory (including the minimal energy states) are four times degenerated, with two bosonic and two fermionic states of same energy.

The energy levels of the system are given by

$$E_{b,n} = \frac{1}{2}(n-\alpha), \qquad E_{f,n} = \frac{1}{2}(n+\alpha+1), \quad n = 0,1,2,\ldots. \tag{78}$$

$E_{b,n}$ ($E_{f,n}$) are the energy levels of the bosonic (fermionic) states (they coincide for $\alpha = -\frac{1}{2}$).

The energy of the degenerate vacua is

$$E_{b,vac} = -\frac{1}{2}\alpha, \quad (\alpha \leq -\frac{1}{2}) \quad ; \quad E_{f,vac} = \frac{1}{2}(\alpha+1), \quad (\alpha \geq -\frac{1}{2}). \tag{79}$$

the fermionic degenerate vacua ($\alpha < -\frac{1}{2}$) $E_{b,n}$ applies to the bosonic states, $E_{f,n}$ to the fermionic degenerate vacua.

The scaling dimension α can be regarded as an external control parameter of the theory, so that the vacuum energy can be interpreted as a Casimir energy. The Casimir energy of the $(1,4,3)$ $D(2,1;\alpha)$ (un)deformed oscillator admits a very nice expression in terms of α, being simply given by

$$E_{vac} = \frac{1}{4}(1+|2\alpha+1|). \tag{80}$$

This expression should be compared with the much more complicated expression of the vacuum energy in terms of the Calogero coupling constant g and derived from (72). This result suggests that the scaling dimension α has a more direct physical interpretation of the Calogero coupling constant g. One should also note that, contrary to g, α directly enters the spectrum-generating superalgebra $D(2,1;\alpha)$.

6.2 The $\mathcal{N}=2$ $(2,2,0)$ trigonometric model with $sl(2|1)$ invariance

As in the parabolic case, we obtain from quantization the non-vanishing (anti)commutators

$$[y^*, p_{y*}] = [y, p_y] = i\hbar, \qquad \{\chi, \chi^\dagger\} = \frac{\hbar}{\omega C}, \tag{81}$$

with $\chi = \sqrt{\frac{\hbar}{\omega C}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\chi^\dagger = \sqrt{\frac{\hbar}{\omega C}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We work with $\hbar = 1$, $C = \frac{1}{2}$, $\omega = 2$.

The fermionic operators $\hat{Q}^{(I)}_{\pm}$, $I = 1, 2$, entering $sl(2|1)$ are

$$\hat{Q}^{(1)}_{\pm} = ie^{\mp it} \begin{pmatrix} 0 & -A_{\pm} \\ B_{\pm} & 0 \end{pmatrix}, \quad \hat{Q}^{(2)}_{\pm} = e^{\mp it} \begin{pmatrix} 0 & A_{\pm} \\ B_{\pm} & 0 \end{pmatrix}, \tag{82}$$

where, using the polar coordinates as in the parabolic case, we have

$$\begin{aligned} A_{\pm} &= -\frac{i}{2} e^{i2\lambda\theta} \left(\partial_r - \frac{i}{r}\partial_\theta + \frac{2\lambda+1}{2r} \pm r \right), \\ B_{\pm} &= -\frac{i}{2} e^{-i2\lambda\theta} \left(\partial_r + \frac{i}{r}\partial_\theta + \frac{2\lambda+1}{2r} \pm r \right). \end{aligned} \tag{83}$$

In the trigonometric case the Hamiltonian $\mathcal{H}$ is the Cartan generator $\hat{D}$, given by

$$\hat{D} = -[\frac{1}{2}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2) + i\frac{(2\lambda+1)}{2r^2}\sigma_z\partial_\theta + \frac{(2\lambda+1)^2}{8r^2} + \frac{r^2}{2}]\mathbb{I}_2. \tag{84}$$

In the r.h.s. σ_z is the diagonal Pauli matrix.

The three remaining bosonic symmetry operators which close the $sl(2|1)$ superalgebra are

$$\begin{aligned} \hat{L}_{\pm 1} &= i\frac{e^{\mp 2it}}{2}[-\frac{1}{2}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2) + i\frac{(2\lambda+1)}{2r^2}\sigma_z\partial_\theta + \frac{(2\lambda+1)^2}{8r^2} - \frac{r^2}{2} \pm (r\partial_r + 1)]\mathbb{I}_2, \\ \hat{J} &= -\frac{i}{2}\mathbb{I}_2\partial_\theta - \frac{2\lambda-1}{4}\sigma_z. \end{aligned} \tag{85}$$

One can easily check that the $sl(2|1)$ superalgebra is recovered from the (anti)commutators of the operators (82,84,85).

The differential equation for the radial part of the eigenfunctions $\psi = e^{im\theta}R_{\pm}(r)e_{\pm}$ of $\hat{D}$, where $e_+ = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $e_- = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, is

$$[-\frac{1}{2}(\partial_r^2 + \frac{1}{r}\partial_r) + \frac{1}{2r^2}(m \pm \frac{2\lambda+1}{2})^2 + \frac{r^2}{2} - E]R_{\pm}(r) \;\; = \;\; 0. \tag{86}$$

E is the energy. In [3] the same equation is found and solved for the problem of three bodies in a line. Furthermore, the issue of selfadjointness of the differential operator acting on $R_{\pm}$ was investigated in [36]; since $\sqrt{\left(m \pm \frac{2\lambda+1}{2}\right)^2} \geq 0$, the existence of a selfadjoint extension for the Halmiltonian (84) is ensured.

The requirement of single-valuedness for the operators $\hat{Q}_{\pm}^{(I)}$ on the $\mathbb{R}^2$-plane implies, from the exponents in (83), that the constraint $4\lambda\pi = 2k\pi$, with k integer, must be satisfied. Therefore the scaling dimension λ has to be quantized, either $\lambda = \frac{1}{2} + \mathbb{Z}$ or $\lambda = \mathbb{Z}$. We discuss in detail the half-integer case, with side remarks about the models with integer values of λ.

One should note that at $\lambda = -\frac{1}{2}$ one obtains (two copies of) the Hamiltonian of the undeformed two-dimensional bosonic oscillator.

For half-integer λ the $\hat{Q}_{\pm}^{(I)}$ operators act as raising/lowering operators. Let us take, e.g., $\hat{Q}_{\pm}^{(2)}$; it follows, from the commutators $[\hat{D}, \hat{Q}_{\pm}^{(2)}] = \mp\hat{Q}_{\pm}^{(2)}$, that an energy eigenstate ψ with eigenvalue E_n is mapped into an eigenstate $\hat{Q}_{\pm}^{(2)}\psi$ with eigenvalue $E_n \mp 1$ (provided that $E_n \mp 1 \neq 0$):

$\hat{D}\psi = E_n\psi \rightarrow \hat{D}\hat{Q}_{\pm}^{(2)}\psi = (E_n \mp 1)\,\hat{Q}_{\pm}^{(2)}\psi$.

Therefore, starting from a lowest weight state satisfying $\hat{Q}_{+}^{(2)}\psi = 0$, an infinite tower of higher energy eigenstates are constructed by repeatedly applying $\hat{Q}_{-}^{(2)}$. The solutions of the lowest weight equation $\hat{Q}_{+}^{(2)}\psi = 0$ are given by the eigenfunctions

$$\begin{aligned}
\psi_{m+}(r,\theta) &= A_m r^{(m-\frac{2\lambda+1}{2})} e^{-r^2} e^{im\theta} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
\psi_{m-}(r,\theta) &= B_m r^{-(m+\frac{2\lambda+1}{2})} e^{-r^2} e^{im\theta} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\end{aligned} \tag{87}$$

where A_m, B_m are normalization constants given by

$$\begin{aligned}
A_m &= 2^{\frac{\alpha+1}{2}} \frac{1}{\sqrt{\pi\Gamma(\alpha+1)}}, \quad \alpha = m - \frac{2\lambda+1}{2}, \\
B_m &= 2^{\frac{\beta+1}{2}} \frac{1}{\sqrt{\pi\Gamma(\beta+1)}}, \quad \beta = -(m + \frac{2\lambda+1}{2})
\end{aligned} \tag{88}$$

and Γ is the gamma function.

In order to have finite lowest weight eigenfunctions at the origin, the integer m is constrained. From the bosonic states the necessary condition is

$$m \;\; \geq \;\; \frac{2\lambda+1}{2}, \tag{89}$$

while from the fermionic states the necessary condition is

$$m \;\; \leq \;\; -\frac{2\lambda+1}{2}. \tag{90}$$

The energy eigenvalue equation of the bosonic and fermionic lowest weight eigenstates is respectively given by

$$\begin{aligned}\hat{D}\psi_{m+} &= (1+m-\frac{2\lambda+1}{2})\psi_{m+},\\ \hat{D}\psi_{m-} &= (1-(m+\frac{2\lambda+1}{2}))\psi_{m-}.\end{aligned} \tag{91}$$

Two minimal vacua, one bosonic and the other fermionic, are obtained with vacuum energy 1. They are recovered from the "saturated" bosonic and fermionic lowest weight eigenstates with, respectively, $m=\frac{2\lambda+1}{2}$ and $m=-\frac{2\lambda+1}{2}$.

The lowest weight condition obtained from the lowering operator $Q_+^{(1)}$ ($Q_+^{(1)}\psi=0$) produces the same set of lowest weight states (87). The application of the raising operator $Q_-^{(1)}$ produces, up to a phase, the higher energy states obtained from the raising operator $Q_-^{(2)}$.

The theory therefore possesses a degenerate vacuum, one vacuum state being bosonic, the other one fermionic. As discussed in Appendix **A** it is possible to impose a superselection rule, imposed by a projector, which selects half of the states being physical. The superselected theory possesses a unique bosonic vacuum and, for $\lambda=-\frac{1}{2}$, its spectrum coincides with the spectrum of the ordinary two-dimensional (undeformed) oscillator, which can therefore be recovered as the superselected, $\lambda=-\frac{1}{2}$, $sl(2|1)$ acting on $(2,2,0)$, quantum trigonometric model.

We conclude this Section with two important remarks. Contrary to the two vacua of the (not superselected) $\lambda=\frac{1}{2}+\mathbb{Z}$ theory, the $\lambda\in\mathbb{Z}$ quantum deformed oscillators possess four vacuum states (two bosonic and two fermionic states). The construction of the Hilbert space follows the same lines as the half-integer λ case. The main difference lies in the fact that the necessary conditions (89) and (90) for the integer m cannot be satisfied as equalities when $\lambda\in\mathbb{Z}$. It is beyond the scope of this work to present the detailed analysis of the $\lambda\in\mathbb{Z}$ deformed oscillators, which will be presented elsewhere.

The second important remark concerns the fact that, for the superselected $\lambda=\frac{1}{2}+\mathbb{Z}$ theory, the Hilbert space cannot be recovered by repeatedly acting with the $sl(2|1)$ raising operators from the vacuum state. The Hilbert space is decomposed (this point is discussed in Appendix **A**) in a infinite direct sum of the $sl(2|1)$ lowest weight representations. This is in sharp contrast with respect to the one-dimensional harmonic oscillator, whose single irreducible lowest weight representation of the $osp(1|2)$ spectrum-generating superalgebra allows to recover the whole Hilbert space.

One can note, however, that it is possible to construct an extra set of fermionic symmetry operators, $\overline{Q}_\pm^{(I)}$, which also act as raising/lowering operators. The construction goes as follows. At first a discrete symmetry operator $\hat{C}$, playing the role of a charge conjugation operator, is introduced. It is given by

$$\hat{C} = \begin{pmatrix} 0 & e^{i(2\lambda+1)\theta} \\ e^{-i(2\lambda+1)\theta} & 0 \end{pmatrix}. \tag{92}$$

One can verify that $[\hat{D},\hat{C}]=0$ and that $\hat{C}^2=I$. This operator also commutes with the $\hat{L}_{\pm1}$ operators in (85). It does not commute, however, with $\hat{J}$ and the $sl(2|1)$ fermionic operators.

With the help of $\hat{C}$ we can introduce the new symmetry operators

$$\hat{C}\hat{Q}_\pm^{(1)}\hat{C}=\overline{Q}_\pm^{(1)}=ie^{\mp it}\begin{pmatrix} 0 & C_\pm \\ -D_\pm & 0 \end{pmatrix}, \qquad \hat{C}\hat{Q}_\pm^{(2)}\hat{C}=\overline{Q}_\pm^{(2)}=e^{\mp it}\begin{pmatrix} 0 & C_\pm \\ D_\pm & 0 \end{pmatrix}, \tag{93}$$

where

$$\begin{aligned} C_{\pm} &= -\frac{i}{2}e^{i2(\lambda+1)\theta}(\partial_r - \frac{i}{r}\partial_\theta - \frac{2\lambda+1}{2r} \pm r), \\ D_{\pm} &= -\frac{i}{2}e^{-i2(\lambda+1)\theta}(\partial_r + \frac{i}{r}\partial_\theta - \frac{2\lambda+1}{2r} \pm r), \end{aligned} \tag{94}$$

and

$$\hat{C}\hat{J}\hat{C} = \overline{J} = -\tfrac{i}{2}\partial_\theta - \tfrac{2\lambda+3}{4}\sigma_z. \tag{95}$$

As discussed in Appendix **A** (where a schematic presentation in diagrams, the dashed lines, of the action of the $\overline{Q}^I_\pm$ operators is given), $\overline{Q}^I_\pm$ act as raising/lowering operators for the eigenstates of the Hilbert space of the theory. Any given eigenstate can be reached by repeatedly applying to the vacuum both sets of $\hat{Q}^I_+, \overline{Q}^I_+$ raising operators.

In terms of $\hat{C}$ we can also introduce the new quantum operators

$$\mathcal{J} = \hat{J} + \overline{J} = -i\partial_\theta - \tfrac{2\lambda+1}{2}\sigma_z, \quad N_f = \sigma_z = \hat{J} - \overline{J}, \tag{96}$$

which allows us to define the new quantum numbers (used in Appendix **A**, see Figure 4):

$$\hat{D}\,|n,j,\epsilon\rangle = (n+1)\,|n,j,\epsilon\rangle\,, \quad \mathcal{J}\,|n,j,\epsilon\rangle = j\,|n,j,\epsilon\rangle\,, \quad \sigma_z\,|n,j,\epsilon\rangle = \epsilon\,|n,j,\epsilon\rangle\,. \tag{97}$$

7 Conclusions

In this paper we presented a framework for quantizing the large class of classical worldline superconformal σ-models derived from supermultiplets. These systems are defined in [25] (for the parabolic case) and [1] (for the trigonometric case). We applied the quantization prescription to derive explicitly the $\mathcal{N}=4$ $(1,4,3)$ and the $\mathcal{N}=2$ $(2,2,0)$ quantum superconformal mechanics (with $D(2,1;\alpha)$ and $sl(2|1)$ dynamical symmetry, respectively). The parameter $\alpha \neq 0,-1$ is the scaling dimension of the $(1,4,3)$ supermultiplet, while the scaling dimension of the $(2,2,0)$ supermultiplet is quantized and given by $\lambda = \frac{1}{2} + \mathbb{Z}$ or $\lambda \in \mathbb{Z}$.

The results concerning the trigonometric models are particularly relevant. These systems are only "softly supersymmetric", see the discussion in Appendix **C**. As such they have not received much attention like the parabolic models. The trigonometric models correspond to superconformal mechanics in the presence of the DFF damping oscillatorial term; stated otherwise, they are oscillators where Calogero potential terms are possibly present. Their spectrum is discrete and bounded from below.

For the $(1,4,3)$ trigonometric models (i.e., the $D(2,1;\alpha)$ oscillators) we derive the following nice formula for the vacuum energy:

$$E_{vac} = \frac{1}{4}(1+|2\alpha+1|). \tag{98}$$

If α is interpreted as a physical external parameter, then (98) can be interpreted as a Casimir energy.

Concerning the $(2,2,0)$ trigonometric models, at the special value $\lambda = -\frac{1}{2}$ one recovers, after imposing a superselection rule derived by a projector, see Appendix **A**, the spectrum of the ordinary two-dimensional oscillator.

It has been noted very recently, see [37], that the ordinary two-dimensional quantum oscillator possesses an $sl(2|1)$ dynamical symmetry. As a byproduct of our framework we can further

point out that the spectrum of the two-dimensional oscillator is decomposed into an infinite direct sum of $sl(2|1)$ lowest weight representations.

In our approach the existence of $sl(2|1)$ as a dynamical symmetry, not only of the undeformed $\lambda = -\frac{1}{2}$, but also of the deformed ($\lambda \in \frac{1}{2} + \mathbb{Z}$ and $\lambda \in \mathbb{Z}$) two-dimensional oscillators, is a natural consequence of the construction of these models from the $\mathcal{N} = 2$ $(2, 2, 0)$ (trigonometric) supermultiplet. The decomposition of the spectrum in a direct sum of $sl(2|1)$ lowest weight representations comes as a bonus and is not surprising. What is really puzzling and unexpected is another feature, discussed at length in Section **6** and in Appendix **A** and **C**, the presence of the extra fermionic symmetry generators which act as raising and lowering operators. They allow to reach each state belonging to the Hilbert space of the two-dimensional oscillator by repeatedly applying the raising operators to the vacuum state.

This result is very puzzling. It is quite possible that, in order to recover the spectrum of the two-dimensional oscillator from a single, irreducible, lowest weight representation, one needs to extend the concept of superalgebra; possibly by making use of the notion of generalized supersymmetry. In a related context, the appearance of a generalized superalgebra as a symmetry of a dynamical system has been noted in [38].

In a forthcoming paper we will present a detailed investigation of the puzzling properties of the deformed $\lambda \in \frac{1}{2} + \mathbb{Z}$ and $\lambda \in \mathbb{Z}$ two-dimensional oscillators.

Appendix A: Diagrams of the spectrum-generating superalgebra for the $\mathcal{N}=2$, $(2,2,0)$, $\lambda=\frac{1}{2}+\mathbb{Z}$ trigonometric cases.

It is convenient, for the two-dimensional cases based on the $\mathcal{N}=2$ $(2,2,0)$ trigonometric reps, to encode in diagrams the action of the raising and lowering operators of the spectrum-generating superalgebra. We explicitly present three such diagrams, Figures 1, 2 and 3, respectively associated with three values of the scaling dimension, $\lambda=\frac{1}{2}$, $\lambda=-\frac{1}{2}$, $\lambda=-\frac{3}{2}$. In a further diagram the general features of the $\lambda=\frac{1}{2}+\mathbb{Z}$ case are presented.

In the diagrams the bosonic (fermionic) states are denoted by white (black) dots. Grey dots denote the presence of both bosonic and fermionic states. The vertical axis represents the energy level, labeled by n, while the horizontal axis represents the angular momentum, labeled by m. We denote with ϵ the eigenvalues of the Fermion Number operator ($\epsilon=+1$ for bosons, $\epsilon=-1$ for fermions). Solid (dashed) lines represent states connected by $\widehat{Q}_{\pm}^{(I)}$ (respectively, $\overline{Q}_{\pm}^{(I)}$) raising and lowering operators with $I=1,2$, see (82) and (93) (for simplicity we drop here the indices).

The $sl(2|1)$ lowest weight states appear, in the diagrams, as the dots where the solid lines originate (in the upward direction). In Figure 2 and 4 the existence of such lowest weight states is not immediately evident, this is however just a side effect of the condensed notation used (a grey dot being associated with two states).

The operators $\widehat{Q}_{\pm}^{(1)},\widehat{Q}_{\pm}^{(2)}$ (and, similarly, $\overline{Q}_{\pm}^{(1)},\overline{Q}_{\pm}^{(2)}$), applied to a $|n,m,\epsilon\rangle$ state which does not coincide with a lowest weight state produce, apart a normalization factor, the same state. We can write, for $I=1,2$,

$$\begin{aligned}\widehat{Q}_{\pm}^{(I)}\,|n,m,\epsilon\rangle &\propto |n\mp 1,m-\epsilon 2\lambda,-\epsilon\rangle\,,\\ \overline{Q}_{\pm}^{(I)}\,|n,m,\epsilon\rangle &\propto |n\mp 1,m-\epsilon 2\,(\lambda+1)\,,-\epsilon\rangle\,.\end{aligned}\tag{A.1}$$

From the three diagrams, Figures 1, 2 and 3, we can immediately read several important features. In particular, in all three cases, the $n>0$ higher energy states are produced via repeated applications of the $\widehat{Q}$'s, $\overline{Q}$'s raising operators from the two (one bosonic and one fermionic) $n=0$ fundamental level states. As a corollary, we need both types ($\widehat{Q}$'s, $\overline{Q}$'s) of raising operators to recover the Hilbert space of the associated model. This means, stated otherwise, that the Hilbert space is *reducible* with respect to the $sl(2|1)$ superalgebra defined by the $\widehat{Q}_{\pm}^{(I)}$ operators alone. In terms of a $sl(2|1)$ decomposition, an infinite tower (one state at each given integer value n) of lowest weight states need to be introduced to recover the Hilbert space of the theory. Therefore, in order to have an irreducible description, the $\overline{Q}_{+}^{(I)}$ operators need to enter the picture.

One shoud note that the $\lambda=-\frac{1}{2}$ case corresponds to the undeformed (namely, without the extra Calogero potential term) two-dimensional harmonic oscillator. The Hilbert space defined by Figure 2 contains a double degeneracy. Two eigenstates (one bosonic, the other one fermionic) are associated with each n,m pair of eigenvalues. The introduction of a suitable projection allows to remove the double degeneracy and recover the Hilbert space of the ordinary two-dimensional harmonic oscillator. The superselection rule is defined in terms of the projection operator $\hat{P}$ ($\hat{P}^2=\mathbb{I}$), given by

$$\hat{P} \;=\; N_f e^{i\pi\mathcal{H}},\tag{A.2}$$

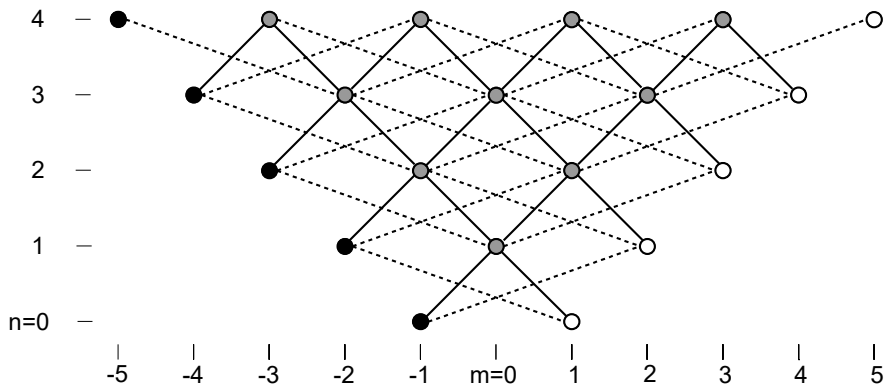

Figure 1: $\lambda = \frac{1}{2}$ diagram of $\widehat{Q}$'s, $\overline{Q}$'s raising and lowering operators.

where N_f is the fermion number operator and $\mathcal{H} = \hat{D}$ is the Hamiltonian (its eigenvalues are the non-negative integers n). The

$$\hat{P}|\Psi\rangle \quad = \quad |\Psi\rangle \tag{A.3}$$

superselection rule implies that the Hilbert space of the superselected theory is given by bosonic states at even energy eigenvalues ($n = 2k$, with $k = 0, 1, 2, \ldots$) and fermionic states at odd energy eigenvalues ($n = 2k + 1$).

The superselection removes, in particular, the degeneracy of the vacuum, the single vacuum state being now bosonic. The spectrum of the ordinary two-dimensional harmonic oscillator is therefore recovered from the *superselected* $\mathcal{N} = 2$ $(2, 2, 0)$ model at scaling dimension $\lambda = -\frac{1}{2}$.

For any half-integer value $\lambda = \frac{1}{2} + \mathbb{Z}$ the Hilbert space of the two-dimensional deformed (due to the presence, besides the quadratic potential, of a Calogero potential term) harmonic oscillator, can be formally recovered from the $\lambda = -\frac{1}{2}$ Figure 2 diagram, by replacing the angular momentum m with the j eigenvalues of the $\mathcal{J}$ operator introduced in (96) (this is also true for the $\lambda = \frac{1}{2}, -\frac{3}{2}$ cases explicitly introduced in Figure 1 and 3).

Let us introduce the basis defined by the quantum numbers

$$\hat{D}\,|n, j, \epsilon\rangle = (n + 1)\,|n, j, \epsilon\rangle\,; \quad \hat{\mathcal{J}}\,|n, j, \epsilon\rangle = j\,|n, j, \epsilon\rangle\,, (j \in \mathbb{Z}); \quad N_f\,|n, j, \epsilon\rangle = \epsilon\,|n, j, \epsilon\rangle\,, (\epsilon = \pm 1).$$

In this basis the action of $\hat{Q}^{(I)}_{\pm}$, $\overline{Q}^{(I)}_{\pm}$ on a state which does not coincide with a lowest weight state, reads as follows

$$\hat{Q}^{(I)}_{\pm}\,|n, j, \epsilon\rangle \propto |n \mp 1, j + \epsilon, -\epsilon\rangle\,, \quad \overline{Q}^{(I)}_{\pm}\,|n, j, \epsilon\rangle \propto |n \mp 1, j - \epsilon, -\epsilon\rangle\,. \tag{A.4}$$

The $\lambda = \frac{1}{2} + \mathbb{Z}$ associated diagrams are presented in Figure 4.

This makes clear that the superselection rule induced by (A.2) can be imposed on any $\lambda = \frac{1}{2} + \mathbb{Z}$ deformed oscillator, guaranteeing in all these cases the existence of a Hilbert space with a single bosonic vacuum.

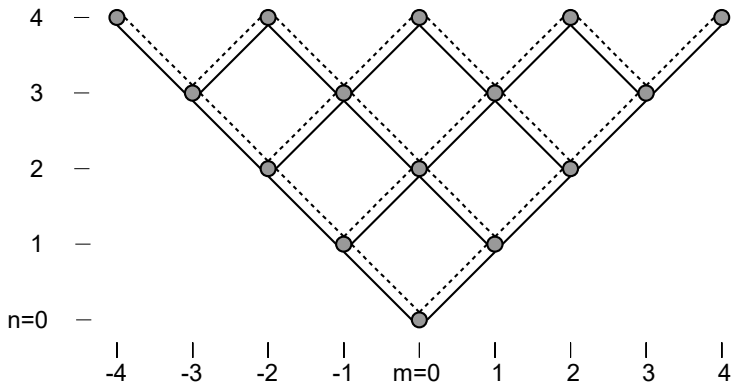

Figure 2: $\lambda = -\frac{1}{2}$ diagram of $\widehat{Q}$'s, $\overline{Q}$'s raising and lowering operators..

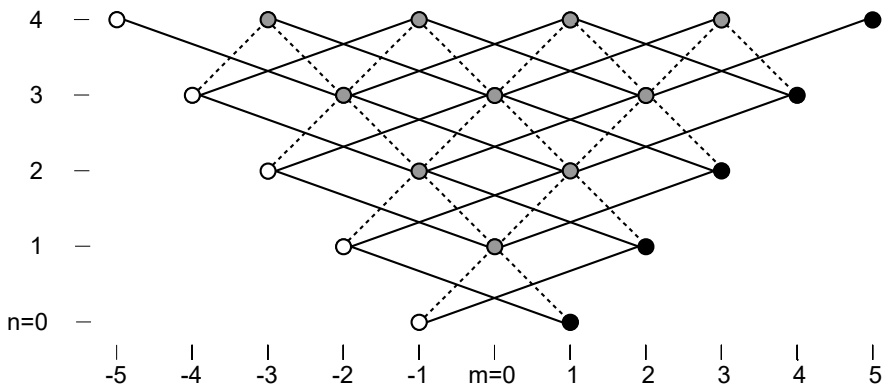

Figure 3: $\lambda = -\frac{3}{2}$ diagram of $\widehat{Q}$'s, $\overline{Q}$'s raising and lowering operators.

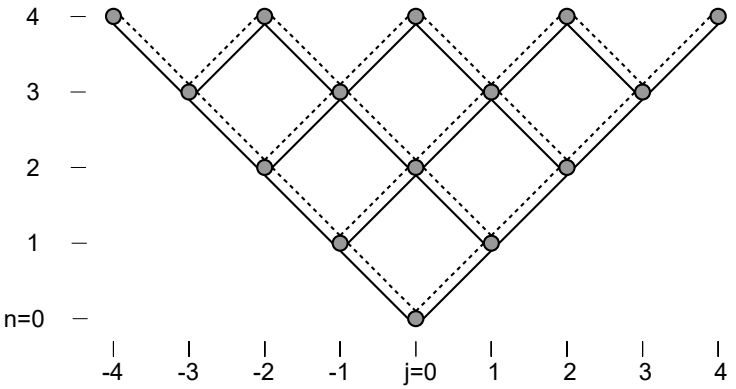

Figure 4: the $\lambda = \frac{1}{2} + \mathbb{Z}$ general diagram.

Appendix B: The classical $(2,2,0)$ $sl(2|1)$-invariant models.

We present, for completeness, the construction of the $sl(2|1)$-invariant classical actions obtained from, respectively, the parabolic and the trigonometric D-module reps acting on the $(2,2,0)$ supermultiplet.

The parabolic D-module rep is given by the transformations

$$\begin{aligned}
&L_n x_i = t^n \left(t\dot{x}_i + (n+1)\,\lambda x_i\right), && L_n\psi_i = t^n(t\dot{\psi}_i + (n+1)\,(\frac{2\lambda+1}{2})\psi_i),\ n=0,\pm1;\\
&Jx_i = -\lambda\epsilon_{ij}x_j, && J\psi_i = -\frac{2\lambda-1}{2}\epsilon_{ij}\psi_j,\\
&Q^1_\pm x_i = t^{\frac{1\pm1}{2}}\epsilon_{ij}\psi_j, && Q^1_\pm\psi_i = -it^{\frac{1\pm1}{2}}\epsilon_{ij}\left(t\dot{x}_j + (1\pm1)\,\lambda x_j\right),\\
&Q^2_\pm x_i = t^{\frac{1\pm1}{2}}\psi_i, && Q^2_\pm\psi_i = it^{\frac{1\pm1}{2}}\left(t\dot{x}_i + (1\pm1)\,\lambda x_i\right),
\end{aligned} \tag{B.1}$$

where the x_i's $(i=1,2)$ are the propagating bosons and the ψ_i's the fermionic fields.

The above transformations close the $sl\,(2|1)$ superalgebra.

The $sl(2|1)$-invariant action is obtained from the Lagrangian $\mathcal{L} = Q^2_+Q^1_+(\frac{1}{2}F\epsilon_{ij}\psi_i\psi_j)$, with the operators Q^2_+, Q^1_+ acting on the prepotential $F = C(x_ix_i)^{-\frac{2\lambda+1}{2\lambda}}$ (C is a normalization constant). Explicitly, the invariant action of the classical $(2,2,0)$ parabolic model is

$$\mathcal{S} = \int dt\mathcal{L} = \int dt(F(\dot{x}_i\dot{x}_i - i\dot{\psi}_i\psi_i) - iF_i\dot{x}_j\psi_i\psi_j). \tag{B.2}$$

The trigonometric D-module rep is given by the transformations

$$\begin{aligned}
&L_n x_i = \frac{e^{-in\omega t}}{-i\omega}\left(\dot{x}_i - in\lambda\omega x_i\right), && L_n\psi_i = \frac{e^{-in\omega t}}{-i\omega}(\dot{\psi}_i - in(\frac{2\lambda+1}{2})\omega\psi_i),\ n=0,\pm1;\\
&Jx_i = -\lambda\epsilon_{ij}x_j, && J\psi_i = -\frac{2\lambda-1}{2}\epsilon_{ij}\psi_j,\\
&Q^1_\pm x_i = e^{\mp i\frac{\omega}{2}t}\epsilon_{ij}\psi_j, && Q^1_\pm\psi_i = \frac{e^{\mp i\frac{\omega}{2}t}}{i\omega}\epsilon_{ij}\left(\dot{x}_j \mp i\lambda\omega x_j\right),\\
&Q^2_\pm x_i = e^{\mp i\frac{\omega}{2}t}\psi_i, && Q^2_\pm\psi_i = \frac{e^{\mp i\frac{\omega}{2}t}}{-i\omega}\left(\dot{x}_i \mp i\lambda\omega x_i\right).
\end{aligned} \tag{B.3}$$

Without loss of generality we can set $\omega = 1$. The classical action, $sl(2|1)$-invariant under the (B.3) trigonometric transformations, is therefore given by

$$\mathcal{S} = \int dt\mathcal{L} = \int dt(F(\dot{x}_i\dot{x}_i - i\dot{\psi}_i\psi_i) - iF_i\dot{x}_j\psi_i\psi_j + C\lambda^2\,(x_ix_i)^{-\frac{1}{2\lambda}}). \tag{B.4}$$

Appendix C: On the "soft" supersymmetry of the oscillators.

We make here some comments on the role of superalgebras applied to oscillators (either the ordinary quantum oscillators or the oscillators which are "deformed" by the presence of a Calogero potential term).

The starting point is the famous work of Wigner [39]. In modern terms, after the concept of superalgebra was introduced in mathematics, Wigner's results can be reinterpreted (see [40]) according to the following lines. For the ordinary quantum oscillator, with creation/annihilation operators a, $a^\dagger$ (satisfying $[a, a^\dagger] = 1$) and symmetrized Hamiltonian $\mathcal{H} = \{a, a^\dagger\}$, we can assign odd-grading to the operators $a, a^\dagger$, so that they belong to a set of 5 operators, $a, a^\dagger, a^2, (a^\dagger)^2, \mathcal{H} = \{a, a^\dagger\}$, closing the $osp(1|2)$ superalgebra under (anti)commutations. The last three (bosonic) operators close the $sl(2)$ subalgebra. Under this construction we have an alternative point of view for describing the computation of the the spectrum of the ordinary (one-dimensional) harmonic oscillator: we can state that, instead of deriving it from the Fock vacuum $|0\rangle$, annihilated by a ($a|0\rangle = 0$), the spectrum is obtained from a lowest weight representation of $osp(1|2)$, the Hamiltonian being the Cartan element. By adopting this viewpoint the superalgebra $osp(1|2)$ becomes a spectrum-generating superalgebra for the ordinary quantum oscillator, with its Hilbert space being recovered from a single, irreducible, $osp(1|2)$ lowest weight representation.

One should note that the bosonic $sl(2)$ subalgebra also acts as a spectrum-generating algebra for the harmonic oscillator. The Hilbert space of the harmonic oscillator is, however, reducible under the $sl(2)$ decomposition. It is given by the direct sum of two irreducible $sl(2)$ lowest-weight representations. The first lowest state is the vacuum of the theory (proportional to the gaussian e^{-x^2} under proper conventions and normalization). The other lowest state is the first excited state, with eigenfunction proportional to xe^{-x^2} and having odd-parity with respect to the $x \mapsto -x$ transformation. The two $sl(2)$ lowest weight reps correspond to, respectively, the even-parity and the odd-parity energy eigenstates. The role of the fermionic operators in $osp(1|2)$ consists in connecting energy eigenstates of even and odd parity.

After the introduction and the subsequent classification of simple Lie superalgebras [41, 42], the Wigner's approach was advocated in [43], with special emphasis on parastatistics, prompting a series of investigations on lowest weight representations of simple Lie superalgebras (for a recent review see, e.g., [44]).

On a separate development the DFF "trick" of introducing oscillator damping potentials in conformal mechanics relates oscillators (with/without the Calogero potential term) to conformal algebras.

It was recognized in [28] that, due to the DFF "trick", the introduction of new potentials for conformal mechanics became possible. The two aspects, superalgebra versus conformal algebra, were reconciled in [1]. The notion of parabolic versus trigonometric/hyperbolic D-module reps of superconformal algebras was pointed out, with the latter class describing the (deformed or undeformed) oscillators and bounded from below potentials in the trigonometric case.

The main property shared by the two big classes of superconformal theories, parabolic versus trigonometric, is that at the classical level their respective actions are superconformally invariant. Concerning their differences:
i) the parabolic models are, both classically and quantum, superconformal and supersymmetric. The supersymmetry implies the existence of a symmetry operator $\mathcal{Q}$ which is the "square root" of the Hamiltonian $\mathcal{H}$, namely $\mathcal{Q}^2 = \mathcal{H}$;
ii) the trigonometric models, on the other hand, despite being superconformally invariant, are not supersymmetric. In this case symmetry operators $\mathcal{Q}$, $\mathcal{Z}$ exist such that $\mathcal{Q}^2 = \mathcal{Z}$. The key point is that the operator $\mathcal{Z}$ does not coincide with the Hamiltonian: $\mathcal{Z} \neq \mathcal{H}$.

One can easily say that the trigonometric models are "intermediate" between the supersymmetric and the non-supersymmetric theories. This "intermediate notion of supersymmetry", namely $\mathcal{Q}^2 = \mathcal{Z} \neq \mathcal{H}$, has no special name in the literature. In [1] the notion of "weak supersymmetry" was employed, borrowing the term from a construction described in [45] which shares a similar feature. The use of the term "weak supersymmetry", however, could be misleading

since the models in [45] are not based on superconformal algebras. In that paper a "weak supersymmetric oscillator" is discussed that has no relation with the oscillators derived from the trigonometric D-module reps of superconformal algebras.

For this reason it seems more appropriate to denote this important class of trigonometric models (which include, as shown in this paper, the ordinary one-dimensional and two-dimensional harmonic oscillators) as "softly supersymmetric". As far as we know the term "soft supersymmetry" has not been employed in a different context, making this term both suitable and available to describe the special properties of the trigonometric superconformal mechanics.

The softly supersymmetric trigonometric models are characterized by
i) classical superconformal invariance of the action;
ii) spontaneous breaking of the superconformal invariance. Indeed, in the simplest application, the Fock vacuum $|0\rangle$ of the harmonic oscillator is annihilated by a and not by the hermitian operator $a + a^\dagger$: $(a + a^\dagger)|0\rangle \neq 0$;
iii) in the quantum case the role of the superconformal algebra is that of a spectrum-generating superalgebra.

Concerning the last point, we indeed proved, see Appendix **A**, that the spectrum of the ordinary two-dimensional oscillator is decomposed into an infinite tower of $sl(2|1)$ irreducible lowest weight representations. The puzzling presence of the extra fermionic generators (93) which connect eigenstates belonging to different lowest weight reps reminds the role, just discussed above, played by the $osp(1|2)$ fermionic generators in connecting the two $sl(2)$ lowest weight reps of the one-dimensional oscillator.

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