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Exact solution of a generalized two-sites Bose-Hubbard model

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Abstract. I introduce a new parametrization of a bosonic Lax operator for the algebraic Bethe ansatz method with the $gl(2)$-invariant R-matrix and use it to present the exact solution of a generalized two-sites Bose-Hubbard model with asymmetric tunnelling. In the no interaction limit I show that the Bethe ansatz equations can be written as a S^{N-1} sphere, where N is the total number of atoms in the condensate.

1. Introduction

The first experimental verification of the Bose-Einstein condensation (BEC) [1–3] occurred more then seven decades after its theoretical prediction [4, 5], and a great deal of progress has been in the theoretical and experimental study of this many body physical phenomenon [6–11]. In this direction the algebraic Bethe ansatz method has been used to solve and study models that may describe BEC [12–14]. The quantum phase transitions and classical analysis of some of these models have been studied in [15–17]. We are considering here a generalized issue of the two-sites Bose-Hubbard model, also known as the *canonical Josephson Hamiltonian* [7], that has been an useful model in understanding tunnelling phenomena using two BEC [18–24]. The model that we will study is more general that the model [7, 25] in the sense that we introduce the on-well energies and asymmetric tunnelling. Here we will discuss its integrability and exact solution. The generalized model is described by the Hamiltonian

$$\hat{H} = \sum_{i,j=1}^{2} K_{ij}\hat{N}_i\hat{N}_j - \sum_{i=1}^{2}(U_i - \mu_i)\hat{N}_i - \sum_{i\neq j}^{2}\Omega_{ij}\hat{a}_i^{\dagger}\hat{a}_j, \qquad (1)$$

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where, $\hat{a}_i^\dagger(\hat{a}_i)$, denote the single-particle creation (annihilation) operators and $\hat{N}_i = \hat{a}_i^\dagger \hat{a}_i$ are the corresponding boson number operators in each condensate. The boson operator total number of particles, $\hat{N} = \hat{N}_1 + \hat{N}_2$, is a conserved quantity, $[\hat{H}, \hat{N}] = 0$. The couplings K_{ij}, with $K_{ij} = K_{ji}$ $(i \neq j)$, provides the interaction strength between the bosons and they are proportional to the s-wave scattering length, Ω_{ij} are the amplitude of tunnelling, μ_i are the external potentials and $U_i = K_{ii} - \kappa_i$ are the on-well energies per particle, with κ_i the kinetic energies in each condensate.

The Hamiltonian (1) is invariant under the discrete $\mathbb{Z}_2$ mirror transformation, $\hat{a}_j \to -\hat{a}_j$, and under the global $U(1)$ gauge transformation, $\hat{a}_j \to e^{i\alpha}\hat{a}_j$, where α is an arbitrary c-number and, $\hat{a}_j^\dagger \to e^{-i\alpha}\hat{a}_j^\dagger$, $\;j = 1, 2$. For $\alpha = \pi$ we get again the $\mathbb{Z}_2$ symmetry.

For the particular choice of the couplings parameters we can get some Hamiltonians, as for example by the choices $K_{ii} = \frac{K}{8}$, $K_{12} = K_{21} = -\frac{K}{8}$, $\mu_1 = -\mu_2 = \mu$, $U_i = 0$, and $\Omega_{12} = \Omega_{21} = \frac{\mathcal{E}_J}{2}$ we get the canonical Josephson Hamiltonian studied in [7]. The case with $K_{12} = K_{21} = 0$, $K_{ii} = U_i = U$, $\mu_1 = -\mu_2 = \mu$, and $\Omega_{12} = \Omega_{21} = t$ was used to study the interplay between disorder and interaction [26]. For these models we have symmetric tunnelling if $\Delta\mu = 0$ and when we turn on $\Delta\mu$ we break the symmetry. For the symmetric case we also can put $\mu_1 = \mu_2 = \mu$ and change the deep of both wells at the same time. In the antisymmetric case $U_1 - \mu_1 \neq U_2 - \mu_2$ we have asymmetric tunnelling with the bias of one well increasing the on-well energy. In this case it is called a tilted two-wells potential [27] and an experimental set up was made to study the distillation of a Bose-Einstein condensate, providing a model system for metastability in condensates, a test for quantum kinetic theories of condensate formation [28] and atomtronic devices [29, 30]. The on-well energies is determined by the internal states of the atoms in the condensates and/or by the kinetic (thermal) energy of the atoms.

In the Fig. 1 we represent the two BEC by a two-wells potential for the case $U_1 = U_2$ and $\Delta\mu \neq 0$, with asymmetric tunnelling $\Omega_{12} \neq \Omega_{12}$. The tunnelling amplitudes Ω_{ij} are related to the barrier height V_0 [31].

2. The algebraic Bethe ansatz method

The spectrum of the Hamiltonian (1), with $U_i = 0$ and symmetric tunnelling, has been appeared in different papers [24, 25, 32–35] and the Bethe states in [36]. To be complete I will shortly describe the algebraic Bethe ansatz method [25,33] and present the exact solution for the general case of $U_i \neq 0$ and asymmetric tunnelling $\Omega_{12} \neq \Omega_{12}$. We begin with the $gl(2)$-invariant R-matrix, depending on the spectral parameter u,

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

with $b(u) = u/(u+\eta)$, $c(u) = \eta/(u+\eta)$ and $b(u) + c(u) = 1$. Above, η is an arbitrary parameter, to be chosen later.

It is easy to check that $R(u)$ satisfies the Yang-Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v), \tag{3}$$

where $R_{jk}(u)$ denotes the matrix acting non-trivially on the j-th and the k-th spaces and as the identity on the remaining space.

Next we define the monodromy matrix $\hat{T}(u)$,

$$\hat{T}(u) = \begin{pmatrix} \hat{A}(u) & \hat{B}(u) \\ \hat{C}(u) & \hat{D}(u) \end{pmatrix}, \tag{4}$$

such that the Yang-Baxter algebra is satisfied

$$R_{12}(u-v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)R_{12}(u-v). \tag{5}$$

In what follows we will choose a realization for the monodromy matrix $\pi(\hat{T}(u)) = \hat{L}(u)$ to obtain a solution for the two-sites BEC model (1). In this construction, the Lax operator $\hat{L}(u)$ have to satisfy the algebra

$$R_{12}(u-v)\hat{L}_1(u)\hat{L}_2(v) = \hat{L}_2(v)\hat{L}_1(u)R_{12}(u-v), \tag{6}$$

where we use the notation,

$$\hat{L}_1 = \hat{L}(u) \otimes \hat{I} \quad \text{and} \quad \hat{L}_2 = \hat{I} \otimes \hat{L}(u). \tag{7}$$

Then, defining the transfer matrix, as usual, through

$$\hat{t}(u) = \mathrm{Tr}\ \pi(\hat{T}(u)) = \pi(\hat{A}(u) + \hat{D}(u)), \tag{8}$$

it follows from (5) that the transfer matrix commutes for different values of the spectral parameter; i. e.,

$$[\hat{t}(u), \hat{t}(v)] = 0, \qquad \forall\ u,\ v. \tag{9}$$

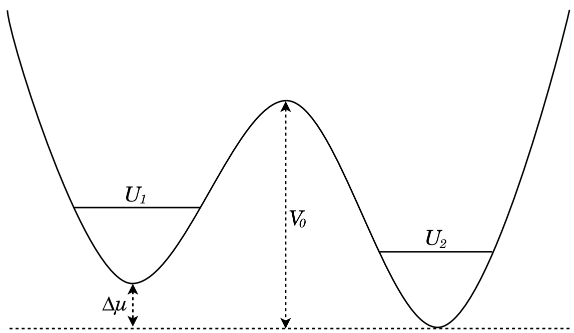

FIGURE 1. Two-wells potential showing the asymmetric tunnelling for the case $U_1 = U_2$, $\Delta\mu \neq 0$ and barrier height V_0.

Consequently, the models derived from this transfer matrix will be integrable. Another consequence is that the coefficients $\hat{\mathcal{C}}_k$ in the transfer matrix $\hat{t}(u)$,

$$\hat{t}(u) = \sum_k \hat{\mathcal{C}}_k u^k, \tag{10}$$

are conserved quantities or simply c-numbers, with

$$[\hat{\mathcal{C}}_j, \hat{\mathcal{C}}_k] = 0, \qquad \forall\ j,\ k. \tag{11}$$

If the transfer matrix $\hat{t}(u)$ is a polynomial function in u, with $k \geq 0$, it is easy to see that,

$$\hat{\mathcal{C}}_0 = \hat{t}(0) \quad \text{and} \quad \hat{\mathcal{C}}_k = \frac{1}{k!}\left.\frac{d^k \hat{t}(u)}{du^k}\right|_{u=0}. \tag{12}$$

We are using a new solution of the equation (6), a new parametrization of a well known [24, 33] Lax operator,

$$\hat{L}_i(u) = \begin{pmatrix} \lambda_i(u\hat{I} + \eta\hat{N}_i) & \alpha_i \hat{a}_i \\ \beta_i \hat{a}_i^\dagger & \alpha_i\beta_i\gamma_i\eta^{-1}\hat{I} \end{pmatrix} \qquad i = 1, 2, \tag{13}$$

for the boson operators $\hat{a}_i^\dagger$, $\hat{a}_i$, and $\hat{N}_i$ and with $\lambda_i\gamma_i = 1$. The parameters α_i and β_i are arbitrary. These operators obey the canonical boson commutation rules

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, \qquad [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}\hat{I}, \tag{14}$$

$$[\hat{N}_i, \hat{a}_j] = -\delta_{ij}\hat{a}_j, \qquad [\hat{N}_i, \hat{a}_j^\dagger] = +\delta_{ij}\hat{a}_j^\dagger. \tag{15}$$

The $\hat{I}$-operator is the identity operator.

Using the co-multiplication property of the Lax operators (13) we get the following realization for the monodromy matrix,

$$\pi(\hat{T}(u)) = \hat{L}_1(u + \omega_1)\hat{L}_2(u - \omega_2), \tag{16}$$

whose entries are,

$$\begin{aligned} \pi(\hat{A}(u)) &= \lambda_1\lambda_2(u+\omega_1)(u-\omega_2)\hat{I} + \lambda_1\lambda_2(u+\omega_1)\eta\hat{N}_2 \\ &+ \lambda_1\lambda_2(u-\omega_2)\eta\hat{N}_1 + \lambda_1\lambda_2\eta^2\hat{N}_1\hat{N}_2 + \beta_2\alpha_1\hat{a}_2^\dagger\hat{a}_1, \end{aligned} \tag{17}$$

$$\pi(\hat{B}(u)) = \lambda_1\alpha_2[(u+\omega_1)\hat{I} + \eta\hat{N}_1]\hat{a}_2 + \alpha_1\alpha_2\beta_2\gamma_2\eta^{-1}\hat{a}_1, \tag{18}$$

$$\pi(\hat{C}(u)) = \beta_1\lambda_2[(u-\omega_2)\hat{I} + \eta\hat{N}_2]\hat{a}_1^\dagger + \alpha_1\beta_1\beta_2\gamma_1\eta^{-1}\hat{a}_2^\dagger, \tag{19}$$

$$\pi(\hat{D}(u)) = \beta_1\alpha_2\hat{a}_1^\dagger\hat{a}_2 + \alpha_1\alpha_2\beta_1\beta_2\gamma_1\gamma_2\eta^{-2}\hat{I}. \tag{20}$$

Hereafter we will use the same symbol for the operators and its respective realization, so we define $\pi(\hat{O}(u)) \equiv \hat{O}(u)$ for any operator in the entries of the monodromy matrix (4).

The transfer matrix $\hat{t}(u)$ is,

$$\begin{aligned}\hat{t}(u) &= \lambda_1\lambda_2(u+\omega_1)(u-\omega_2)\hat{I} + \lambda_1\lambda_2(u+\omega_1)\eta\hat{N}_2 \\ &+ \lambda_1\lambda_2(u-\omega_2)\eta\hat{N}_1 + \lambda_1\lambda_2\eta^2\hat{N}_1\hat{N}_2 + \beta_2\alpha_1\hat{a}_2^\dagger\hat{a}_1 \\ &+ \beta_1\alpha_2\hat{a}_1^\dagger\hat{a}_2 + \alpha_1\alpha_2\beta_1\beta_2\gamma_1\gamma_2\eta^{-2}\hat{I}. \end{aligned} \tag{21}$$

We can write the transfer matrix (21) using (10)

$$\hat{t}(u) = \hat{\mathcal{C}}_0 + \hat{\mathcal{C}}_1 u + \hat{\mathcal{C}}_2 u^2, \tag{22}$$

with the conserved quantities

$$\begin{aligned}\hat{\mathcal{C}}_0 &= \lambda_1\lambda_2(\omega_1\hat{N}_2 - \omega_2\hat{N}_1)\eta + \lambda_1\lambda_2\eta^2\hat{N}_1\hat{N}_2 \\ &+ \beta_2\alpha_1\hat{a}_2^\dagger\hat{a}_1 + \beta_1\alpha_2\hat{a}_1^\dagger\hat{a}_2 + (\alpha_1\alpha_2\beta_1\beta_2\gamma_1\gamma_2\eta^{-2} - \lambda_1\lambda_2\omega_1\omega_2)\hat{I}, \quad (23)\\ \hat{\mathcal{C}}_1 &= \lambda_1\lambda_2[(\omega_1-\omega_2)\hat{I} + \eta\hat{N}], \quad (24)\\ \hat{\mathcal{C}}_2 &= \lambda_1\lambda_2\hat{I}. \quad (25)\end{aligned}$$

We can rewrite the Hamiltonian 1 using these conserved quantities,

$$\hat{H} = \xi_0\hat{\mathcal{C}}_0 + \xi_1\hat{\mathcal{C}}_1^2 + \xi_2\hat{\mathcal{C}}_2, \tag{26}$$

with the following identification for the parameters,

$$\begin{aligned}\xi_2 &= -\xi_0(\alpha_1\alpha_2\beta_1\beta_2\gamma_1^2\gamma_2^2\eta^{-2} - \omega_1\omega_2) \\ &- \xi_1(\omega_1-\omega_2)^2\lambda_1\lambda_2, \quad (27)\\ K_{11} &= K_{22} = \xi_1\lambda_1^2\lambda_2^2\eta^2, \quad (28)\\ K_{12} &= K_{21} = (\xi_0 + 2\xi_1\lambda_1\lambda_2)\lambda_1\lambda_2\eta^2, \quad (29)\\ \mu_1 - U_1 &= [2\xi_1\lambda_1\lambda_2\omega_1 - (\xi_0 + 2\xi_1\lambda_1\lambda_2)\omega_2]\lambda_1\lambda_2\eta, \quad (30)\\ \mu_2 - U_2 &= [(\xi_0 + 2\xi_1\lambda_1\lambda_2)\omega_1 - 2\xi_1\lambda_1\lambda_2\omega_2]\lambda_1\lambda_2\eta, \quad (31)\\ \Omega_{12} &= -\xi_0\beta_1\alpha_2, \quad (32)\\ \Omega_{21} &= -\xi_0\beta_2\alpha_1, \quad (33)\end{aligned}$$

with $\xi_i \neq 0,\ i = 0, 1, 2$.

Now it is straightforward to check that the Hamiltonians (1) and (26) are related to the transfer matrix $\hat{t}(u)$ (21) through

$$\hat{H} = \xi_0\hat{t}(u) + \xi_1\hat{\mathcal{C}}_1^2 - \xi_0\hat{\mathcal{C}}_1 u - (\xi_0 u^2 - \xi_2)\hat{\mathcal{C}}_2, \tag{34}$$

and from 26 or 34 that $[\hat{H}, \hat{t}(u)] = 0$. Notice that the spectral parameter u appearing in the Hamiltonian (34) is canceled. The Hamiltonian parameters in (1) or (34) are real numbers. The transfer matrix parameters in (21) can be complex numbers, but in this case the transfer matrix is not Hermitian. We will only consider the Hermitian case.

We can apply the algebraic Bethe ansatz method, using the Fock vacuum as the pseudo-vacuum $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$, to find the BAE

$$\frac{\lambda_1\lambda_2[v_i^2 + (\omega_1-\omega_2)v_i - \omega_1\omega_2]}{\alpha_1\alpha_2\beta_1\beta_2\gamma_1\gamma_2\eta^{-2}} = \prod_{j\neq i}^{N} \frac{v_i - v_j - \eta}{v_i - v_j + \eta}, \qquad i, j = 1, \ldots, N, \tag{35}$$

and the energies of the Hamiltonian

$$\begin{aligned}E(\{v_i\}) &= \xi_0\lambda_1\lambda_2[u^2+(\omega_1-\omega_2)u-\omega_1\omega_2]\prod_{i=1}^{N}\left(1+\frac{\eta}{v_i-u}\right)\\&+ \xi_1\lambda_1^2\lambda_2^2(\omega_1-\omega_2+\eta N)^2-\xi_0\lambda_1\lambda_2(\omega_1-\omega_2+\eta N)u\\&- \xi_0\lambda_1\lambda_2u^2+\xi_2\lambda_1\lambda_2+\xi_0\alpha_1\alpha_2\beta_1\beta_2\gamma_1\gamma_2\eta^{-2}\prod_{i=1}^{N}\left(1-\frac{\eta}{v_i-u}\right).\end{aligned} \tag{36}$$

Fortunately this expression is a function of the spectral parameter u, which can be chosen arbitrarily. For asymmetric tunnelling, $\Omega_{12}\neq\Omega_{21}$, we can consider $\alpha_1\beta_2=\eta$ and $\beta_1\alpha_2=\kappa\eta$ and in the limit of no interaction, $K_{ij}\to 0$ with $\eta\ll 1$, we can write the Bethe ansatz equation (35) as

$$\sum_{i=1}^{N}\left[v_i+\frac{1}{2}(\omega_1-\omega_2)\right]^2=R_N^2. \tag{37}$$

The Eq. (37) is the equation of a complex manifold in $\mathbb{C}^N$ with

$$R_N=\sqrt{\left[\frac{1}{4}(\omega_1-\omega_2)^2+\frac{(\kappa+\lambda_1^2\lambda_2^2\omega_1\omega_2)}{\lambda_1^2\lambda_2^2}\right]N}. \tag{38}$$

If all Bethe roots $\{v_i\}$ are real numbers, in $\mathbb{R}^N$ the surface is a S^{N-1} sphere with radii R_N and center in

$$v_i=-\frac{1}{2}(\omega_1-\omega_2),\qquad \forall i=1,\ldots,N. \tag{39}$$

For $u=0$ and $K_{ij}\to 0$ we can write the eigenvalues as

$$\begin{aligned}E(\{v_i\}) &= \xi_1\lambda_1^2\lambda_2^2(\omega_1-\omega_2+\eta N)^2+\xi_2\lambda_1\lambda_2+\xi_0(\kappa\gamma_1\gamma_2-\lambda_1\lambda_2\omega_1\omega_2)\\&- \xi_0(\lambda_1\lambda_2\omega_1\omega_2+\kappa\gamma_1\gamma_2)\eta\sum_{i=1}^{N}\frac{1}{v_i}.\end{aligned} \tag{40}$$

When we consider symmetric tunnelling we just put $\kappa=1$ and $\omega_1=-\omega_2$ to get

$$\begin{aligned}E(\{v_i\}) &= \xi_1\lambda_1^2\lambda_2^2(2\omega_1+\eta N)^2+\xi_2\lambda_1\lambda_2+\xi_0(\gamma_1\gamma_2+\lambda_1\lambda_2\omega_1^2)\\&- \xi_0(\gamma_1\gamma_2-\lambda_1\lambda_2\omega_1^2)\eta\sum_{i=1}^{N}\frac{1}{v_i}.\end{aligned} \tag{41}$$

3. Summary

I have introduced a new parametrization of a bosonic Lax operator and explicitly calculated the spectrum of a generalized two-sites Bose-Hubbard model with asymmetric tunnelling by the algebraic Bethe ansatz method using the $gl(2)$-invariant R-matrix and showed that in the no interaction limit the Bethe ansatz equations can be written as a S^{N-1} sphere in $\mathbb{R}^N$, where N is the total number of atoms in the condensate.

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