$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie Symmetries of the Lévy-Leblond Equations

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Abstract

The first-order differential Lévy-Leblond equations (LLE's) are the non-relativistic analogs of the Dirac equation, being square roots of $(1+d)$-dimensional Schrödinger or heat equations. Just like the Dirac equation, the LLE's possess a natural supersymmetry. In previous works it was shown that non supersymmetric PDE's (notably, the Schrödinger equations for free particles or in the presence of a harmonic potential), admit a natural $\mathbb{Z}_2$-graded Lie symmetry.

In this paper we show that, for a certain class of supersymmetric PDE's, a natural $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie symmetry appears. In particular, we exhaustively investigate the symmetries of the $(1+1)$-dimensional Lévy-Leblond Equations, both in the free case and for the harmonic potential. In the free case a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra, realized by first and second-order differential symmetry operators, is found. In the presence of a non-vanishing quadratic potential, the Schrödinger invariance is maintained, while the $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$- graded extensions are no longer allowed.

The construction of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie symmetry of the $(1+2)$-dimensional free heat LLE introduces a new feature, explaining the existence of first-order differential symmetry operators not entering the super Schrödinger algebra.

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1 Introduction

In this paper we prove the existence of a finite $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie symmetry, realized by first-order and second-order differential operators, of the free generalized Lévy-Leblond equations (the $(1+1)$- and $(1+2)$-dimensional cases are here explicitly discussed).

It was pointed out in [1] that finite Lie superalgebras can be symmetries for a certain class of purely bosonic partial differential equations (including the cases of the free particle and of the harmonic oscillator)*. The recognition that a symmetry superalgebra is present was later applied [3] in the context of Conformal Galilei Algebras to identify new bosonic invariant partial differential equations.

It is therefore natural to pose the question: what happens in the case of a supersymmetric system, namely one which already possesses a $\mathbb{Z}_2$-graded structure? Is, in that case, a second $\mathbb{Z}_2$-gradation present? To give an answer we started investigating the generalized Lévy-Leblond equations, which are the non-relativistic analogs of the Dirac equations, being the square roots of the heat or of the Schrödinger equation (with or without potential terms). The original Lévy-Leblond equation [4] is the non-relativistic wave equation of a spin-$\frac{1}{2}$ particle in the ordinary $(1+3)$-dimensional space-time and possesses a natural supersymmetry. In Section **2** we discuss at length the generalized Lévy-Leblond equations, induced by first-order matrix differential operators, and their construction from the relevant Clifford algebras.

This investigation about graded symmetries requires a preliminary understanding of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie (super)algebra structures. Generalizations of Lie and super-Lie algebras were introduced and named *color* algebras (for certain resemblances to parastatistics) in [5, 6]. They were further investigated in [7, 8]. Nowadays there is a small body of literature about these structures, dealing with possible physical applications (see, e.g., [9, 10, 11, 12]) and a larger number of works devoted to their mathematical development (for more recent papers see, e.g., [13, 14] and references therein). The absence of a spin-statistics connection in the relativistic setting (since we are working in a non-relativistic setting, this is not a concern for us) has been the major impediment for the full development of the topic of generalized supersymmetries. To make this paper self-contained, the relevant properties of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded color Lie (super)algebras are presented in an Appendix.

Our investigation requires the further notion, following [15], of *symmetry operator*. We consider two classes of symmetries operators; they can be recovered either from commutators or from anticommutators, see equations (14) and, respectively, (15). Quite interestingly, we need symmetry operators belonging to both classes in order to produce a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry.

In this work we computed the complete list of symmetry operators for the Lévy-Leblond square root of the free heat equation in $(1+1)$-dimensions (a 2×2 matrix operator) and the square root of the free Schrödinger equation in $(1+1)$-dimensions (a 4×4 matrix operator in the real counting). We also computed the symmetry operators recovered from commutators for the Lévy-Leblond square root of the heat equation with quadratic potential (a 4×4 matrix operator). Once identified the symmetry operators, we investigated the closed, finite (graded) Lie symmetry algebras induced by them. We proved, in particular, that the Lévy-Leblond square roots of the $(1+1)$-dimensional free heat and free Schrödinger equations possess a super Schrödinger symmetry algebra with maximal $\mathcal{N}=1$ extension (super Schrödinger algebras were discussed in [16, 17]). The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie symmetry $\mathcal{G}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ of first and second-order

*The fact that a Lie superalgebra appears even in a purely bosonic setting is not so surprising. Indeed, for the harmonic oscillator, the old results of [2] can be expressed, in modern language, by stating that the Fock vacuum of creation/annihilation operators can be replaced by a lowest weight representation of an $osp(1|2)$ spectrum-generating superalgebra.

differential operators, spanned by the 13 generators in (53), is found.

The situation is quite different for the square root of the heat equation with quadratic potential. A Schrödinger symmetry algebra is still present. We proved that it can not, on the other hand, be extended to a super Schrödinger symmetry algebra.

In Appendix **B** we present the construction of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie symmetry superalgebra of the Lévy-Leblond operator associated with the $(1+2)$-dimensional free heat equation. A new feature emerges. The $\mathbb{Z}_2\times\mathbb{Z}_2$-graded symmetry includes first-order differential symmetry operators which do not belong to the two-dimensional super Schrödinger algebra.

We postpone to the Conclusions a more detailed summary of our results, with comments and a discussion of their implications.

The scheme of the paper is as follows. In Section **2** we introduce the generalized Lévy-Leblond operators and their relation to Clifford algebras. In Section **3** we introduce, following [15], the notion of *symmetry operators*. In Section **4** the full list of symmetry operators of the Lévy-Leblond square roots of the free heat and free Schrödinger equation in $(1+1)$-dimensions is presented. Finite (graded) Lie algebras induced by these symmetry operators (including the $\mathcal{N}=1$ super Schrödinger algebra and the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded symmetry) are presented in Section **5**. Section **6** is devoted to the symmetry of the Lévy-Leblond square root of the $(1+1)$-dimensional heat equation with quartic potential. A summary of our results is given in the Conclusions. A self-contained presentation of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded color Lie (super)algebras (and their relation to quaternions and split-quaternions) is given in Appendix **A**. We present in Appendix **B** the construction of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded symmetry of the free $(1+2)$-dimensional Lévy-Leblond equation. In Appendix **C** we discuss the possibility of introducing different $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie (super)algebras, based on different assignments of gradings to given operators.

2 On Clifford algebras and generalized Lévy-Leblond equations

The original Lévy-Leblond equation [4] is the square root of the Schrödinger equation in $1+3$ dimensions. Generalized Lévy-Leblond equations are square roots of heat or Schrödinger equations in an arbitrary number of space dimensions; they can be systematically constructed from Clifford algebras irreducible representations. We introduce here the basic ingredients for this general scheme, focusing on issues such as complex structure, introduction of a potential term and so on.

We remind that the irreducible representations (over $\mathbb{R}$) of the $Cl(p,q)$ Clifford algebras (the enveloping algebras whose gamma-matrix generators satisfy $\gamma_i\gamma_j+\gamma_j\gamma_i=2\eta_{ij}$, where η_{ij} is a diagonal matrix with p positive $(+1)$ and q negative (-1) entries) can be obtained by tensoring four 2×2 matrices, see e.g. [18]. It is convenient to follow the presentation of [19]. The 2×2 matrices can be identified with (four) letters. General gamma matrices can be expressed, since no ambiguity arises, as words in a 4-letter alphabet by dropping the symbol of tensor product "$\otimes$". By expressing

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad X=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad Y=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad A=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\tag{1}$$

the split-quaternions, see Appendix **A**, can be represented as

$$\widetilde{e}_0=I,\quad \widetilde{e}_1=Y,\quad \widetilde{e}_2=X,\quad \widetilde{e}_3=A,\tag{2}$$

where X,Y,A are the gamma matrices realizing the $Cl(2,1)$ Clifford algebra.

The quaternions can be realized as 4×4 real matrices. With the adopted convention of dropping the tensor product symbol they can be presented as

$$e_0 = II, \quad e_1 = AI, \quad e_2 = XA, \quad e_3 = YA. \tag{3}$$

Some comments are in order:
i) a matrix is block-diagonal if, in its associated word, the first letter is either I or X. Conversely, it is block-antidiagonal if the first letter is Y or A,
ii) a matrix is (anti)symmetric, depending on the number (even or odd) of A's in its word,
iii) in real form the complex structure is defined by a real matrix J such that $J^2 = -\mathbb{I}$.
The complex-structure preserving matrices commute with J. If the complex structure is preserved, complex numbers can be used. A possible choice for the 4×4 real matrices complex structure is setting $J = IA$, since $(IA)^2 = -II = -\mathbb{I}_4$. Therefore, the eight 4×4 complex-structure preserving matrices are $II, XI, YI, AA, IA, XA, YA, AI$.

Depending on the (p, q) signature, the maximal number $p + q$ of Clifford gamma-matrix generators that can be accommodated in $2^n \times 2^n$ real matrices is

$$\begin{array}{rcl} 2\times 2 & : & Cl(2,1), \\ 4\times 4 & : & Cl(3,2), \quad Cl(0,3), \\ 8\times 8 & : & Cl(4,3), \quad Cl(5,0), \quad Cl(1,4), \quad Cl(0,7), \end{array} \tag{4}$$

and so on.

Clifford algebras can be used to introduce Supersymmetric Quantum Mechanics (SQM). In its simplest version (the one-dimensional, $\mathcal{N} = 2$ supersymmetry, with x as a space coordinate) the two supersymmetry operators Q_1, Q_2 need to be block anti-diagonal, hermitian and complex structure preserving first-order real differential operators. Moreover, they have to satisfy the $\mathcal{N} = 2$ SQM algebra

$$\{Q_i, Q_j\} = 2\delta_{ij} H, \qquad [H, Q_i] = 0, \tag{5}$$

where H is the Hamiltonian.

The irreducible representation (in real counting) requires 4×4 matrices. By setting the complex structure to be $J = IA$, the most general solution, for an arbitrary function $f(x)$ (the prepotential), can be expressed as

$$\begin{array}{rcl} Q_1 & = & AI\partial_x + YIf(x), \\ Q_2 & = & YA\partial_x + AAf(x), \\ H & = & II(-\partial_x^2 + f(x)^2) + XIf_x(x) = \mathbb{I}_4(-\partial_x^2 + f(x)^2) + N_f f_x(x). \end{array} \tag{6}$$

The hamiltonian H is diagonal. In its upper block the potential is $V_+(x) = f(x)^2 + f_x(x)$, while in its lower block the potential is $V_-(x) = f(x)^2 - f_x(x)$.

XI defines the fermion number operator N_f ($N_f = XI$). All operators act on two real component bosonic and two real component fermionic fields. For energy eigenvalues $E > 0$ the $H_\pm = -\partial_x^2 + f(x)^2 \pm f_x(x)$ Hamiltonians share the same spectrum. The free case is obtained by taking $f(x) = 0$; the harmonic oscillator case is recovered by taking $f(x) = cx$.

Since Q_1, Q_2, H commute with J and preserve the complex structure, they can also be expressed as 2×2 *complex* matrices.

The construction of a (generalized) Lévy-Leblond real differential operator requires the linear combination of two gamma matrices (one with a positive and the other one with a negative square) in order to produce a nilpotent operator which eliminates the ∂_t^2 term present in the relativistic Klein-Gordon equation.

The original Lévy-Leblond equation (the square root of the Schrödinger equation in $1+3$ dimensions), can be recovered from the $Cl(1,4)$ Clifford algebra, whose matrices act on 8 real component fields. Since $Cl(1,4)$ has a quaternionic (and therefore, *a fortiori*, complex) structure [18], the 8×8 real matrices preserving the complex structure can be described as 4×4 complex matrices acting on a multiplet of 4 component complex fields.
We call a first-order real matrix differential operator a "generalized Lévy-Leblond operator" if it is the square root of either the heat or the Schrödinger equation in $1+d$ dimensions (with or without a potential term). Further properties can be imposed. It could be required the operator to be block anti-diagonal and anticommuting with the fermion number operator, so that it mutually exchanges bosons into fermions.

The minimal matrix size to accommodate a Lévy-Leblond square root of the $1+1$ heat equation is 2. Indeed, we can set

$$\overline{\Omega}\Psi(x,t)=0,\quad \overline{\Omega}=\frac{1}{2}(A+Y)\partial_t+\frac{1}{2}(A-Y)\lambda+X\partial_x \quad\rightarrow\quad \overline{\Omega}^2=\mathbb{I}_2(-\lambda\partial_t+\partial_x^2), \qquad (7)$$

with λ an arbitrary real number.

One should note that $\overline{\Omega}$ is neither block-antidiagonal nor complex structure preserving. The minimal solution to have a block antidiagonal (and complex-structure preserving) operator requires 4×4 real matrices. A convenient basis for the five 4×4 $Cl(3,2)$ gamma matrices is given by AA, AX, AY, XI, YI. The complex structure can be defined by $J=AI$. The three complex-structure preserving matrices AA, AX, AY satisfy the $p=1$, $q=2$ gamma-matrix relations.

A block antidiagonal, complex structure preserving, square root of the free heat equation in $1+1$ dimensions is given by

$$\begin{aligned}\Omega_{heat,free} &= \frac{1}{2}(AA+AY)\partial_t+\frac{1}{2}(AA-AY)\lambda+AX\partial_x,\\ \Omega^2_{heat,free} &= II(\lambda\partial_t-\partial_x^2)=\mathbb{I}_4(\lambda\partial_t-\partial_x^2).\end{aligned} \qquad (8)$$

The introduction of a potential term requires the use of the extra antidiagonal gamma matrix YI. Therefore, in the presence of a non-vanishing potential, the 4×4 Lévy-Leblond operator cannot preserve the complex-structure. We have

$$\begin{aligned}\Omega_{heat} &= \frac{1}{2}(AA+AY)\partial_t+\frac{1}{2}(AA-AY)\lambda+AX\partial_x+YIf(x),\\ \Omega^2_{heat} &= II(\lambda\partial_t-\partial_x^2+f(x)^2)+XXf_x(x).\end{aligned} \qquad (9)$$

Due to the mentioned property, $\Omega_{heatfree}$ in (8) can be represented by 2×2 complex matrices, while this is not true for Ω_{heat} in (9) if $f(x)$ is non-vanishing.

The (free) Schrödinger equation requires the complex structure. The square root of the free Schrödinger equation can be obtained by modifying $\Omega_{heat,free}$ so that

$$\Omega_{Sch,free} = \frac{1}{2}(AA+AY)\cdot AI\partial_t+\frac{1}{2}(AA-AY)\lambda+AX\partial_x, \qquad (10)$$

where "$\cdot$" denotes the ordinary matrix multiplication.

Its square gives

$$\Omega^2_{Sch,free} = AI\lambda\partial_t-II\partial_x^2, \qquad (11)$$

namely a Schrödinger equation on 4-component real fields (which can be equivalently expressed as an equation for 2-component complex fields).

The introduction of a non-vanishing potential for the $1+1$ Schrödinger equation requires a Lévy-Leblond operator Ω_{Sch} acting on at least 8 real component fields. With the adopted conventions Ω_{Sch} can be expressed as

$$\begin{aligned}\Omega_{Sch} &= \frac{1}{2}(AAA+AAY)\cdot AII\partial_t+\frac{1}{2}(AAA-AAY)\lambda+AYI\partial_x+AAXf(x),\\ \Omega_{Sch}^2 &= -AII\lambda\partial_t+III(-\partial_x^2+f(x)^2)+IXXf_x(x),\end{aligned} \tag{12}$$

the complex structure being defined by $J=AII$.

It is straightforward to systematically construct, following this scheme and with the tools in [19], generalized Lévy-Leblond operators in $(1+d)$ dimensions.

3 On symmetries of matrix partial differential equations

Our investigation heavily relies on the notion of *symmetry operator* defined in [15]. We recall that a symmetry operator Z of a matrix partial differential equation induced by an operator Ω maps a given solution $\Psi(\vec{x})$ into another solution $Z\Psi(\vec{x})$, according to

$$\Omega\Psi(\vec{x})=0 \quad\Rightarrow\quad \Omega(Z\Psi(\vec{x}))|_{\Omega\Psi=0}=0. \tag{13}$$

Z can be any kind of operator. In the traditional Lie viewpoint the symmetry group of a differential equation is generated by a subset of operators which are closed under commutation. This condition is relaxed for superalgebras or $\mathbb{Z}_2\times\mathbb{Z}_2$ graded Lie algebras. Based on the grading of the symmetry operators, the closure requires both commutators and anticommutators.

If we restrict Z to be a differential operator of finite order, Z can be called a *symmetry operator* [15] if the following sufficient condition for symmetry is fulfilled:

either

$$[\Omega,Z] \;=\; \Phi_Z(\vec{x})\Omega, \tag{14}$$

or

$$\{\Omega,Z\} \;=\Phi_Z(\vec{x})\Omega, \tag{15}$$

where $\Phi_Z(\vec{x})$ is a $n\times n$ matrix-valued function of the $\vec{x}$ space(time) coordinates.

If there is no ambiguity (certain operators, like the identity $\mathbb{I}$, satisfy both (14) and (15)) throughout the text we denote with Σ's the symmetry operators satisfying (14) and with Λ's the symmetry operators satisfying (15).

In our applications we consider, initially, first-order differential symmetry operators. Second-order differentials symmetry operators are also constructed by taking suitable anticommutators of the previous operators.

It is important to note that, if Σ_1,Σ_2 are two operators satisfying (14), then by construction the identity

$$[\Omega,[\Sigma_1,\Sigma_2]] \;=\; \Phi_{[\Sigma_1,\Sigma_2]}\Omega \tag{16}$$

is satisfied. This does not imply, however, that the commutator $[\Sigma_1,\Sigma_2]$ is a symmetry operator according to the given definition. Indeed, $\Phi_{[\Sigma_1,\Sigma_2]}$ can be a differential operator and not necessarily a matrix-valued function. In the following, see e.g. (38,39), some examples are given. A Lie algebra of symmetry operators requires $\Phi_{[\Sigma_1,\Sigma_2]}$ to be a matrix-valued function.

4 Symmetries of the Lévy-Leblond square root of the free heat and Schrödinger equations

As discussed in Section **2**, the minimal realization of the Lévy-Leblond square root of the $1+1$-dimensional free heat (Schrödinger) equation requires 2×2 (and, respectively, 4×4) matrices.

We produce here the exhaustive list of its symmetry operators.

With respect to Section **2** we slightly change the notations in order to easily extrapolate the results from the heat to the Schrödinger equation. It is convenient to set, as 2×2 matrices,

$$\gamma_\pm = \frac{1}{2}(Y\pm A) \quad , \quad \gamma_3 = X, \tag{17}$$

where X, Y, A and the identity $I\equiv \mathbb{I}$ are introduced in (1). The following relations are satisfied

$$\gamma_\pm^2 = 0, \quad \gamma_3^2 = \mathbb{I}, \quad \gamma_\pm\gamma_\mp = \tfrac{1}{2}(\mathbb{I}\pm\gamma_3), \quad \gamma_3\gamma_\pm = \pm\gamma_\pm = -\gamma_\pm\gamma_3. \tag{18}$$

Our analysis of the symmetries only uses the (18) relations; therefore, the results below are representation-independent and can be extended to other representations (in particular to the case of 4×4 matrices). For 4×4 matrices we can introduce, following the convention of Section **2**, the $Cl(1,2)$ Clifford algebra generators

$$\widetilde{\gamma}_1 = AA, \quad \widetilde{\gamma}_2 = AY, \quad \widetilde{\gamma}_3 = AX. \tag{19}$$

They define the complex structure J given by

$$J \;=\; \widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3 = -AI, \qquad (J^2 = -II = -\mathbb{I}). \tag{20}$$

A realization of the (18) algebra in terms of 4×4 matrices is given by setting

$$\gamma_\pm = \pm\tfrac{1}{2}J\cdot(\widetilde{\gamma}_1\pm\widetilde{\gamma}_2), \quad \gamma_3 = J\cdot\widetilde{\gamma}_3. \tag{21}$$

By construction $\gamma_\pm, \gamma_3$ defined in (21) commute with J ($[J,\gamma_\pm] = [J,\gamma_3] = 0$).

The Lévy-Leblond operator Ω of the free heat equation is introduced (both in the 2×2 and in the 4×4 representations) as

$$\Omega \;=\; \gamma_+\partial_t + \gamma_-\lambda + \gamma_3\partial_x. \tag{22}$$

Its square is

$$\Omega^2 \;=\; \mathbb{I}\cdot(\lambda\partial_t + \partial_x^2). \tag{23}$$

According to the convention introduced in Section **3** we denote with Σ's the symmetry operators arising from commutators and with Λ's the symmetry operators arising from anticommutators

$$[\Sigma,\Omega] \;=\; \Phi_\Sigma\cdot\Omega, \tag{24}$$
$$\{\Lambda,\Omega\} \;=\; \Phi_\Lambda\cdot\Omega, \tag{25}$$

for some given matrix-valued function Φ_Σ or Φ_Λ in the x, t variables.

The computation of the symmetry operators is tedious but straightforward. In the 2×2 case the complete list (up to normalization) of first-order differential operators satisfying (24) is given by

$$
\begin{aligned}
H &= \mathbb{I}\partial_t,\\
D &= -(\mathbb{I}(t\partial_t+\frac{1}{2}x\partial_x+\frac{1}{2})+\frac{1}{4}\gamma_3),\\
K &= -(\mathbb{I}(t^2\partial_t+tx\partial_x-\frac{\lambda}{4}x^2+t)-\frac{1}{2}\gamma_+x+\frac{1}{2}\gamma_3 t),\\
P_+ &= \mathbb{I}\partial_x,\\
P_- &= \mathbb{I}(t\partial_x-\frac{\lambda}{2}x)-\frac{1}{2}\gamma_+,\\
C &= \mathbb{I},\\
\Omega_{z(x,t)} &= z(x,t)\Omega = z(x,t)(\gamma_+\partial_t+\gamma_-\lambda+\gamma_3\partial_x).
\end{aligned}
\tag{26}
$$

The class of symmetry operators $\Omega_{z(x,t)}$ depends on an unconstrained function $z(x,t)$ (Ω is recovered by setting $z(x,t)=1$).

The complete set (up to normalization) of 2×2 first-order differential operators satisfying (25) is given by

$$
\begin{aligned}
\Lambda_1 &= \mathbb{I},\\
\Lambda_2 &= \gamma_+\partial_t-\lambda\gamma_-,\\
\Lambda_3 &= \gamma_+(t\partial_t+\frac{1}{2})-\gamma_-\lambda t+\frac{\lambda}{2}x\mathbb{I},\\
\Lambda_4 &= \gamma_3\partial_t-2\gamma_-\partial_x,\\
\Lambda_5 &= \gamma_3(t\partial_t+\frac{1}{4})+\gamma_-(-2t\partial_x+\frac{\lambda}{2})-\gamma_+\frac{x}{2}+\frac{1}{2}\mathbb{I},\\
\Lambda_6 &= \gamma_3(\frac{1}{2}t^2\partial_t+\frac{t}{4})+\gamma_-(-t^2\partial_x+\frac{\lambda}{2}tx)+\gamma_+(-\frac{1}{2}tx\partial_t-\frac{x}{4})+(\frac{t}{2}-\frac{\lambda x^2}{8})\mathbb{I},\\
\widetilde{\Lambda}_{w(x,t)} &= w(x,t)(\gamma_+\partial_x-\frac{\lambda}{2}(\mathbb{I}+\gamma_3)).
\end{aligned}
\tag{27}
$$

The $\widetilde{\Lambda}_{w(x,t)}$ class of operators depend on an unconstrained function $w(x,t)$.

The computation of the associated matrix-valued functions $\Phi_\Sigma(x,t),\Phi_\Lambda(x,t)$ is left as a simple exercise for the Reader.

As explained above, formulas (26) and (27) are representation-independent and provide symmetry operators for the 4×4 matrix case as well. For this matrix representation, since the complex structure operator J commutes with $\gamma_\pm,\gamma_3$, the number of symmetry operators is doubled with respect to the 2×2 matrix representation. Indeed, for any given symmetry operator Σ or Λ entering (26) or (27), $J\cdot\Sigma$ ($J\cdot\Lambda$) is a symmetry operator satisfying (24) (and, respectively, (25)).

We explicitly verified that no further symmetry operator is encountered, in the 4×4 representation, besides the operators given in (26,27) and their J-doubled counterparts.

The presence of the complex structure J in the 4×4 matrix case allows to induce, from the Lévy-Leblond square root of the free heat equation, the Lévy-Leblond square root of the free Schrödinger equation in $1+1$ dimensions. The most straightforward way consists in replacing, in the formulas above, $\lambda\mapsto\beta J$, for a real β. This substitution is allowed because J commutes with all matrices entering (22,26,27).

We therefore get, for the $1+1$-dimensional free Schrödinger case,

$$
\begin{aligned}
\overline{\Omega} &= \gamma_+\partial_t+\gamma_-\cdot J\beta+\gamma_3\partial_x,\\
\overline{\Omega}^2 &= \mathbb{I}\cdot(J\beta\partial_t+\partial_x^2).
\end{aligned}
\tag{28}
$$

5 The graded Lie symmetry algebras of the free equations

We investigate here the properties of the closed graded Lie (super)algebras recovered from the symmetry operators entering (26) and (27).

We start by observing that the first six operators in (26) span the one-dimensional Schrödinger algebra $sch(1)$. Their non-vanishing commutators are

$$\begin{array}{rcl}[D,H] &=& H,\\ [D,K] &=& -K,\\ [H,K] &=& 2D,\\ [D,P_\pm] &=& \pm\frac{1}{2}P_\pm,\\ [H,P_-] &=& P_+,\\ [K,P_+] &=& P_-,\\ [P_+,P_-] &=& -\frac{\lambda}{2}C.\end{array} \qquad (29)$$

D is the grading operator, the grading z being defined by the commutator

$$[D,Z] \;=\; zZ, \qquad (30)$$

for some given generator Z.

H, D, K close an $sl(2)$ algebra with D as the Cartan element.

$P_\pm$, together with the central charge C, realize the $h(1)$ Lie-Heisenberg subalgebra. Since $P_\pm$ have half-integer grading, it is convenient in the following to introduce the notation

$$P_\pm \equiv P_{\pm\frac{1}{2}}. \qquad (31)$$

$P_{\pm\frac{1}{2}}$ induce a closed $osp(1|2)$ algebra, whose even generators are the second-order differential operators $P_{\pm 1}, P_0$, introduced via the anticommutators (for δ and ϵ taking values ± 1)

$$P_{(\delta+\epsilon)\frac{1}{2}} \;=\; \{P_{\delta\frac{1}{2}}, P_{\epsilon\frac{1}{2}}\}, \qquad (32)$$

while $P_{\pm\frac{1}{2}}$ belong to the odd sector of $osp(1|2)$.

One should note that $[D, P_s] = sP_s$. Since

$$[\Omega, P_s] \;=\; 0, \quad \forall s = \pm 1, \pm\frac{1}{2}, 0, \qquad (33)$$

the $osp(1|2)$ Lie superalgebra spanned by the P_s's operators is a symmetry superalgebra of the Lévy-Leblond operator Ω.

A closed symmetry superalgebra $u(1)\oplus sl(2)\ni\!\!\!\!\!\!\;\;\; osp(1|2)$, with 2 odd and 7 even generators, is spanned by H, D, K, C and the five P_s operators.

We note, as a remark, that if we express the free heat equation

$$\Omega^2\Psi(x,t) = (\lambda\partial_t + \partial_x^2)\Psi(x,t) = 0 \qquad (34)$$

in the equivalent form

$$-\lambda\partial_t\Psi(x,t) = \partial_x^2\Psi(x,t), \qquad (35)$$

the operator $P_{\frac{1}{2}}$ is the square root of the r.h.s. operator in (35).

The Lévy-Leblond operator $\Omega \equiv \Omega_{z(x,t)=1}$ possesses half-integer grading:

$$[D,\Omega] \;=\; \frac{1}{2}\Omega. \tag{36}$$

The commutator with K produces an operator with $-\frac{1}{2}$ grading,

$$\begin{aligned} [K,\Omega] &= t\Omega \equiv \Omega_{z(x,t)=t}, \\ [D,t\Omega] &= -\frac{1}{2}t\Omega. \end{aligned} \tag{37}$$

It is convenient to set $\Omega \equiv \Omega_{\frac{1}{2}}$, $t\Omega \equiv \Omega_{-\frac{1}{2}}$.

The commutator

$$[\Omega,t\Omega] \;=\; \gamma_+\Omega \tag{38}$$

gives an operator, $\gamma_+\Omega$, which does not belong to the class of symmetry operators here considered. The reason is that its commutator with Ω can be expressed as

$$[\Omega,\gamma_+\Omega] \;=\; (-\lambda\gamma_3+2\gamma_+\partial_x)\Omega \equiv T\Omega, \tag{39}$$

where $T=-\lambda\gamma_3+2\gamma_+\partial_x$ is not a matrix-valued function, but a differential operator.

On the other hand, $\Omega_{\pm\frac{1}{2}}$ can be taken as the odd sector of an $osp(1|2)$ superalgebra whose even generators are the second-order differential operators $\Omega_{\pm1},\Omega_0$ given by the anticommutators (for δ and ϵ taking values ±1)

$$\Omega_{(\delta+\epsilon)\frac{1}{2}} \;=\; \{\Omega_{\delta\frac{1}{2}},\Omega_{\epsilon\frac{1}{2}}\}. \tag{40}$$

The non-vanishing commutators of this second $osp(1|2)$ superalgebra are given by

$$\begin{aligned} [\Omega_0,\Omega_s] &= -2s\lambda\Omega_s, \qquad s=\pm\frac{1}{2},\pm1,0, \\ [\Omega_1,\Omega_{-1}] &= 4\lambda\Omega_0, \\ [\Omega_{\pm\frac{1}{2}},\Omega_{\mp1}] &= \pm2\lambda\Omega_{\mp\frac{1}{2}}. \end{aligned} \tag{41}$$

It follows, as a corollary, that the second-order differential operators $\Omega_{\pm1},\Omega_0$ are symmetry operators (since, e.g., $[\Omega_{-1},\Omega_{\frac{1}{2}}]=-2\lambda t\Omega_{\frac{1}{2}}$).

It is worth pointing out that in the Schrödinger case, when λ is replaced by βJ, see the discussion before equation (28), the five symmetry operators closing the $osp(1|2)$ superalgebra are $\Omega_{+\frac{1}{2}}, J\cdot\Omega_{-\frac{1}{2}},\Omega_{\pm1},J\cdot\Omega_0$.

The action of the $sch(1)$ Schrödinger generators $H,D,K,C,P_{\pm\frac{1}{2}}$ on the five operators Ω_s produces the semidirect superalgebra $sch(1)\ni\!\!\!\!\supset osp(1|2)$ since the only non-vanishing commutators involving the Ω_s operators and the Schrödinger generators are

$$\begin{aligned} &[D,\Omega_s]=s\Omega_s, \qquad s=\pm\frac{1}{2},\pm1,0, \\ &[H,\Omega_{-\frac{1}{2}}]=\Omega_{\frac{1}{2}}, \quad [H,\Omega_{-1}]=2\Omega_0, \quad [H,\Omega_0]=\Omega_1, \\ &[K,\Omega_{\frac{1}{2}}]=-\Omega_{-\frac{1}{2}}, \quad [K,\Omega_0]=\Omega_{-1}, \quad [K,\Omega_1]=2\Omega_0. \end{aligned} \tag{42}$$

Due to the fact that, for any $s, s' = \pm\frac{1}{2}, \pm 1, 0$,

$$[P_s, \Omega_{s'}] \;=\; 0, \tag{43}$$

a first $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra, whose vector space is spanned by the P_s and $\Omega_{s'}$ symmetry operators, is obtained. This $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra is decomposed (see Appendix **A**, equations (A.1,A.2,A.4)) according to

$$\mathcal{G}_{00} = \{P_{\pm 1}, P_0, \Omega_{\pm 1}, \Omega_0\}, \quad \mathcal{G}_{01} = \{P_{\pm\frac{1}{2}}\}, \quad \mathcal{G}_{10} = \{\Omega_{\pm\frac{1}{2}}\}, \quad \mathcal{G}_{11} = \{\emptyset\}, \tag{44}$$

with an empty $\mathcal{G}_{11}$ sector.

A super Schrödinger symmetry algebra is obtained. In order to recover it, we need to introduce the symmetry operators arising from the Λ-sector defined by the equation (25). The presence of a super Schrödinger algebra requires the existence of symmetry operators which are square roots, up to a sign, of H, K entering (26,29). In the 4×4 representation, due to the existence of the complex structure J, the sign can be flipped (if Q is an operator such that $Q^2 = \pm H$, then $J \cdot Q$ is a symmetry operator such that $(J \cdot Q)^2 = \mp H$). Our analysis proves the existence of a super Schrödinger algebra with at most an $\mathcal{N} = 1$ supersymmetric extension, the only supersymmetric roots being the symmetry operators $Q_\pm$ given by

$$Q_+ \;=\; \frac{1}{\sqrt{\lambda}}\Lambda_2 = \frac{1}{\sqrt{\lambda}}(\gamma_+\partial_t - \lambda\gamma_-), \qquad Q_+^2 = -H, \tag{45}$$

and

$$Q_- \;=\; \frac{1}{\sqrt{\lambda}}(\Lambda_3 + \widetilde{\Lambda}_x) = \frac{1}{\sqrt{\lambda}}(\gamma_+(t\partial_t + x\partial_x + \frac{1}{2}) - \gamma_-\lambda t - \gamma_3\frac{\lambda x}{2}), \qquad Q_-^2 = K. \tag{46}$$

Since $Q_\pm$ have half-integer grading,

$$[D, Q_\pm] \;=\; \pm\frac{1}{2}Q_\pm, \tag{47}$$

as before it is convenient to set $Q_\pm \equiv Q_{\pm\frac{1}{2}}$.

The operators $Q_{\pm\frac{1}{2}}$ induce a third $osp(1|2)$ symmetry superalgebra. Their anticommutators $Q_{(\delta+\epsilon)\frac{1}{2}} = \{Q_{\delta\frac{1}{2}}, Q_{\epsilon\frac{1}{2}}\}$, for $\delta, \epsilon = \pm 1$, produce the even generator $Q_{\pm 1}, Q_0$ which coincides, up to normalization, with the $sl(2)$ generators H, K, D given in (26):

$$\{Q_{+\frac{1}{2}}, Q_{+\frac{1}{2}}\} = -2H, \quad \{Q_{-\frac{1}{2}}, Q_{-\frac{1}{2}}\} = 2K, \quad \{Q_{+\frac{1}{2}}, Q_{-\frac{1}{2}}\} = 2D. \tag{48}$$

The remaining non-vanishing commutators of the third $osp(1|2)$ superalgebra are

$$[H, Q_{-\frac{1}{2}}] = Q_{+\frac{1}{2}} \quad , \quad [K, Q_{+\frac{1}{2}}] = Q_{-\frac{1}{2}}. \tag{49}$$

In the super Schrödinger algebra, the two extra operators $P_{\pm\frac{1}{2}}$ enter the even sector and a further symmetry operator X, obtained from the commutators

$$[P_{\pm\frac{1}{2}}, Q_{\mp\frac{1}{2}}] \;=\; \pm X, \tag{50}$$

enters the odd sector. We have

$$X \;=\; \frac{1}{\sqrt{\lambda}}\widetilde{\Lambda}_{w(x,t)=1} + \frac{\sqrt{\lambda}}{2}\Lambda_1 = \frac{1}{\sqrt{\lambda}}(\gamma_+\partial_x - \gamma_3\frac{\lambda}{2}). \tag{51}$$

The super Schrödinger algebra $ssch(1)$ is decomposed into even and odd sector as

$$ssch(1) = \mathcal{G}_0 \oplus \mathcal{G}_1,$$

with $\mathcal{G}_0 = \{H, D, K, C, P_{\pm\frac{1}{2}}\}$ and $\mathcal{G}_1 = \{Q_{\pm\frac{1}{2}}, X\}$.

The extra non-vanishing (anti)commutators involving the X generator are

$$\begin{aligned} \{X, Q_{\pm\frac{1}{2}}\} &= -P_{\pm\frac{1}{2}}, \\ \{X, X\} &= \frac{\lambda}{2}C. \end{aligned} \tag{52}$$

It is worth pointing out that the operator $Q_{+\frac{1}{2}}$ is the square root of the operator in the l.h.s. of the equation (35).

We recovered three independent $osp(1|2)$ symmetry superalgebras induced, respectively, by the three pairs of half-graded odd generators $P_{\pm\frac{1}{2}}, \Omega_{\pm\frac{1}{2}}, Q_{\pm\frac{1}{2}}$. We already discussed the compatibility of the $osp(1|2)$ superalgebras obtained from the two pairs $P_{\pm\frac{1}{2}}, \Omega_{\pm\frac{1}{2}}$. The last step consists in discussing the compatibility of the $osp(1|2)$ superalgebras induced by the two pairs $Q_{\pm\frac{1}{2}}, \Omega_{\pm\frac{1}{2}}$ and $Q_{\pm\frac{1}{2}}, P_{\pm\frac{1}{2}}$.

It is easily shown that an algebra which presents both pairs of $Q_{\pm\frac{1}{2}}, \Omega_{\pm\frac{1}{2}}$ generators brings us beyond the scheme of symmetry algebras discussed in the present paper. Indeed, their repeated (anti)commutators produce higher derivative operators which do not satisfy equations (24) and (25) with matrix-valued functions on their right hand sides.

On the other hand, requiring the presence of the two pairs of operators $Q_{\pm\frac{1}{2}}, P_{\pm\frac{1}{2}}$, a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra (following the definition in Appendix **A**) of symmetry operators is naturally induced. This superalgebra, $\mathcal{G}_{\mathbb{Z}_2\times\mathbb{Z}_2}$, is decomposed according to $\mathcal{G}_{\mathbb{Z}_2\times\mathbb{Z}_2} = \mathcal{G}_{00}\oplus\mathcal{G}_{01}\oplus\mathcal{G}_{10}\oplus\mathcal{G}_{11}$, with their respective sectors spanned by the generators

$$\begin{aligned} \mathcal{G}_{00} &= \{H, D, K, P_{\pm 1}, P_0\}, \\ \mathcal{G}_{01} &= \{P_{\pm\frac{1}{2}}\}, \\ \mathcal{G}_{10} &= \{Q_{\pm\frac{1}{2}}, X_{\pm\frac{1}{2}}\}, \\ \mathcal{G}_{11} &= \{X\}. \end{aligned} \tag{53}$$

The extra generators we have yet to introduce are the second-order differential operators $X_{\pm\frac{1}{2}}$, obtained from the anticommutators

$$X_{\pm\frac{1}{2}} = \{X, P_{\pm\frac{1}{2}}\}. \tag{54}$$

Some comments are in order. The $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra (53) is closed under (anti)-commutators and, furthermore, the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Jacobi identities are fulfilled, as explicitly verified.

It is important to point out that (53) is a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra of symmetry operators (as defined in Section **3**). The last check consists in verifying that the anticommutators of $X_{\pm\frac{1}{2}}$ with $\Omega\equiv\Omega_{+\frac{1}{2}}$ are vanishing:

$$\{\Omega, X\} = \{\Omega, X_{\pm\frac{1}{2}}\} = 0. \tag{55}$$

For completeness we present the list of the remaining (besides the ones already introduced) non-vanishing (anti)commutators involving the (53) generators. We have

$$
\begin{array}{ll}
[D,P_s]=sP_s, & [D,X_{\pm\frac{1}{2}}]=\pm\frac{1}{2}X_{\pm\frac{1}{2}},\\
[H,P_m]=(1-m)P_{m+1}, & [H,X_{-\frac{1}{2}}]=X_{\frac{1}{2}},\\
[K,P_m]=(1+m)P_{m-1}, & [K,X_{\frac{1}{2}}]=X_{-\frac{1}{2}},\\
[P_1,P_{-1}]=-4\lambda P_0, & [P_{\pm1},P_{\mp\frac{1}{2}}]=\mp2\lambda P_{\pm\frac{1}{2}},\\
[P_{\pm1},Q_{\mp\frac{1}{2}}]=\pm2X_{\pm\frac{1}{2}}, & [P_{\pm1},X_{\mp\frac{1}{2}}]=\mp2\lambda X_{\pm\frac{1}{2}},\\
[P_0,P_s]=2s\lambda P_s, & [P_0,Q_{\pm\frac{1}{2}}]=\mp X_{\pm\frac{1}{2}},\\
[P_0,X_{\pm\frac{1}{2}}]=\pm\lambda X_{\pm\frac{1}{2}}, & [P_{\pm\frac{1}{2}},Q_{\mp\frac{1}{2}}]=\pm X,\\
[P_{\pm\frac{1}{2}},X_{\mp\frac{1}{2}}]=\mp\lambda X, & \{X_{\pm\frac{1}{2}},X_{\pm\frac{1}{2}}\}=\lambda P_{\pm1},\\
\{X_{\frac{1}{2}},X_{-\frac{1}{2}}\}=\lambda P_0, & \{Q_{\pm\frac{1}{2}},X_{\pm\frac{1}{2}}\}=-P_{\pm1},\\
\{Q_{\pm\frac{1}{2}},X_{\mp\frac{1}{2}}\}=-P_0, & \{X,X_{\pm\frac{1}{2}}\}=\lambda P_{\pm\frac{1}{2}}.
\end{array}
\qquad (56)
$$

where $m=\pm1,0$ and $s=\pm\frac{1}{2},\pm1,0$.

It is worth pointing out that adding the identity operator C to the $\mathcal{G}_{00}$ sector, we obtain the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra $u(1)\oplus\mathcal{G}_{\mathbb{Z}_2\times\mathbb{Z}_2}$. The $\mathcal{N}=1$ super Schrödinger algebra $ssch(1)$ is spanned by a subset of the $u(1)\oplus\mathcal{G}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ generators. Even so, $ssch(1)$ is not a $u(1)\oplus\mathcal{G}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ subalgebra (since for $ssch(1)$ the brackets involving, e.g., $P_{\pm\frac{1}{2}}$ are defined with commutators).

6 Symmetries of the Lévy-Leblond square root of the heat equation with quadratic potential

The Schrödinger algebra is the maximal kinetic symmetry algebra of the heat or of the Schrödinger equation, see [20, 21, 1], for three type of potentials, constant, linear or quadratic. The last case corresponds to the harmonic oscillator. We investigate here the symmetries of the the associated Lévy-Leblond equation for the heat equation with quadratic potential. On the basis of the construction of Section **2**, the minimal matrix differential operator is expressed as a 4×4 matrix. The quadratic potential is recovered from the linear function $f(x)=\omega x$ entering (9). Without loss of generality we can set $\lambda=\omega=1$ and take the block-antidiagonal Lévy-Leblond operator $\widetilde{\Omega}$ to be given by

$$\widetilde{\Omega} \;=\; (e_{14}-e_{32})\partial_t+(e_{13}-e_{24}-e_{31}+e_{42})\partial_x+x(e_{13}+e_{24}+e_{31}+e_{42})-e_{23}+e_{41}, \quad (57)$$

where e_{ij} denotes the matrix with entry 1 at the cross of the i-th row and j-th column and 0 otherwise.

In this basis its squared operator $\widetilde{\Omega}^2$ is

$$\widetilde{\Omega}^2 \;=\; (e_{11}+e_{22}+e_{33}+e_{44})(\partial_t-\partial_x^2+x^2)+e_{11}+e_{44}-e_{22}-e_{33}. \quad (58)$$

The exhaustive list of first-order differential symmetry operators Σ's satisfying

$$[\Sigma,\widetilde{\Omega}] \;=\; \Phi_\Sigma(x,t)\cdot\widetilde{\Omega}, \quad (59)$$

for some 4×4 matrix-valued functions $\Phi_\Sigma(x,t)$ in the x,t coordinates, can be computed with lengthy but straightforward methods. We present here the complete list of symmetry operators (for simplicity we leave as an exercise for the Reader the computation of their associated matrix-valued functions $\Phi_\Sigma(x,t)$'s).

The symmetry operators can be split into the two big classes of block-diagonal and block-antidiagonal operators. In both cases we have 12 fixed symmetry operators (up to normalization) plus two extra sets of operators depending on an unconstrained function (denoted as $z(x,t)$) of the x,t coordinates.

The 12 fixed block-diagonal symmetry operators can be conveniently presented as

$$\begin{array}{rcl}
\Sigma_1 &=& e^{4t}((e_{22}+e_{33}-2xe_{34})\partial_t+(e_{21}-e_{43}+2x(e_{22}+e_{44}))\partial_x+\\
&& xe_{21}+2x^2e_{22}+4e_{33}-8xe_{34}-xe_{43}+2(x^2+1)e_{44}),\\
\Sigma_2 &=& e^{4t}((e_{11}+e_{44}+2xe_{34})\partial_t+(e_{43}-e_{21}+2x(e_{11}+e_{33}))\partial_x+\\
&& (2+2x^2)e_{11}-xe_{21}+2x^2e_{33}+xe_{43}),\\
\Sigma_3 &=& e^{-4t}((e_{22}+e_{33}+2xe_{34})\partial_t+(e_{21}-e_{43}-2x(e_{22}+e_{44}))\partial_x+\\
&& -3xe_{21}-2x^2e_{22}+(4x^2-2)e_{33}-xe_{43}+2x^2e_{44}),\\
\Sigma_4 &=& e^{-4t}((e_{11}+e_{44}-2xe_{34})\partial_t+(e_{34}-e_{21}-2x(e_{11}+e_{33}))\partial_x+\\
&& (2x^2-4)e_{11}-8xe_{12}+3xe_{21}+(4x^2-2)e_{22}-2x^2e_{33}+xe_{43}),\\
\Sigma_5 &=& e^{2t}((e_{11}+e_{22}+e_{33}+e_{44})(\partial_x+x)-2e_{34}),\\
\Sigma_6 &=& e^{-2t}((e_{11}+e_{22}+e_{33}+e_{44})(\partial_x-x)+2e_{12}),\\
\Sigma_7 &=& (e_{11}+e_{44})\partial_t+(e_{43}-e_{21})\partial_x+x(e_{21}+e_{43}),\\
\Sigma_8 &=& (e_{22}+e_{33})\partial_t+(e_{21}-e_{43})\partial_x-x(e_{21}+e_{43}),\\
\Sigma_9 &=& e_{11}+e_{22},\\
\Sigma_{10} &=& e_{33}+e_{44},\\
\Sigma_{11} &=& e^{2t}(e_{34}\partial_t-(e_{22}+e_{44})\partial_x-e_{21}+2e_{34}-x(e_{22}+e_{44})),\\
\Sigma_{12} &=& e^{-2t}(e_{34}\partial_t-(e_{22}+e_{44})\partial_x-e_{21}+x(e_{44}+2e_{33}-e_{22})).
\end{array}\qquad (60)$$

The 12 fixed block-antidiagonal symmetry operators can be conveniently presented as

$$\begin{array}{rcl}
\Sigma_{13} &=& e^{6t}((e_{13}-e_{41}-4xe_{32})\partial_t+(4x(e_{13}-e_{31})-(e_{23}+e_{41}))\partial_x+\\
&& (8x^2+4)e_{13}-(12x+8x^3)e_{14}-3xe_{23}+(4x^2+2)e_{24}+4x^2e_{31}+xe_{41}),\\
\Sigma_{14} &=& e^{-6t}((e_{24}-e_{31}-4xe_{32})\partial_t+(e_{23}+e_{41}+4x(e_{42}-e_{24}))\partial_x+\\
&& -3xe_{23}+4x^2e_{24}+4(1-x^2)e_{31}+(12x-8x^3)e_{32}+xe_{41}+2e_{42}),\\
\Sigma_{15} &=& e^{4t}(-e_{32}\partial_t+(e_{13}-e_{31})\partial_x+3xe_{13}-(4x^2+2)e_{14}-e_{23}+2xe_{24}+xe_{31}),\\
\Sigma_{16} &=& e^{-4t}(e_{32}\partial_t+(e_{24}-e_{42})\partial_x+e_{23}-xe_{24}+2xe_{31}+(4x^2-2)e_{32}+xe_{42}),\\
\Sigma_{17} &=& e^{2t}\left((e_{24}-e_{31})\partial_t+e_{23}\partial_x+x(e_{23}+e_{41})-2e_{31}\right),\\
\Sigma_{18} &=& e^{2t}((e_{13}-e_{42}-2xe_{32})\partial_t-(e_{23}+e_{41}+2x(e_{31}-e_{13}))\partial_x+x(e_{41}-e_{23})+2x^2(e_{13}+e_{31})),\\
\Sigma_{19} &=& e^{2t}(e_{13}+e_{24}-2xe_{14}),\\
\Sigma_{20} &=& e^{-2t}((e_{13}-e_{42})\partial_t-(e_{23}+e_{41})\partial_x-2e_{13}+x(e_{23}+e_{41})),\\
\Sigma_{21} &=& e^{-2t}\left((e_{24}-e_{31}-2xe_{32})\partial_t+(-2x(e_{24}-e_{42})+e_{23}+e_{41})\partial_x+2x^2(e_{24}+e_{42})+x(e_{41}-e_{23})\right),\\
\Sigma_{22} &=& e^{-2t}(e_{31}+e_{42}+2xe_{32}),\\
\Sigma_{23} &=& (e_{13}+e_{24}-e_{31}-e_{42})\partial_x+(e_{13}-e_{24}+e_{31}-e_{42})x,\\
\Sigma_{24} &=& e_{32}\partial_t+(e_{24}-e_{42})\partial_x+e_{23}-x(e_{24}+e_{42}).
\end{array}\qquad (61)$$

The 4 (two block-diagonal and two block-antidiagonal) extra sets of symmetry operators depending on the unconstrained functions $z_i(x,t)$, $i=1,2,3,4$, are

$$\begin{aligned}
\widetilde{\Sigma}_{1,z_1(x,t)} &= z_1(x,t)((e_{12}+e_{34})\partial_t+(e_{11}-e_{22}+e_{33}-e_{44})\partial_x-(e_{21}+e_{43})-x(e_{11}+e_{22}-e_{33}-e_{44})),\\
\widetilde{\Sigma}_{2,z_2(x,t)} &= z_2(x,t)((e_{12}-e_{34})\partial_x+e_{11}-e_{33}+x(e_{12}+e_{34})),\\
\widetilde{\Sigma}_{3,z_3(x,t)} &= z_3(x,t)((e_{14}-e_{32})\partial_t+(e_{13}-e_{24}-e_{31}+e_{42})\partial_x-e_{23}+e_{41}+x(e_{13}+e_{24}+e_{31}+e_{42})),\\
\widetilde{\Sigma}_{4,z_4(x,t)} &= z_4(x,t)((e_{14}+e_{32})\partial_x+e_{13}+e_{31}+x(e_{32}-e_{14})).
\end{aligned} \tag{62}$$

The Lévy-Leblond operator $\widetilde{\Omega}$ is recovered from $\widetilde{\Sigma}_{3,z_3(x,t)}$ by setting $z_3(x,t)=1$. We have, indeed,

$$\widetilde{\Omega} = \widetilde{\Sigma}_{3,z_3(x,t)=1}. \tag{63}$$

Some of the block-diagonal symmetry operators have special meaning: the identity $\mathbb{I}_4$ is given by the combination $\mathbb{I}_4=\Sigma_9+\Sigma_{10}$, while the fermion-number symmetry operator N_f is the difference $N_f=\Sigma_9-\Sigma_{10}$. The time-derivative is a symmetry operator and, as it is the case for the Schrödinger's equation of the harmonic oscillator, it can be used to define a grading operator D. We can set

$$D = \frac{1}{4}(\Sigma_7+\Sigma_8)=\mathbb{I}_4\cdot\frac{1}{4}\partial_t. \tag{64}$$

A natural Schrödinger symmetry algebra is encountered. The identification (up to normalization) of its symmetry generators at degree $\pm1,\pm\frac{1}{2},0$ is given by

$$+1:\Sigma_1+\Sigma_2\equiv H,\quad +\tfrac{1}{2}:\Sigma_5\equiv P_{+\frac{1}{2}},\quad 0:D,C\equiv\mathbb{I}_4,\quad -\tfrac{1}{2}:\Sigma_6\equiv P_{-\frac{1}{2}},\quad -1:\Sigma_3+\Sigma_4\equiv K. \tag{65}$$

$\Sigma_5\equiv P_{\frac{1}{2}},\Sigma_6\equiv P_{-\frac{1}{2}}$ are, respectively, the creation/annihilation operators, while H,D,K are the $sl(2)$ subalgebra generators.

A first difference with respect to the symmetry algebra of the free Lévy-Leblond equation is already encountered at this level. In the free case the Lévy-Leblond operator possesses half-integer grading ($=\frac{1}{2}$) with respect to the $sl(2)$ Cartan generator of the Schrödinger subalgebra. In the present case $\widetilde{\Omega}$ has zero grading

$$[D,\widetilde{\Omega}] = 0. \tag{66}$$

This implies, as a a corollary, that for the quadratic potential there is no $osp(1|2)$ symmetry superalgebra induced by $\Omega_{\pm\frac{1}{2}}$.

The extra question to be investigated is whether, in the presence of a non-trivial potential, the Lévy-Leblond operator admits a super-Schrödinger invariance.

In principle one can repeat the same steps as in the free case and compute the most general solution obtained from the anti-commutators $\{\Lambda,\widetilde{\Omega}\}=\Phi_\Lambda(x,t)\cdot\widetilde{\Omega}$. On the other hand, since these computations are rather cumbersome, it is preferable to address the question with a different procedure, based on the key observation that the existence of the super-Schrödinger symmetry requires the presence of at least one operator which is the square root (up to normalization) of $H=\Sigma_1+\Sigma_2$. Requiring $Q^2=kH$ for some real $k\neq0$ produces the most general solution

$$Q = e^{2t}\begin{pmatrix} 0 & \partial_t+2x\partial_x+2x^2 & r_3 & -2r_3x\\ r_1r_2 & 0 & 0 & -r_1r_3\\ 0 & 0 & 2r_2x & r_1\partial_t+2r_1x\partial_x+(2r_1-4r_2)x^2+2r_1\\ 0 & 0 & r_2 & -2r_2x\end{pmatrix}, \tag{67}$$

depending on three real parameters r_i $(i = 1, 2, 3)$ and with the identification $k = r_1 r_2$. Therefore, both r_1, r_2 need to be non-vanishing. Once identified the most general operator Q, we need to verify whether it belongs to the class of Lévy-Leblond symmetry operators. This means that a matrix-valued function $\Phi(x,t)$ should be found for

$$either \quad [Q, \widetilde{\Omega}] = \Phi(x,t) \cdot \widetilde{\Omega} \quad or \quad \{Q, \widetilde{\Omega}\} = \Phi(x,t) \cdot \widetilde{\Omega}. \tag{68}$$

It is easily checked that, for $r_{1,2} \neq 0$, no such matrix-valued function exists, implying that the $r_{1,2} \neq 0$ operators in (67) are not symmetry generators of the Lévy-Leblond operator (57) with non-trivial potential.

In the presence of a non-trivial potential there is no analog of the $osp(1|2)$ symmetry generated by the $Q_{\pm\frac{1}{2}}$ symmetry operators of the free case.

In the free case we discussed three separated $osp(1|2)$ symmetry superalgebras, induced by $P_{\pm\frac{1}{2}}, Q_{\pm\frac{1}{2}}, \Omega_{\pm\frac{1}{2}}$, respectively. In the presence of the non-trivial potential we obtain a single $osp(1|2)$ symmetry from the (anti)commutators of the creation/annihilation operators $P_{\pm\frac{1}{2}}$ entering (65), with the even sector being given by $P_{\pm 1} = 2P^2_{\pm\frac{1}{2}}$, $P_0 = \{P_{+\frac{1}{2}}, P_{-\frac{1}{2}}\}$. All these operators commute with $\widetilde{\Omega}$:

$$[\widetilde{\Omega}, P_s] \;\; = \;\; 0, \qquad s = 0, \pm\frac{1}{2}, \pm 1. \tag{69}$$

Together with the extra generator H, D, K, C in (65), we therefore find a symmetry superalgebra for $\widetilde{\Omega}$, given by $u(1) \oplus sl(2) \ni\!\!\!\!+ osp(1|2)$. It contains two odd generators $P_{\pm\frac{1}{2}}$, while the 7 remaining generators span the even sector.

The investigation and identification of other closed finite symmetry (super)algebras recovered from the (60,61,62) symmetry operators is beyond the scope of the present paper.

7 Conclusions

We highlight the main results of the paper.

We proved, for the Lévy-Leblond square root of the free heat or the free Schrödinger equation, the existence of a symmetry of first-order and second-order differential operators closing a finite $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra. Let's take, for the sake of clarity, the $1+1$-dimensional Schrödinger equation. It can be equivalently written in two forms, either

$$(i\partial_t + \partial_x^2)\Psi(x,t) \;\; = \;\; 0, \tag{70}$$

or

$$i\partial_t \Psi(x,t) \;\; = \;\; -\partial_x^2 \Psi(x,t), \tag{71}$$

where 2×2 diagonal operators (the identity $\mathbb{I}_2$ is dropped for simplicity) act upon the 2-column complex vector $\Psi(x,t)$.

Three independent sets of $osp(1|2)$ symmetry superalgebras are encountered. They are generated by the first-order, odd, symmetry differential operators $\Omega_{\pm\frac{1}{2}}$, $P_{\pm\frac{1}{2}}$, $Q_{\pm\frac{1}{2}}$, respectively.

$\Omega_{+\frac{1}{2}}$ is the Lévy-Leblond operator we initially started with; it is the square root of the operator entering (70). $P_{+\frac{1}{2}}$ ($Q_{+\frac{1}{2}}$) is the square root of the operator entering the right hand side (respectively, the left hand side) of equation (71). We investigated the mutual compatibility conditions for these $osp(1|2)$ superalgebras. Since the $\Omega_{\pm\frac{1}{2}}$ operators commute with the $P_{\pm\frac{1}{2}}$

operators, a first $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra, given by (44), is encountered. Its $\mathcal{G}_{11}$ sector is empty, while its vector space is spanned by the P_s and $\Omega_{s'}$ symmetry operators ($s, s' = \pm\frac{1}{2}, \pm 1, 0$).

The $\Omega_{\pm\frac{1}{2}}$ operators, together with the $Q_{\pm\frac{1}{2}}$ operators, produce an algebra containing higher-order (degree ≥ 3) differential operators.

A finite, closed, non-trivial $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra (spanned by the 13 generators recovered from (53)) is discovered by requiring the presence of the two pairs of operators, $P_{\pm\frac{1}{2}}$ and $Q_{\pm\frac{1}{2}}$.

The (70,71) system also possesses a (maximally $\mathcal{N} = 1$) $\mathbb{Z}_2$-graded super Schrödinger algebra symmetry.

We completed the $(1+1)$-dimensional investigation by looking at the symmetries of the Lévy-Leblond square root of the $(1+1)$-dimensional heat equation with quadratic potential. This system, defined by the Lévy-Leblond operator $\widetilde{\Omega}$, possesses a symmetry closing the Schrödinger algebra $sch(1)$. We proved, on the other hand, by exhaustive computations, that the non-vanishing potential does not allow a graded extension (neither $\mathbb{Z}_2$- nor $\mathbb{Z}_2 \times \mathbb{Z}_2$-) of the Schrödinger algebra. The main difference, with respect to the free case, is due to the fact that $\widetilde{\Omega}$ commutes with the Cartan generator D of the $sl(2) \subset sch(1)$ subalgebra, see (66). Measured by D, $\widetilde{\Omega}$ is a 0-grade operator. In the free case, the Lévy-Leblond operator $\Omega_{+\frac{1}{2}}$ possesses the half-integer $+\frac{1}{2}$-grade when measured by the corresponding D operator, see (36).

Some comments are in order. To our knowledge, this is the first time that a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie (super)algebra is encountered in the context of symmetries of partial differential equations. This feature could prove to be significant for the community of physicists (searching for applications of) and mathematicians (developing the mathematical structure of) graded color Lie (super)algebras. It opens the possibility, e.g., to use them as spectrum-generating algebras for suitable dynamical systems.

The constructions of (both $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-) graded extensions of Lie symmetries require two types of symmetry operators, the ones obtained from the commutator condition (14) and those obtained from the anticommutator condition (15).

In Appendix **B** we constructed a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra symmetry of the Lévy-Leblond operator associated with the free heat equation in $(1 + 2)$-dimensions. A new feature appears. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry allows to explain the existence of first-order differential symmetry operators which do not belong to the two-dimensional super Schrödinger algebra.

It is easily realized that our results have a more general validity, so that (70,71) and (B.1) are the prototypes of a class of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded invariant equations. The formal demonstration, based on the systematic Clifford algebra tools discussed in section **2**, of the existence of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra symmetry for the Lévy-Leblond square roots of the free heat or Schrödinger equations in $(1 + d)$-dimension, for an arbitrary number of space dimensions d, will be presented elsewhere. We are reporting the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry of the original Lévy-Leblond equation (the square root of the $(1 + 3)$-dimensional free Schrödinger equation) in [22], a paper based on a talk given at the 2016 ICGTMP (Group 31 Colloquium).

Finally, as discussed in Appendix **C**, future investigations of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetries of partial differential equations should systematically study the different possible graded Lie (super)algebras recovered from different assignment of gradings to the symmetry operators.

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Appendix A: On $\mathbb{Z}_2\times\mathbb{Z}_2$-graded color Lie (super)algebras and their relations with quaternions and split-quaternions

To make the paper self-contained we collect here the main properties of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded color Lie algebras and superalgebras induced by a composition law; we make also explicit their relations with both the division algebra of quaternions and its split version.

A $\mathbb{Z}_2\times\mathbb{Z}_2$-graded color(super)algebra $\mathcal{G}$ is decomposed according to

$$\mathcal{G} \quad = \quad \mathcal{G}_{00}\oplus\mathcal{G}_{01}\oplus\mathcal{G}_{10}\oplus\mathcal{G}_{11}. \tag{A.1}$$

For the generators $x_{\vec{\alpha}}, x_{\vec{\beta}}\in\mathcal{G}$, with $\vec{\alpha},\vec{\beta}$ labeling the grading, the $\mathcal{G}\times\mathcal{G}\rightarrow\mathcal{G}$ brackets are

$$(x_{\vec{\alpha}}, x_{\vec{\beta}}) \quad = \quad x_{\vec{\alpha}}x_{\vec{\beta}}-(-1)^{(\vec{\alpha}\cdot\vec{\beta})}x_{\vec{\beta}}x_{\vec{\alpha}}, \tag{A.2}$$

where $x_{\vec{\alpha}}x_{\vec{\beta}}$ is a composition law and, for the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded color algebra, the inner product is defined by

$$(\vec{\alpha}\cdot\vec{\beta}) \quad = \quad \alpha_1\beta_2-\alpha_2\beta_1, \tag{A.3}$$

while, for the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded color superalgebra, the inner product is given by

$$(\vec{\alpha}\cdot\vec{\beta}) \quad = \quad \alpha_1\beta_1+\alpha_2\beta_2. \tag{A.4}$$

For both $\mathbb{Z}_2\times\mathbb{Z}_2$-graded color algebra and superalgebra, the grading $deg(x_{\vec{\alpha}}, x_{\vec{\beta}})$ of the $(x_{\vec{\alpha}}, x_{\vec{\beta}})$ bracket is

$$deg(x_{\vec{\alpha}}, x_{\vec{\beta}}) \quad = \quad \vec{\alpha}+\vec{\beta}. \tag{A.5}$$

$\mathbb{Z}_2\times\mathbb{Z}_2$-graded Jacobi identities are imposed, in both algebra and superalgebra cases, through

$$(-1)^{(\vec{\alpha}\cdot\vec{\gamma})}(x_{\vec{\alpha}},(x_{\vec{\beta}},x_{\vec{\gamma}}))+(-1)^{(\vec{\beta}\cdot\vec{\alpha})}(x_{\vec{\beta}},(x_{\vec{\gamma}},x_{\vec{\alpha}}))+(-1)^{(\vec{\gamma}\cdot\vec{\beta})}(x_{\vec{\gamma}},(x_{\vec{\alpha}},x_{\vec{\beta}})) \quad = \quad 0. \tag{A.6}$$

Some comments are in order. In both algebra and superalgebra cases, the $\mathcal{G}_{00}$-graded sector of $\mathcal{G}$ is singled out with respect to the three other sectors. For what concerns the three remaining sectors we have that

- for the color algebra, all three sectors $\mathcal{G}_{01}$, $\mathcal{G}_{10}$ and $\mathcal{G}_{11}$ are on equal footing while,
- for the color superalgebra, the $\mathcal{G}_{11}$ sector is singled out with respect to the $\mathcal{G}_{01}$ and $\mathcal{G}_{10}$ sectors (which are equivalent and can be interchanged).

The basic examples of $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie algebras and superalgebras are induced by the quaternions and the split-quaternions. We recall here their basic features.

The division algebra of the quaternions (over $\mathbb{R}$) is given by the generators e_0 (the identity) and the three imaginary roots e_i, $i=1,2,3$, with composition law

$$e_i\cdot e_j \quad = \quad -\delta_{ij}e_0+\epsilon_{ijk}e_k, \tag{A.7}$$

for the totally antisymmetric tensor $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$.

One should note that the three imaginary quaternions are on equal footing.

The split-quaternions (over $\mathbb{R}$) are given, see [23], by the generators $\widetilde{e}_0$ (the identity) and $\widetilde{e}_i$, $i = 1, 2, 3$, with composition law

$$\widetilde{e}_i \cdot \widetilde{e}_j = -\eta_{ij}\widetilde{e}_0 + \widetilde{\epsilon}_{ijk}\widetilde{e}_k. \tag{A.8}$$

The metric η_{ij} is diagonal and satisfies

$$\eta_{11} = \eta_{22} = -\eta_{33} = 1, \tag{A.9}$$

while the totally antisymmetric tensor $\widetilde{\epsilon}_{ijk}$ is

$$\widetilde{\epsilon}_{123} = -\widetilde{\epsilon}_{231} = -\widetilde{\epsilon}_{312} = -1. \tag{A.10}$$

For split-quaternions, only the generators $\widetilde{e}_1$, $\widetilde{e}_2$ are on equal footing and can be interchanged.

Different Lie (super)algebras or color Lie (super)algebras can be defined in terms of both quaternions and split-quaternions; their brackets are given by either commutators or anti-commutators obtained from the composition laws ((A.7) for quaternions and (A.8) for split-quaternions). In all cases the graded Jacobi identities are satisfied.

Starting from either the quaternions or the split-quaternions, the following algebraic structures can be defined:
i) a Lie-algebra structure with brackets defined by the commutators;
ii) $\mathbb{Z}_2$-graded Lie algebra structures with brackets defined by the appropriate (anti)commutators;
iii) a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded color Lie algebra structure defined by the (anti)commutators given by (A.2) and (A.3);
iv) $\mathbb{Z}_2\times\mathbb{Z}_2$-graded color Lie superalgebras structures defined by the appropriate (anti)commutators given by (A.2) and (A.4) .
In all cases the identity (either e_0 or $\widetilde{e}_0$) belongs to the even sector ($\mathcal{G}_{00}$ for the $\mathbb{Z}_2\times\mathbb{Z}_2$ gradings).

An important remark is that in the quaternionic case, since all three imaginary roots are on equal footing, there exists a unique assignment of the graded (super)algebras in all four cases above. Without loss of generality we can assign, for the *ii* case, $e_0, e_3 \in \mathcal{G}_0$ and $e_1, e_2 \in \mathcal{G}_1$; for both *iii* and *iv* cases the assignment, without loss of generality, can be assumed to be $e_1 \in \mathcal{G}_{01}, e_2 \in \mathcal{G}_{10}, e_3 \in \mathcal{G}_{11}$.

For split-quaternions, since $\widetilde{e}_3$ is singled out with respect to $\widetilde{e}_1, \widetilde{e}_2$, a unique assignment is only present in the Lie algebra cases *i* and *iii*. In the supercases *ii* and *iv* the assignment of the grading depends on a θ-angle. We have indeed that
- for the $\mathbb{Z}_2$ superalgebra case, inequivalent consistent assignments are given by $\widetilde{e}_0, \cos\theta\widetilde{e}_3 + \sin\theta\widetilde{e}_1 \in \mathcal{G}_0$ and $\widetilde{e}_3, -\sin\theta\widetilde{e}_3 + \cos\theta\widetilde{e}_1 \in \mathcal{G}_1$;
- for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ color superalgebra case, inequivalent consistent assignments are given by $\cos\theta\widetilde{e}_3 + \sin\theta\widetilde{e}_1 \in \mathcal{G}_{11}$, $-\sin\theta\widetilde{e}_3 + \cos\theta\widetilde{e}_1 \in \mathcal{G}_{01}$, $\widetilde{e}_2 \in \mathcal{G}_{10}$.

Appendix B: $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie symmetry of the free $(1+2)$-dimensional Lévy-Leblond equation

We present, for completeness, the construction of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra symmetry of the Lévy-Leblond equation associated with the free $(1+2)$-dimensional heat equation.

For this case the Lévy-Leblond operator is realized by 4×4 matrices. Five 4×4 gamma-matrices γ_μ ($\mu = 1, 2, \ldots, 5$), satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{I}$, with $\eta_{\mu\nu}$ a diagonal matrix with diagonal entries $(+1,+1,+1,-1,-1)$, can be introduced. We have $\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -\gamma_4^2 = -\gamma_5^2 = \mathbb{I}$. We also set $\gamma_\pm = \frac{1}{2}(\gamma_3 \pm \gamma_4)$. Expressed in the notations introduced in Section **2**, an explicit representation is given, e.g., by $\gamma_1 = YX, \gamma_2 = YX, \gamma_3 = XI, \gamma_4 = AI, \gamma_5 = YA$.

The $(1+2)$-dimensional free Lévy-Leblond equation, defined by the operator Ω, is given by

$$\Omega\Psi(x_1,x_2,t) = 0, \qquad \Omega = \gamma_+\partial_t + \lambda\gamma_- + \gamma_i\partial_i. \tag{B.1}$$

In the above formula $i = 1,2$, the Einstein convention over repeated indices is understood and $\partial_i = \partial_{x_i}$. In the following we also make use of the antisymmetric tensor ϵ_{ij}, with the normalization convention $\epsilon_{12} = -\epsilon_{21} = 1$.

The free heat equation reads

$$\Omega^2\Psi(x_1,x_2,t) = (\lambda\partial_t + \partial_i^2)\Psi(x_1,x_2,t) = 0. \tag{B.2}$$

The following list of first-order differential symmetry operators, satisfying the (14) commutation property with Ω, is easily derived

$$\begin{aligned}
H &= \mathbb{I}\partial_t,\\
D &= -\mathbb{I}(t\partial_t + \frac{1}{2}x_i\partial_i + \frac{1}{2}) - \frac{1}{2}\gamma_+\gamma_-,\\
K &= -\mathbb{I}(t^2\partial_t + tx_i\partial_i - \frac{\lambda}{4}x_i^2 + t) - \frac{1}{2}x_i\gamma_+\gamma_i - t\gamma_+\gamma_-,\\
P_{+i} &= \mathbb{I}\partial_i,\\
P_{-i} &= \mathbb{I}(t\partial_i - \frac{1}{2}\lambda x_i) + \frac{1}{2}\gamma_+\gamma_i,\\
J &= \epsilon_{ij}(\mathbb{I}x_i\partial_j + \frac{1}{8}[\gamma_i,\gamma_j]),\\
\widetilde{X} &= \epsilon_{ij}(\gamma_+\gamma_i\partial_j + \frac{\lambda}{8}[\gamma_i,\gamma_j]),\\
C &= \mathbb{I}.
\end{aligned} \tag{B.3}$$

D, K are the only operators of the above list which do not commute with Ω ($[D,\Omega] = \frac{1}{2}\Omega$, $[K,\Omega] = t\Omega$).

D is the scaling operator. For a given operator Z its scaling dimension z is defined via the commutator $[D,Z] = zZ$. The scaling dimensions of the above operators are given by $z(H) = +1, z(K) = -1, z(P_{\pm i}) = \pm\frac{1}{2}$, $z(D) = z(J) = z(\widetilde{X}) = z(C) = 0$.

The operators D,H,K close a $sl(2)$ symmetry algebra; the operators $P_{\pm i}, C$ close the 2-dimensional Heisenberg-Lie algebra $h(2)$. J is the generator of the $SO(2)$ rotational symmetry. These $3+5+1 = 9$ operators span the 2-dimensional Schrödinger symmetry algebra $sch(2)$.

The physical interpretation of the remaining operator $\widetilde{X}$ entering (B.3) is better grasped when introducing the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded extension. For that we need to introduce symmetry operators satisfying the (15) anticommutation property with Ω. We get

$$\begin{aligned}
Q_+ &= \gamma_+\partial_t - \lambda\gamma_-,\\
Q_- &= \gamma_+(t\partial_t + x_i\partial_i + 1) - \frac{1}{2}\lambda x_i\gamma_i - \lambda t\gamma_-,\\
X_i &= \gamma_+\partial_i - \frac{1}{2}\lambda\gamma_i.
\end{aligned} \tag{B.4}$$

The operators Q_+, X_i anticommute with Ω ($\{Q_+,\Omega\} = \{X_i,\Omega\} = 0$), while $\{Q_-,\Omega\} = -\gamma_+\Omega$.

Their scaling dimension (measured by D) are $z(Q_\pm) = \pm\frac{1}{2}$, $z(X_i)$=0.

The operators $Q_\pm$ belong to the odd sector of a $osp(1|2)$ symmetry superalgebra whose even sector is the $sl(2)$ algebra spanned by H, D, K. We get, indeed,

$$\{Q_+, Q_+\} = -2\lambda H, \quad \{Q_+, Q_-\} = 2\lambda D, \quad \{Q_-, Q_-\} = 2\lambda K. \tag{B.5}$$

The closure of the full set of $osp(1|2)$ anticommutation relations follows immediately.

We obtain, as symmetry Lie superalgebra for Ω, the $\mathcal{N}=1$ supersymmetric extension $ssch(2)$ of the two-dimensional Schrödinger algebra by adding, to the even generators spanning $sch(2)$, an odd sector containing the operators $Q_\pm$. This requires, for consistency, to add the operators X_i to the odd sector, since they are recovered from the commutators

$$[P_{\pm i}, Q_\mp] = \pm X_i. \tag{B.6}$$

The $\mathcal{N}=1$ Schrödinger symmetry superalgebra $ssch(2)$ is decomposed according to

$$\begin{aligned} ssch(2) &= ssch(2)_0 \oplus ssch(2)_1, \\ ssch(2)_0 &= \{H, D, K, P_{\pm i}, C, J\}, \\ ssch(2)_1 &= \{Q_\pm, X_i\}. \end{aligned} \tag{B.7}$$

It is a straightforward exercise to check the closure of the $\mathbb{Z}_2$-graded (anti)commutation relations derived for the operators entering $ssch(2)$.

One should note that the operator $\widetilde{X}$ entering (B.3) is still unaccounted for. The operator $\widetilde{X}$, on the other hand, naturally appears in the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded extension, being obtained from the X_i's commutators; we have indeed

$$[X_i, X_j] = \lambda\epsilon_{ij}\widetilde{X}. \tag{B.8}$$

We recall that, in the $\mathbb{Z}_2$-graded $ssch(2)$ algebra, the X_i's brackets are given by the anticommutators

$$\{X_i, X_j\} = \frac{1}{2}\lambda^2\delta_{ij}C. \tag{B.9}$$

We therefore see that the introduction of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded structure offers a proper interpretation to all first-order differential operators entering (B.3) and (B.4).

The construction of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra goes as follows. It requires the introduction of extra first-order and second-order differential operators $P_{s,ij}$ (with $s=\pm1,0$) and $\overline{X}_{\pm ij}$, introduced via the anticommutators

$$P_{1,ij} = \{P_{+i}, P_{+j}\}, \quad P_{0,ij} = \{P_{+i}, P_{-j}\}, \quad P_{-1,ij} = \{P_{-i}, P_{-j}\} \tag{B.10}$$

and

$$\overline{X}_{\pm,ij} = \{P_{\pm i}, X_j\}. \tag{B.11}$$

Their scaling dimensions are, respectively, $z(P_{s,ij}) = s$ and $z(\overline{X}_{\pm,ij}) = \pm\frac{1}{2}$.

We observe that the anticommutators $\{Q_\pm, X_i\} = -\lambda P_{\pm i}$ produce the first-order differential operators $P_{\pm i}$. New first-order differential operators, not entering (B.3) and (B.4), are also obtained. We have, for instance,

$$[Q_+, \widetilde{X}] = \{P_{+2}, X_1\} - \{P_{+1}, X_2\} = \overline{X}_{+,21} - \overline{X}_{+,12} = \lambda(\gamma_2\partial_1 - \gamma_1\partial_2). \tag{B.12}$$

The $\mathbb{Z}_2\times\mathbb{Z}_2$-graded symmetry structure is introduced as a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra, with brackets, the (anti)commutators, defined in (A.2) and based on the (A.4) inner product.

The vector space of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra $\mathcal{L}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ is decomposed according to

$$\mathcal{L}_{\mathbb{Z}_2\times\mathbb{Z}_2} \quad=\quad \mathcal{L}_{00}\oplus\mathcal{L}_{01}\oplus\mathcal{L}_{10}\oplus\mathcal{L}_{11}, \tag{B.13}$$

with the different sectors respectively spanned by the operators

$$\begin{aligned}\mathcal{L}_{00} &= \{H,D,K,J,\widetilde{X},P_{s,ij}\},\\ \mathcal{L}_{01} &= \{P_{\pm i}\},\\ \mathcal{L}_{10} &= \{Q_\pm,\overline{X}_{\pm,ij}\},\\ \mathcal{L}_{11} &= \{X_i\}.\end{aligned} \tag{B.14}$$

With lenghty and tedious, but straighforward, computations the following three main properties are proven:
i) all operators in (B.14), including the second-order differential operators, are symmetry operators, satisfying either the (14) or the (15) conditions with respect to the operator Ω given in (B.1),
ii) the (A.2,A.4) brackets, defined by the respective (anti)commutators, are closed on the (B.14) generators and, finally,
iii) the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Jacobi identities (A.6) are satisfied by the (B.14) generators.

These three main results prove that $\mathcal{L}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ is a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra symmetry of the Lévy-Leblond square root of the $(1+2)$-dimensional free heat equation.

By adding the identity operator C to the $\mathcal{L}_{00}$ sector of $\mathcal{L}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ we obtain the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra $\mathcal{L}'_{\mathbb{Z}_2\times\mathbb{Z}_2}$, given by

$$\mathcal{L}'_{\mathbb{Z}_2\times\mathbb{Z}_2} \quad=\quad \mathcal{L}_{\mathbb{Z}_2\times\mathbb{Z}_2}\oplus u(1). \tag{B.15}$$

Appendix C: On different grading assignments for the symmetry operators

In Appendix **A** we showed, with examples taken from quaternions and split-quaternions, that different $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie (super)algebras can be defined on the same vector space, based on a different grading assignment for the same given set of operators.

It is tempting to investigate this feature for the symmetry operators of the free Lévy-Leblond equation. Since, in particular, the natural finite $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra (53) is induced from the repeated (anti)commutators of the operators $P_{\pm\frac{1}{2}}\in\mathcal{G}_{01}$, $Q_{\pm\frac{1}{2}}\in\mathcal{G}_{10}$, one can ask which is the output if $P_{\pm\frac{1}{2}}$ (given in (26,31)) are kept in the $\mathcal{G}_{01}$ sector, while $Q_{\pm\frac{1}{2}}$ (given in (45,46)) are assigned to the $\mathcal{G}_{11}$ sector of a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra. The consistency condition requires to derive the operator $X\in\mathcal{G}_{10}$ from the commutators $[\pm P_\pm,Q_\pm]=X$. The commutator $[Q_+,Q_-]=\overline{Q}_0$ produces the first-order differential operator $\overline{Q}_0\in\mathcal{G}_{00}$. The further commutators $[P_{\pm\frac{1}{2}},\overline{Q}_0]$ produce the extra (41) pair of $\Omega_{\pm\frac{1}{2}}$ operators. One should note that $\Omega_{\pm\frac{1}{2}}$ are assigned to $\mathcal{G}_{01}$ and that $\Omega_{+\frac{1}{2}}$ is the original Lévy-Leblond operator introduced in equation (22).

The construction has to be continued. The new grading assignment produces a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra which, unlike (53), is infinite-dimensional and spanned by differential operators of any order. To check this statement it is sufficient to compute the repeated commutators $[Q_{+\frac{1}{2}}, [Q_{+\frac{1}{2}}, \ldots [Q_{+\frac{1}{2}}, \overline{Q}_0] \ldots]]$.

For completeness we produce here a finite $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra, defined abstractly in terms of (anti)commutators and whose consistency is implied by the fact that the graded Jacobi identities are satisfied. This algebra cannot be realized by the Lévy-Leblond symmetry operators. It shares, nevertheless, some features in common. We impose, in particular, the existence of two $osp(1|2)$ subalgebras, whose generators (as in Section **5**) are expressed as $\{P_0, P_{\pm\frac{1}{2}}, P_{\pm 1}\}$ and $\{H, D, K, Q_{\pm\frac{1}{2}}\}$. Their respective (anti)commutators are given by equations (29,32,47,48,49). Equation (50) introduces the extra generator X. An extra set of generators, $R_0, R_{\pm 1}, Y, Z, C$, are introduced through the positions

$$R_0 = [Q_{\frac{1}{2}}, Q_{-\frac{1}{2}}], \quad R_{\pm 1} = \{Q_{\pm\frac{1}{2}}, P_{\pm\frac{1}{2}}\}, \quad Y = \{Q_{-\frac{1}{2}}, P_{\frac{1}{2}}\}, \quad Z = \{Q_{\frac{1}{2}}, P_{-\frac{1}{2}}\}, \tag{C.1}$$

while C is the center of the algebra.

A consistent $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra assignment is realized by spanning the graded vector spaces as follows:

$$\begin{aligned}
\mathcal{G}_{00} &= \{H, D, K, P_{\pm 1}, P_0, R_0, C\},\\
\mathcal{G}_{01} &= \{P_{\pm\frac{1}{2}}\},\\
\mathcal{G}_{10} &= \{X, R_{\pm 1}, Y, Z\},\\
\mathcal{G}_{11} &= \{Q_{\pm\frac{1}{2}}\}.
\end{aligned} \tag{C.2}$$

The remaining non-vanishing (anti)commutators which define the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra structure of (C.2) are given by

$$\begin{aligned}
&[H, Y] = R_1, && [H, Z] = R_1\\
&[H, R_{-1}] = Y + Z, && [D, R_{\pm 1}] = \pm R_{\pm 1},\\
&[K, R_1] = Y + Z, && [K, Y] = R_{-1},\\
&[K, Z] = R_{-1}, && [R_0, X] = Y - Z,\\
&\{X, X\} = C, && \{X, R_{\pm 1}\} = -P_{\pm 1},\\
&\{X, Y\} = -P_0, && \{X, Z\} = -P_0,\\
&\{X, Q_{\pm\frac{1}{2}}\} = -P_{\pm\frac{1}{2}}. &&
\end{aligned} \tag{C.3}$$

These (anti)commutators guarantee the closure of the graded Jacobi identities for (C.2).

Unlike (53), the finite $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra (C.2) is not a symmetry algebra of the Lévy-Leblond equation. It is an open question whether it appears as the symmetry algebra of some dynamical system.

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