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Convergence and applications of some solutions of the confluent Heun equation

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Abstract

We study the convergence of a group of solutions in series of confluent hypergeometric functions for the confluent Heun equation. These solutions are expansions in two-sided infinite series (summation from minus to plus infinity) which are interpreted as a modified version of expansions proposed by Leaver [E. W. Leaver, J. Math. Phys. **27**, 1238 (1986)]. We show that the two-sided solutions yield two nonequivalent groups of one-sided series solutions (summation from zero to plus infinity). In the second place, we find that one-sided solutions of one of these groups can be used to solve an equation which describes a time-dependent two-level system of Quantum Optics. For this problem, in addition to finite-series solutions, we obtain infinite-series wavefunctions which are convergent and bounded for any value of the time t, and vanish when t goes to infinity.

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1 Introductory remarks

Recently we have found a new solution for the ordinary spheroidal wave equation [1]. It is given by an expansion in series of irregular confluent hypergeometric functions $\Psi(a,c;y)$ with parameters a and b, and argument y. That expansion is a one-sided series in the sense that the summation, indicated by the index n, runs from zero to plus infinity ($n \geq 0$). From that solution we have inferred, without affording details, a group of solutions for the confluent Heun equation (CHE), given also by expansions in series of Ψ. Now we provide a detailed derivation of the solutions and their convergence.

Actually, we generalize the previous study by considering a group constituted by sets of three solutions: one in series of regular confluent hypergeometric functions $\Phi(a,c;y)$ and two in series of irregular functions $\Psi(a_i,c_i;y)$ (i=1,2). From an initial set of solutions, other sets follow systematically by means of substitutions of variables which preserve the form of the CHE. The inclusion of the functions $\Phi(a,c;y)$ gives solutions valid near the origin $y=0$. As a further generalization, we introduce an arbitrary parameter into the solutions and get two-sided series expansions ($-\infty < n < \infty$) which are necessary to treat problems where there is no free parameter in the CHE. We will find that these two-sided series can be interpreted as a modified version of Leaver's solutions [2].

We start with the two-sided series solutions and, from these, obtain the one-sided solutions as follows.

- We derive the group of two-sided series solutions and study the convergence of the solutions; then, we establish relations between these solutions and solutions found by Leaver in 1986 [2].
- We truncate the aforementioned two-sided series in order to find two different groups of one-sided series solutions.
- Finally, we apply one-sided solutions to a two-state system which represents the interaction of an atom with a pulse of Lorentzian shape [3].

In this section we write the CHE, present the tests for determining the series convergence and explain the procedure for truncating the two-sided series; then, we outline the structure of the paper. We use the CHE in the form [2]

$$z(z-z_0)\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left[B_3-2\omega\eta(z-z_0)+\omega^2z(z-z_0)\right]U=0, \quad \omega\neq 0, \tag{1}$$

where z_0, B_i, η and ω are constants. If $z_0 \neq 0$, then $z=0$ and $z=z_0$ are regular singular points with indicial exponents $(0, 1+B_1/z_0)$ and $(0, 1-B_2-B_1/z_0)$, respectively. The point $z=\infty$ is an irregular singularity where the solutions behave as [2, 4]

$$U(z) \sim e^{\pm i\omega z}\ z^{\mp i\eta-\frac{B_2}{2}}, \qquad z\to\infty. \tag{2}$$

The CHE is also called generalized spheroidal wave equation [2, 5, 6, 7] but the last terminology sometimes is attached to a particular case of the CHE [8, 9]. A limit of Eq. (1), called reduced CHE, is introduced in section 5.

We deal with solutions whose series coefficients satisfy three-term recurrence relations, and use the theory concerning the three-term relations [10, 11] to study the convergence. The general form of the two-sided series solutions is

$$U(z)=\sum_{n=-\infty}^{\infty} b_n^{\mu}\ h_n^{\mu}(z), \tag{3}$$

where the coefficients b_n^{μ} and the functions $h_n^{\mu}(z)$ depend on the parameters of the CHE and on a (characteristic) parameter μ to be determined – in principle, each set of solutions presents a different parameter denoted as μ_i. By omitting the parameter μ, the form of the recurrence relations for b_n^{μ} is

$$\alpha_n\ b_{n+1}+\beta_n\ b_n+\gamma_n\ b_{n-1}=0, \qquad -\infty<n<\infty, \tag{4}$$

where α_n, β_n and γ_n depend on the parameters of the CHE and on μ. Equivalently,

$$\begin{bmatrix} \cdot & \cdot & \cdot & & & & \\ & \gamma_n & \beta_n & \alpha_n & & & \\ & & \gamma_{n+1} & \beta_{n+1} & \alpha_{n+1} & & \\ & & & \gamma_{n+2} & \beta_{n+2} & \alpha_{n+2} & \\ & & & & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ b_{n-1} \\ b_n \\ b_{n+1} \\ \cdot \end{bmatrix} = \mathbf{0}, \qquad -\infty<n<\infty, \tag{5}$$

where $\mathbf{0}$ denotes the null column vector. This system of homogeneous equations has non-trivial solutions only if the determinant of the above tridiagonal matrix vanishes. This condition affords the possible values for μ if there is no free constant in the CHE. If there is an arbitrary constant (and only in this case), we can attribute any convenient value for μ, whereas the condition on the determinant permits to find values for the arbitrary constant of the CHE.

The convergence of the two-sided series comes from the ratios

$$L_1(z) = \left|\frac{b_{n+1}h_{n+1}(z)}{b_n h_n(z)}\right| \text{ when } n \to \infty, \quad \text{and} \quad L_2(z) = \left|\frac{b_{n-1}h_{n-1}(z)}{b_n h_n(z)}\right| \text{ when } n \to -\infty. \tag{6}$$

By the D'Alembert test the series converges in the intersection of the regions where $L_1 < 1$ and $L_2 < 1$, and diverges otherwise excepting the inconclusive case $L_1 = L_2 = 1$. Sometimes it is possible to decide about the convergence when $L_1 = L_2 = 1$ by means of the Raabe test [12, 13]. In effect, if for some value of z,

$$L_1(z) = 1 + \frac{A}{n} + O\left(\frac{1}{n^2}\right), \qquad L_2(z) = 1 + \frac{B}{|n|} + O\left(\frac{1}{n^2}\right): \qquad |n| \to \infty, \tag{7}$$

where A and B are constants, then the Raabe test states that the series converges if $A < -1$ and $B < -1$, and diverges otherwise (for $A = B = -1$ the test is inconclusive). In general, the limits of L_1 and L_2 afford different regions of convergence. Since the one-sided infinite series require only one of these limits, their domain of convergence may be larger than the domain of the corresponding two-sided series.

In fact, from two-sided series we will obtain two groups of one-sided series by taking into account that: (i) the series begins at $n = N + 1$ if $\alpha_{n=N} = 0$, where N is an integer (see page 171 of [14]); (ii) the series terminates at $n = M$ if $\gamma_{n=M+1} = 0$ (page 146 of [14]). We write this as

$$\alpha_{n=N} = 0 \quad \Rightarrow \quad \text{series with } n \geq N + 1; \qquad \gamma_{n=M+1} = 0 \quad \Rightarrow \quad \text{series with } n \leq M. \tag{8}$$

So, to find the two groups of one-sided solutions, we suppose that there is a free constant in the CHE and choose the parameter μ such that

$$\begin{aligned} &\alpha_{-1} = 0 \quad \Rightarrow \quad \text{first group: series with } n \geq 0; \\ &\gamma_1 = 0 \quad \Rightarrow \quad \text{second group: series with } n \leq 0. \end{aligned} \tag{9}$$

The first group includes the solutions found in [1] as particular cases. On the other side, to avoid unusual series with $n \leq 0$, we rearrange the terms of the series in the second group in order to obtain solutions with $n \geq 0$. Alternatively, we consider the Leaver-type solutions whose form is given by

$$\boldsymbol{U}(z) = \sum_{n=-\infty}^{\infty} \boldsymbol{b}_n^{\nu}\, \boldsymbol{h}_n^{\nu}(z), \qquad \text{[Leaver-type solutions]} \tag{10}$$

where the coefficients $\boldsymbol{b}_n^{\nu}$ and the functions $\boldsymbol{h}_n^{\mu}(z)$ depend on the parameters of the CHE and on a parameter ν to be determined. Then, we will find that

$$\boldsymbol{h}_n^{\nu}(z) = h_{-n}^{\mu}(z) \quad \text{if} \quad \nu = \nu(\mu) \quad \text{when} \quad -\infty < n < \infty, \tag{11}$$

where $\nu = \nu(\mu)$ means that we can redefine one parameter in terms of the other; for example, in Eq. (71) $\nu_1 = -\mu_1 - B_2 - B_1/z_0$. Since h_n^{μ} and $\boldsymbol{h}_n^{\nu}$ differ only by the sign of the index n and since n runs from minus to plus infinity in two-sided series, we infer that the expressions (3) and (10) are equivalent to each other, differing at most by a rearrangement of the terms of the series. However, by taking $\mu = \nu = 0$ we find immediately the two nonequivalent groups of one-sided solutions with $n \geq 0$, corresponding to the solutions (9).

In section 2 we deal with the two-sided series solutions and their properties, including only a brief discussion of the Leaver-type solutions (section 2.5). In section 3 we consider the two groups of one-sided series; in particular, we show that it is possible to obtain the two groups by demanding that $n \leq 0$ or $n \geq 0$ in the same two-sided series, as indicated in (9). In section 4 we show how we can apply one-sided solutions of the first group to the aforementioned two-level system, while in section 5 we indicate some open questions concerning the CHE. Solutions omitted in section 2 are presented in appendix A since they are applied in section 4.3. Appendix B supplies power series solutions that are used in section 4.2.

2 Expansions in two-sided series of confluent hypergeometric functions

This section deals with the two-sided series solutions which lead to the one-sided solutions of section 3.

- In section 2.1 we present the confluent hypergeometric functions, give the changes of variables which transform the CHE in itself, and establish some notations.
- In sections 2.2, 2.3 and 2.4, respectively, we write the first set of solutions, show how the set is constructed and study the convergence of each solution.
- In section 2.5 we find the relation between the Leaver-type solutions and the solutions having the form (3).

Other sets, resulting from transformations of the CHE, are given in appendix A.

2.1 Confluent hypergeometric functions and transformations of the CHE

The regular and irregular confluent hypergeometric functions are denoted by $\Phi(a,c;u)$ and $\Psi(a,c;u)$, respectively. They satisfy the confluent hypergeometric equation [15]

$$u\frac{d^2\varphi(y)}{dy^2}+(c-y)\,\frac{d\varphi(y)}{du}-a\varphi(y)=0 \tag{12}$$

which admits the solutions

$$\varphi^1(u)=\Phi(a,c;y),\quad \varphi^2(y)=y^{1-c}\Phi(1+a-c,2-c;y),\quad \varphi^3(y)=\Psi(a,c;y),\quad \varphi^4(y)=e^y y^{1-c}\Psi(1-a,2-c;-y), \tag{13}$$

The functions $\Phi(a,c;y)$ and $\Psi(a,c;y)$ are also denoted by $M(a,c,y)$ and $U(a,c,y)$, respectively [9]. Alternative expressions for (13) follow from the relations

$$\Phi(a,c;y)=e^y\Phi(c-a,c;-y),\qquad \Psi(a,c;y)=y^{1-c}\Psi(1+a-c,2-c;y). \tag{14}$$

The functions φ^1 and φ^2 represent solutions for Eq. (12) satisfactory near the origin ($y=0$), while φ^3 and φ^4 are satisfactory in the neighborhood of infinity.

We will use only φ^1, φ^3 and φ^4 as bases for the series expansions for the solutions of the CHE. If c is not a negative integer, in a common region of validity these three functions are linearly dependent since

$$\frac{\Phi(a,c;y)}{\Gamma(c)}=\frac{e^{\mp i\pi a}}{\Gamma(c-a)}\,\Psi(a,c;y)+\frac{e^{\pm i\pi(c-a)}}{\Gamma(a)}e^y\Psi\left(c-a,c;e^{\pm i\pi}y\right). \tag{15}$$

On the other hand, if $U(z)=U(B_1,B_2,B_3;z_0,\omega,\eta;z)$ denotes one solution of the CHE, the following four transformations T_i leave invariant the form of the CHE [1, 16]:

$$\begin{aligned}
&T_1U(z)=z^{1+\frac{B_1}{z_0}}U(C_1,C_2,C_3;z_0,\omega,\eta;z), && T_2U(z)=(z-z_0)^{1-B_2-\frac{B_1}{z_0}}\,U(B_1,D_2,D_3;z_0,\omega,\eta;z),\\
&T_3U(z)=U(B_1,B_2,B_3;z_0,-\omega,-\eta;z), && T_4U(z)=U(-B_1-B_2z_0,B_2,B_3+2\eta\omega z_0;z_0,-\omega,\eta;z_0-z),
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
&C_1=-B_1-2z_0,\qquad C_2=2+B_2+\tfrac{2B_1}{z_0},\qquad C_3=B_3+\left[1+\tfrac{B_1}{z_0}\right]\left[B_2+\tfrac{B_1}{z_0}\right],\\
&D_2=2-B_2-\tfrac{2B_1}{z_0},\qquad D_3=B_3+\tfrac{B_1}{z_0}\left(\tfrac{B_1}{z_0}+B_2-1\right).
\end{aligned} \tag{17}$$

Then, by using the transformations (16), from an initial set of three solutions it is possible to obtain a group constituted by 16 sets. We use the notation

$$\mathbb{U}_i(z)=\left[U_i(z),U_i^{\infty}(z),\bar{U}_i^{\infty}(z)\right],\qquad [i=1,2,\cdots,16] \tag{18}$$

where the solutions U_i are given by series of regular confluent hypergeometric functions, whereas U_i^{∞} and $\bar{U}_i^{\infty}$ are given by series of irregular confluent hypergeometric functions. We write only the 8 sets, namely,

$$\begin{aligned}
&\mathbb{U}_1(z), && \mathbb{U}_2(z)=T_1\mathbb{U}_1(z), && \mathbb{U}_3(z)=T_2\mathbb{U}_2(z), && \mathbb{U}_4(z)=T_1\mathbb{U}_3(z);\\
&\mathbb{U}_5(z)=T_4\mathbb{U}_1(z), && \mathbb{U}_6(z)=T_1\mathbb{U}_5(z), && \mathbb{U}_7(z)=T_2\mathbb{U}_6(z), && \mathbb{U}_8(z)=T_1\mathbb{U}_7(z),
\end{aligned} \tag{19}$$

since others are obtained by the transformation T_3 which changes (η,ω) by $(-\eta,-\omega)$ in the above sets. In order to write the solutions U_i and U_i^{∞} in terms of the same series coefficients $b_n^{(i)}$, instead of $\Phi(a,b;y)$ we use the function $\tilde{\Phi}(a,b;y)$ defined by [2]

$$\tilde{\Phi}(a,c;y=\frac{\Gamma(c-a)}{\Gamma(c)}\,\Phi(a,c;y)=\frac{\Gamma(c-a)}{\Gamma(c)}\left[1+\frac{a}{c}\frac{y}{1!}+\frac{a(a+1)}{c(c+1)}\frac{y^2}{2!}+\cdots\right]. \tag{20}$$

Since this definition changes the coefficients of $U_i(z)$, in order to use relations (8), it is convenient to rewrite $U_i(z)$ in terms of $\Phi(a,b;y)$ as in Eq. (26). The series coefficients satisfy three-term recurrence relations having the form (4), that is,

$$\alpha_n^{(i)}\,b_{n+1}^{(i)}+\beta_n^{(i)}\,b_n^{(i)}+\gamma_n^{(i)}\,b_{n-1}^{(i)}=0,\quad \bar{\alpha}_n^{(i)}\,\bar{b}_{n+1}^{(i)}+\bar{\beta}_n^{(i)}\,\bar{b}_n^{(i)}+\bar{\gamma}_n^{(i)}\,\bar{b}_{n-1}^{(i)}=0,\quad -\infty<n<\infty \tag{21}$$

where $\bar{b}_n^{(i)}$ denote the coefficients of the solutions $\bar{U}_i^{\infty}$. The coefficients $\alpha_n^{(i)}$, $\beta_n^{(i)}$ and $\gamma_n^{(i)}$ ($\bar{\alpha}_n^{(i)}$, $\bar{\beta}_n^{(i)}$ and $\bar{\gamma}_n^{(i)}$) depend on the parameters of the differential equation as well as on a parameter μ_i. To assure two-sided series, we must exclude certain values of this parameter μ_i: for example, μ_i must be such that $\alpha_n^{(i)}\neq 0$ and $\gamma_n^{(i)}\neq 0$, on the contrary the series truncates on the left ($\alpha_n^{(i)}=0$) or on the right-hand side ($\gamma_n^{(i)}=0$), as already explained. From relations (21) it results a transcendental (characteristic) equation given as a sum of two infinite continued fractions [2], namely,

$$\beta_0=\frac{\alpha_{-1}\gamma_0}{\beta_{-1}-}\frac{\alpha_{-2}\gamma_{-1}}{\beta_{-2}-}\frac{\alpha_{-3}\gamma_{-2}}{\beta_{-3}-}\cdots+\frac{\alpha_0\gamma_1}{\beta_1-}\frac{\alpha_1\gamma_2}{\beta_2-}\frac{\alpha_2\gamma_3}{\beta_3-}\cdots, \tag{22}$$

which is equivalent to the vanishing of the determinant of a tridiagonal matrix having the form given in (5).

In each set, the solutions $U_i(z)$ and $U_i^{\infty}(z)$ are valid only if the parameters of the CHE satisfy certain restrictions which avoid that the hypergeometric functions reduce to polynomials of fixed degree l. In effect, we have the relations [15]

$$\tilde{\Phi}(-l,\alpha+1;y)=(-1)^l\Psi(-l,\alpha+1;y)=l!\ L_l^{\alpha}(y),\quad l=0,1,2,\cdots \tag{23}$$

where the $L_l^{\alpha}(y)$ denote Laguerre polynomials of degree l. Then, if l is fixed (i. e., if l does not depend on the summation index n) there is no sense in using these polynomials as bases for series expansions. In addition, we have to take into account that the functions $\tilde{\Phi}(a,c;y)$ in general are well defined only if c is not zero or negative integer. Further, if the definition (20) is unsuitable because the argument of $\Gamma(c-a)$ is zero or a negative integer, we redefine the series coefficients as in (26).

2.2 The initial set of solutions

The following set of solutions, $(U_1(z),U_1^{\infty}(z),\bar{U}_1^{\infty}(z))$, is constituted by expansions in series of the functions $\varphi^1(y)$, $\varphi^3(y)$ and $\varphi^4(y)$ with $y=-2i\omega z$, $a=i\eta+B_2/2$ and $c=-n-\mu_1-B_1/z_0$. The solutions $U_1(z)$ and $U_1^{\infty}(z)$ are valid only if $i\eta+\frac{B_2}{2}\neq 0,-1,\cdots$.

$$\begin{aligned}
\begin{bmatrix} U_1(z)\\ U_1^{\infty}(z)\end{bmatrix} &= e^{i\omega z}\sum_{n=-\infty}^{\infty} b_n^{(1)}\begin{bmatrix}\tilde{\Phi}\left(i\eta+\frac{B_2}{2},-n-\mu_1-\frac{B_1}{z_0};-2i\omega z\right)\\ \Psi\left(i\eta+\frac{B_2}{2},-n-\mu_1-\frac{B_1}{z_0};-2i\omega z\right)\end{bmatrix}, \qquad i\eta+\tfrac{B_2}{2}\neq 0,-1,-2,\cdots,\\
\bar{U}_1^{\infty}(z) &= e^{-i\omega z}\sum_{n=-\infty}^{\infty}\bar{b}_n^{(1)}\Psi\left(-n-\mu_1-i\eta-\frac{B_1}{z_0}-\frac{B_2}{2},-n-\mu_1-\frac{B_1}{z_0};2i\omega z\right)
\end{aligned} \tag{24}$$

where $b_n^{(1)}$ and $\bar{b}_n^{(1)}$ satisfy the recurrence relations (21) with

$$\begin{aligned}
\alpha_n^{(1)} &= -2i\omega z_0(n+\mu_1+1), \qquad \gamma_n^{(1)}=-\left[n+\mu_1+i\eta+\frac{B_2}{2}+\frac{B_1}{z_0}\right]\left[n+\mu_1-1+B_2+\frac{B_1}{z_0}\right],\\
\beta_n^{(1)} &= [n+\mu_1]\left[n+\mu_1+1+2i\omega z_0+B_2+\frac{2B_1}{z_0}\right]+\left[B_2+\frac{B_1}{z_0}\right]\left[1+\frac{B_1}{z_0}+i\omega z_0\right]+B_3
\end{aligned} \tag{25a}$$

and

$$\begin{aligned}
\bar{\alpha}_n^{(1)} &= -2i\omega z_0(n+\mu_1+1)\left[n+\mu_1+1+i\eta+\frac{B_2}{2}+\frac{B_1}{z_0}\right], \qquad \bar{\beta}_n^{(1)}=\beta_n^{(1)},\\
\bar{\gamma}_n^{(1)} &= -\left[n+\mu_1-1+B_2+\frac{B_1}{z_0}\right], \qquad \bar{b}_n^{(1)}=(-1)^n\Gamma\left[-n-\mu_1-i\eta-\frac{B_1}{z_0}-\frac{B_2}{2}\right]b_n^{(1)}.
\end{aligned} \tag{25b}$$

From the above connection between the coefficients, we get

$$U_1(z) = e^{i\omega z}\sum_{n=-\infty}^{\infty}\frac{(-1)^n\bar{b}_n^{(1)}}{\Gamma\left[-n-\mu_1-(B_1/z_0)\right]}\ \Phi\left(i\eta+\frac{B_2}{2},-n-\mu_1-\frac{B_1}{z_0};-2i\omega z\right), \tag{26}$$

where $-n-\mu_1-(B_1/z_0)$ is not zero or negative integer, by hypothesis. This exhibits a particular case of the fact that coefficients of $U_i(z)$ and $\bar{U}_i^{\infty}(z)$ satisfy the same recurrence relations.

The set (24) is a generalization of a one-sided series solution found in [1], namely,

$$\mathfrak{U}_3(z)=e^{i\omega z}\sum_{n=0}^{\infty}b_n^3\ \Psi\left(i\eta+\frac{B_2}{2},-\frac{B_1}{z_0}-n;-2i\omega z\right),\qquad i\eta+\frac{B_2}{2}\neq 0,-1,\cdots.$$

In fact, if we replace $\Psi\left(i\eta+B_2/2,-B_1/z_0-n;-2i\omega z\right)$ by $\Psi\left(i\eta+B_2/2,-B_1/z_0-n-\mu_1;-2i\omega z\right)$ we get $U_1^{\infty}(z)$. Then, we add the expansion $U_1(z)$ in order to get a solution valid in the neighborhood of $z=0$ since the function $\Phi(a,c;y=-2i\omega z)$ is well defined at $z=0$ when c is not a negative integer. The solution $\bar{U}_1^{\infty}(z)$ formally results as a linear combination of $U_1(z)$ and $U_1^{\infty}(z)$ by means of relation (15), but it is not convergent at $z=0$ or $z=\infty$. This is seen by writing

$$\bar{U}_1^{\infty}(z)\overset{(14,24)}{=}e^{-i\omega z}z^{1+\frac{B_1}{z_0}}\sum_{n=-\infty}^{\infty}\bar{b}_n^{(1)}\,(2i\omega z)^{n+\mu_1}\,\Psi\left(1-i\eta-\frac{B_2}{2},n+\mu_1+2+\frac{B_1}{z_0};2i\omega z\right).$$

Thence, we get

$$\bar{U}_1^{\infty}(z) = e^{-i\omega z} z^{1+\frac{B_1}{z_0}} \sum_{n=-\infty}^{\infty} \bar{b}_n^{(1)} \,(2i\omega z)^{n+\mu_1} \qquad \text{if} \quad 1 - i\eta - \tfrac{B_2}{2} = 0 \tag{27}$$

and, for this particular case, we see that the series does not converge at $z = 0$ and $z = \infty$. Notice that we have disregarded a solution $\bar{U}_1$ in terms of regular confluent hypergeometric functions; it reads

$$\bar{U}_1 = e^{i\omega z} \sum_{n=-\infty}^{\infty} b_n^{(1)} \,[2i\omega z]^{n+\mu_1+1+\frac{B_1}{z_0}}\, \tilde{\Phi}\left[n + \mu_1 + 1 + i\eta + \tfrac{B_2}{2} + \tfrac{B_1}{z_0}, n + \mu_1 + 2 + \tfrac{B_1}{z_0}; -2i\omega z\right]. \tag{28}$$

This expansion may be useful to study solutions for the limit (121) of the CHE.

2.3 Construction of the solutions

We construct only the first set of solutions since the others follow from that by the transformations (16). Thus, to get the solutions (24), first we accomplish the substitutions

$$U(z) = e^{i\omega z}\, V(y), \qquad y = -2i\omega z \tag{29}$$

which, when inserted into (1), lead to

$$\big(y + 2i\omega z_0\big) y \left[\tfrac{d^2V}{dy^2} - \tfrac{dV}{dy}\right] + \big[B_2 y - 2i\omega B_1\big] \tfrac{dV}{dy} + \big[B_3 + i\omega B_1 + 2\eta\omega z_0 - \big(i\eta + \tfrac{B_2}{2}\big)\, y\big]\, V = 0. \tag{30}$$

In the second place, we expand $V(y)$ as

$$V(y) = \sum_{n=-\infty}^{\infty} b_n^{(1)} \,\left(\alpha, -\tfrac{B_1}{z_0} - n - \mu_1; y\right), \quad (a, c; y) = \Psi(a, c; y) \text{ or } (a, c; y) = \tilde{\Phi}(a, c; y) \tag{31}$$

and show that $\alpha = i\eta + (B_2/2)$, while the coefficients $b_n^{(1)}$ satisfy the relations (21) with the coefficients (25a). To this end, we use the equations [9, 2]

$$\begin{aligned}
& y\tfrac{d^2(a,c;y)}{dy^2} - y\tfrac{d(a,c;y)}{dy} = -c\tfrac{d(a,c;y)}{dy} + a(a, c; y), \qquad \text{[see Eq. (12)]},\\
& \tfrac{d(a,c;y)}{dy} = (a, c; y) - (a, c + 1; y),\\
& y\tfrac{d(a,c;y)}{dy} = [1 - c](a, c; y) + [c - a - 1](a, c - 1; y),\\
& y(a, c + 1; y) = [a + 1 - c](a, c - 1; y) + [y + c - 1](a, c; y).
\end{aligned} \tag{32}$$

Thence, putting $a = \alpha$, $c = -(B_1/z_0) - n - \mu_1$ and ${}_n(y) = [\alpha, -(B_1/z_0) - n - \mu_1; y]$, we find

$$\begin{aligned}
& y\tfrac{d^2{}_n}{dy^2} - y\tfrac{d_n}{dy} = \left(n + \mu_1 + \tfrac{B_1}{z_0}\right) \tfrac{d_n}{dy} + \alpha_n,\\
& \tfrac{d_n}{dy} = {}_n - {}_{n-1}, \qquad y\tfrac{d_n}{dy} = [1 - c]\,{}_n - \left[n + \mu_1 + \alpha + 1 + \tfrac{B_1}{z_0}\right]{}_{n+1},\\
& y_{n-1} = \left(n + \mu_1 + 1 + \alpha + \tfrac{B_1}{z_0}\right){}_{n+1} + \left(y - \tfrac{B_1}{z_0} - n - \mu_1 - 1\right){}_n.
\end{aligned} \tag{33}$$

Inserting $V(y) = \sum_{n=-\infty}^{\infty} b_n^{(1)}\,{}_n(y)$ into (30) and using (33), we find

$$\sum_{n=-\infty}^{\infty} \left[\alpha_{n-1}^{(1)}\, b_n^{(1)}\,{}_{n-1}(y) + \beta_n^{(1)}\, b_n^{(1)}\,{}_n(y) + \gamma_{n+1}^{(1)}\, b_n^{(1)}\,{}_{n+1}(y) \;+\; \big(\alpha - i\eta - \tfrac{B_2}{2}\big)\, b_n^{(1)} y_n(y)\right] = 0, \tag{34}$$

where

$$\begin{aligned}
& \alpha_n^{(1)} = -2i\omega z_0 \,[n + \mu_1 + 1], \qquad \gamma_n^{(1)} = -\left[n + \mu_1 + \alpha + \tfrac{B_1}{z_0}\right]\left[n + \mu_1 - 1 + \tfrac{B_1}{z_0} + B_2\right],\\
& \beta_n^{(1)} = 2i\omega z_0 \left[n + \mu_1 + \alpha + \tfrac{B_1}{z_0}\right] + \left[n + \mu_1 + 1 + \tfrac{B_1}{z_0}\right]\left[n + \mu_1 + \tfrac{B_1}{z_0} + B_2\right] + 2\eta\omega z_0 - i\omega B_1 + B_3.
\end{aligned}$$

Now we determine the constant α by equating to zero the last term of Eq. (34), that is,

$$\big(\alpha - i\eta - \tfrac{B_2}{2}\big)\, b_n^{(1)} y_n(y) = 0 \quad \Rightarrow \quad \alpha = i\eta + \tfrac{B_2}{2}.$$

In this manner we recover the coefficients (25a), while Eq. (34) becomes

$$\sum_{n=-\infty}^{\infty}\left[\alpha_{n-1}^{(1)}\, b_n^{(1)}\,{}_{n-1}(y)+\beta_n^{(1)}\, b_n^{(1)}\,{}_n(y)+\gamma_{n+1}^{(1)}\, b_n^{(1)}\,{}_{n+1}(y)\right]=0, \tag{35}$$

which, for two-sided infinite series, can be factorized as

$$\sum_{n=-\infty}^{\infty}\left[\alpha_n^{(1)}\, b_{n+1}^{(1)}+\beta_n^{(1)}\, b_n^{(1)}+\gamma_n^{(1)}\, b_{n-1}^{(1)}\right]{}_n(y)=0.$$

This equation is fulfilled by the recurrence relations (21) for $b_n^{(1)}$.

To find the solutions $\bar{U}_1^{\infty}(z)$, first we carry out the substitutions

$$U(z)=\bar{U}_1^{\infty}(y)=e^{-i\omega z}F(y),\qquad y=2i\omega z, \tag{36a}$$

which transform the CHE (1) into

$$(y-2i\omega z_0)y\left[\frac{d^2F}{dy^2}-\frac{dF}{dy}\right]+\left[B_2y+2i\omega B_1\right]\frac{dF}{dy}+\left[B_3-i\omega B_1+2\eta\omega z_0+\left(i\eta-\frac{B_2}{2}\right)y\right]F=0. \tag{36b}$$

In the second place, we expand $F(y)$ as

$$F(y)=\sum_{n=-\infty}^{\infty}\bar{b}_n^{(1)}\,\psi_n(y),\quad \psi_n=\Psi\left[-n-\mu_1-i\eta-\frac{B_1}{z_0}-\frac{B_2}{2},-n-\mu_1-\frac{B_1}{z_0};2i\omega z\right] \tag{37}$$

and use the relations [9, 2]

$$\begin{aligned}
&y\frac{d^2\Psi(a,c;y)}{dy^2}-y\frac{d\Psi(a,c;y)}{dy}=-c\frac{d\Psi(a,c;y)}{dy}+a\Psi(a,c;y),\qquad \text{[see Eq. (12)]},\\
&\frac{d\Psi(a,c;y)}{dy}=-a\Psi(a+1,c+1;y),\quad y\frac{d\Psi(a,c;y)}{dy}=[1-c+y]\Psi(a,c;y)-\Psi(a-1,c-1;y),\\
&ay\Psi(a+1,c+1;y)+[y-c+1]\Psi(a,c;y)-\Psi(a-1,c-1;y)=0.
\end{aligned} \tag{38}$$

Thence, we find

$$\begin{aligned}
&y\frac{d^2\psi_n(y)}{dy^2}-y\frac{d\psi_n(y)}{dy}=\left(n+\mu_1+\frac{B_1}{z_0}\right)\frac{d\psi_n(y)}{dy}-\left(n+\mu_1+i\eta+\frac{B_1}{z_0}+\frac{B_2}{2}\right)\psi_n(y),\\
&\frac{d\psi_n}{dy}=\left(n+\mu_1+i\eta+\frac{B_1}{z_0}+\frac{B_2}{2}\right)\psi_{n-1},\qquad y\frac{d\psi_n}{dy}=\left(n+1+\mu_1+\frac{B_1}{z_0}+y\right)\psi_n-\psi_{n+1},\\
&\left(n+\mu_1+i\eta+\frac{B_1}{z_0}+\frac{B_2}{2}\right)y\psi_{n-1}(y)-\left(n+\mu_1+1+\frac{B_1}{z_0}+y\right)\psi_n(y)+\psi_{n+1}(y)=0.
\end{aligned} \tag{39}$$

Then, by inserting (37) into (36b) and using the above relations we obtain

$$\sum_{n=-\infty}^{\infty}\left[\bar{\alpha}_{n-1}^{(1)}\,\bar{b}_n^{(1)}\,\psi_{n-1}(y)+\beta_n^{(1)}\,\bar{b}_n^{(1)}\,\psi_n(y)+\bar{\gamma}_{n+1}^{(1)}\,\bar{b}_n^{(1)}\,\psi_{n+1}(y)\right]=0, \tag{40}$$

which, for two-sided infinite series, can be rewritten as

$$\sum_{n=-\infty}^{\infty}\left[\bar{\alpha}_n^{(1)}\,\bar{b}_{n+1}^{(1)}+\beta_n^{(1)}\,\bar{b}_n^{(1)}+\bar{\gamma}_n^{(1)}\,\bar{b}_{n-1}^{(1)}\right]\psi_n(y)=0.$$

This equation gives the recurrence relations (21) for $\bar{b}_n^{(1)}$.

2.4 Convergence of the solutions

The function $\Psi(a,c;y)$ is satisfactory in the neighborhood of infinity but it may become inappropriate at $y=0$ (in particular, it presents logarithmic terms at $y=0$ if $c=0,1,2$ [9]). The ratio tests do not detect this fact; on account of this, we suppose that the U_i^{∞} are not valid when $y=0$. Further, we will see that the ratios between successive terms of the series hold only for finite values of z. For this reason, before deciding on the convergence at $z=\infty$, it is necessary to examine the behavior of each solution when $|z|\to\infty$.

The regions of convergence of the one-sided series will depend on how the two-sided series are truncated: on the left-hand ($n\geq 0$) or on the right-hand side ($n\leq 0$). Then, in the following it is important to distinguish between the results coming from the ratio test when $n\to-\infty$ and those coming from $n\to\infty$. We find that:

- The solutions $U_i(z)$ in series of regular confluent hypergeometric functions converge for any finite value of z, but the ratio test does not allow to decide on the convergence at $z=\infty$ – see limit (51).
- When $n\to\infty$ the ratio test for the $U_i^{\infty}(z)$ does not exclude any finite value of z but the functions $\Psi(a,c;y)$ are not appropriate for $y=0$ and, so, the U_i^{∞} are not valid at $z=0$ (if $i=1,2,3,4$) and at $z=z_0$ (if $i=5,6,7,8$); the U_i^{∞} converge as well at $z=\infty$ under the conditions (41). On the other side, when $n\to-\infty$ the test implies that the U_i^{∞} converge for (finite) values of z such that $|z|>|z_0|$ $(i=1,2,3,4)$ or $|z-z_0|>|z_0|$ $(i=5,6,7,8)$; they converge also at $|z|=|z_0|$, or at $|z-z_0|=|z_0|$ under the conditions given in (42) and (43).
- By the ratio test when $n\to\infty$, the solutions $\bar{U}_i^{\infty}(z)$ do not converge at $z=\infty$. By the test when $n\to-\infty$, the $\bar{U}_i^{\infty}$ converge only for values z such that $|z|>|z_0|$ $(i=1,2,3,4)$ or $|z-z_0|>|z_0|$ $(i=5,6,7,8)$; they converge also at $|z|=|z_0|$, or at $|z-z_0|=|z_0|$ under the restrictions given in (42) and (43).

From the Raabe test when $n\to\infty$, we will find that the U_i^{∞} converge at $z=\infty$ if

$$\begin{aligned} n\to\infty:\quad & \mathrm{Re}\left[i\eta+\tfrac{B_2}{2}-1\right]<0 \text{ for } U_1^{\infty},\ U_5^{\infty}; \quad \mathrm{Re}\left[i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}\right]<0 \text{ for } U_2^{\infty},\ U_6^{\infty};\\ & \mathrm{Re}\left[i\eta-\tfrac{B_2}{2}+1\right]<0 \text{ for } U_3^{\infty},\ U_7^{\infty}; \quad \mathrm{Re}\left[i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}\right]<0 \text{ for } U_4^{\infty},\ U_8^{\infty}. \end{aligned} \tag{41}$$

The following conditions come from the Raabe test when $n\to-\infty$:

$$n\to-\infty:\ |z|\geq|z_0|,\ \text{sign ``='' if } \begin{cases} \mathrm{Re}\left[B_2+\frac{B_1}{z_0}\right]<1 \text{ for } (U_1^{\infty},\bar{U}_1^{\infty}),\ (U_2^{\infty},\bar{U}_2^{\infty});\\ \mathrm{Re}\left[B_2+\frac{B_1}{z_0}\right]>1 \text{ for } (U_3^{\infty},\bar{U}_3^{\infty}),\ (U_4^{\infty},\bar{U}_4^{\infty}); \end{cases} \tag{42}$$

$$n\to-\infty:\ |z-z_0|\geq|z_0|,\ \text{sign ``='' if } \begin{cases} \mathrm{Re}\left[\frac{B_1}{z_0}\right]>-1 \text{ for } (U_5^{\infty},\bar{U}_5^{\infty}),\ (U_8^{\infty},\bar{U}_8^{\infty}),\\ \mathrm{Re}\left[\frac{B_1}{z_0}\right]<-1 \text{ for } (U_6^{\infty},\bar{U}_6^{\infty}).\ (U_7^{\infty},\bar{U}_7^{\infty}). \end{cases} \tag{43}$$

It is sufficient to consider explicitly only the first set, $\mathbb{U}_1=(U_1,U_1^{\infty},\bar{U}_1^{\infty})$; the results for the other sets follow by the transformations (16) applied according to (19).

2.4.1 Convergence of U_1 and U_1^{∞}

We write U_1 and U_1^{∞} as

$$\left[U_1(z),U_1^{\infty}(z)\right]=e^{i\omega z}\sum_{n=-\infty}^{\infty}b_n^{(1)}{}_n(y), \qquad y=-2i\omega z, \tag{44a}$$

where ${}_n(y)$ is defined by

$${}_n(y)=\left[\tilde{\Phi}\left(i\eta+\tfrac{B_2}{2},-n-\mu_1-\tfrac{B_1}{z_0};y\right),\Psi\left(i\eta+\tfrac{B_2}{2},-n-\mu_1-\tfrac{B_1}{z_0};y\right)\right]. \tag{44b}$$

For these solutions the ratios (6) read $(y=-2i\omega z)$

$$L_1(z)=\left|\frac{b_{n+1}^{(1)}{}_{n+1}(y)}{b_n^{(1)}{}_n(y)}\right| \ \text{ for } n\to\infty, \qquad L_2(z)=\left|\frac{b_{n-1}^{(1)}{}_{n-1}(y)}{b_n^{(1)}{}_n(y)}\right| \ \text{ for } n\to-\infty. \tag{45}$$

From relations (21) for $b_n^{(1)}$, when $n\to\pm\infty$ we find

$$2i\omega z_0\left[1+\frac{1+\mu_1}{n}\right]\frac{b_{n+1}^{(1)}}{b_n^{(1)}}-\left[n+2\mu_1+1+2i\omega z_0+B_2+\frac{2B_1}{z_0}\right]+\left[n+2\mu_1-1+i\eta+\frac{2B_1}{z_0}+\frac{3B_2}{2}\right]\frac{b_{n-1}^{(1)}}{b_n^{(1)}}=0,$$

which is satisfied by $(n\to\pm\infty)$

$$\begin{aligned} &\frac{b_{n+1}^{(1)}}{b_n^{(1)}}\sim1+\frac{1}{n}\left(i\eta+\frac{B_2}{2}-2\right) \quad\Leftrightarrow\quad \frac{b_{n-1}^{(1)}}{b_n^{(1)}}\sim1-\frac{1}{n}\left(i\eta+\frac{B_2}{2}-2\right) \quad \text{or}\\ &\frac{b_{n+1}^{(1)}}{b_n^{(1)}}\sim\frac{n}{2i\omega z_0}\left[1+\frac{1}{n}\left(\mu_1+B_2+\frac{2B_1}{z_0}\right)\right] \quad\Leftrightarrow\quad \frac{b_{n-1}^{(1)}}{b_n^{(1)}}\sim\frac{2i\omega z_0}{n}\left[1+\frac{1}{n}\left(1-\mu_1-B_2-\frac{2B_1}{z_0}\right)\right]. \end{aligned}$$

Then, for the minimal solutions [10] for $b_n^{(1)}$ we find

$$\frac{b_{n+1}^{(1)}}{b_n^{(1)}}\sim1+\frac{1}{n}\left(i\eta+\frac{B_2}{2}-2\right) \text{ if } n\to\infty, \qquad \frac{b_{n-1}^{(1)}}{b_n^{(1)}}\sim\frac{2i\omega z_0}{n}\left[1+\frac{1}{n}\left(1-\mu_1-B_2-\frac{2B_1}{z_0}\right)\right] \text{ if } n\to-\infty. \tag{46}$$

On the other hand, from the last equation given in (33) we obtain ($y=-2i\omega z$)

$$\left[n+\mu_1+1+i\eta+\frac{B_1}{z_0}+\frac{B_2}{2}\right]\frac{{}_{n+1}(y)}{{}_{n}(y)}-\left[n+\mu_1+1+2i\omega z+\frac{B_1}{z_0}\right]+2i\omega z\frac{{}_{n-1}(y)}{{}_{n}(y)}=0. \tag{47}$$

If z is bounded ($2i\omega z/n\to 0$), when $n\to\pm\infty$ Eq. (47) is satisfied by two expressions for ${}_{n+1}/{}_{n}$ or ${}_{n-1}/{}_{n}$, namely,

$$\begin{aligned}&\frac{{}_{n+1}}{{}_{n}}\sim 1-\frac{1}{n}\left(i\eta+\frac{B_2}{2}\right) &&\Leftrightarrow&& \frac{{}_{n-1}}{{}_{n}}\sim 1+\frac{1}{n}\left(i\eta+\frac{B_2}{2}\right),\\ &\frac{{}_{n+1}}{{}_{n}}\sim \frac{2i\omega z}{n}\left[1-\frac{1}{n}\left(2+\mu_1+\frac{B_1}{z_0}\right)\right] &&\Leftrightarrow&& \frac{{}_{n-1}}{{}_{n}}\sim \frac{n}{2i\omega z}\left[1+\frac{1}{n}\left(1+\mu_1+\frac{B_1}{z_0}\right)\right].\end{aligned} \tag{48}$$

If the ${}_{n}$ represent the functions $\tilde{\Phi}$, we select

$$\frac{{}_{n+1}}{{}_{n}}\sim 1-\frac{1}{n}\left(i\eta+\frac{B_2}{2}\right)\ \text{if}\ n\to\infty,\qquad \frac{{}_{n-1}}{{}_{n}}\sim 1+\frac{1}{n}\left(i\eta+\frac{B_2}{2}\right)\ \text{if}\ n\to-\infty\quad [{}_{n}=\tilde{\Phi}]$$

because these choices are consistent with the fact that $\lim_{c\to\infty}=\Phi(a,c;y)=1$ if a and y remain fixed and bounded [15]. Thence, using (46), we get the ratios

$$\frac{b^{(1)}_{n+1}{}_{n+1}}{b^{(1)}_{n}{}_{n}}\sim 1-\frac{2}{n},\ \text{if}\ n\to\infty;\qquad \frac{b^{(1)}_{n-1}{}_{n-1}}{b^{(1)}_{n}{}_{n}}\sim\frac{2i\omega z_0}{n},\ \text{if}\ n\to-\infty\quad[\text{solution}\ U_1(z)]. \tag{49}$$

Therefore, the Raabe test assures that $U_1(z)$ converges for any finite value of z. To examine the behavior of U_1 when $z\to\infty$ we use [9]

$$\tilde{\Phi}(a,c;y)\sim\frac{\Gamma(c-a)}{\Gamma(a)}\,e^{y}y^{a-c}+e^{\pm i\pi a}y^{-a},\quad [|y|\to\infty,\ a\neq 0,-1,\cdots;\ c-a\neq 0,-1,\cdots] \tag{50}$$

where the upper sign holds for $-\pi/2<\arg y<3\pi/2$ and the lower sign, for $-3\pi/2<\arg y\leq-\pi/2$. Thence, for $z\to\infty$ we find

$$U_1(z)\sim e^{i\omega z}\ z^{-i\eta-\frac{B_2}{2}}\sum_{n=-\infty}^{\infty}b^{(1)}_n+e^{-i\omega z}\ z^{i\eta+\mu_1+\frac{B_1}{z_0}+\frac{B_2}{2}}\sum_{n=-\infty}^{\infty}\Gamma_n\,b^{(1)}_n\,(-2i\omega z)^n,\qquad \Gamma_n=\Gamma\left(-\frac{B_1}{z_0}-\frac{B_2}{2}-n-\mu_1-i\eta\right).$$

This limit does not assure convergence at $z=\infty$ because the ratio between the terms of the second series becomes indefinite when $n\to+\infty$ since

$$\frac{\Gamma_{n+1}\,b^{(1)}_{n+1}\,(-2i\omega z)^{n+1}}{\Gamma_n\,b^{(1)}_n\,(-2i\omega z)^{n}}\overset{(46)}{\sim}\frac{2i\omega z}{n},\qquad [z\to\infty,\ n\to\infty].$$

To select the ratios for ${}_{n}=\Psi$ we use the fact that, if $|c|\to\infty$ while a and y remain fixed and bounded, then [15]

$$\Psi(a,c;y)=c^{-a}\left[(-1)^{-a}+\frac{\sqrt{2\pi}}{\Gamma(a)}\left(\frac{c}{ey}\right)^{c+a-\frac{3}{2}}y^{a-\frac{1}{2}}\ e^{y+a-\frac{3}{2}}\right]\left[1+O\left(\frac{1}{|c|}\right)\right],$$
$$|c|\to\infty,\qquad a\neq 0,-1,-2,\cdots,\qquad |\arg(\pm c)|<\pi. \tag{51}$$

Remembering that in this case ${}_{n}(y)=\Psi(a,c;y)=\Psi\left(i\eta+B_2/2,-n-\mu_1-B_1/z_0;-2i\omega z\right)$ we see that the first term on the right-hand side of (51) is the only term relevant as $n\to\infty$, while the second term is the only one relevant as $n\to-\infty$. Thus,

$$\frac{{}_{n+1}}{{}_{n}}\sim 1-\frac{1}{n}\left(i\eta+\frac{B_2}{2}\right)\ \ \text{if}\ n\to\infty,\qquad \frac{{}_{n-1}}{{}_{n}}\sim\frac{n}{2i\omega z}\left[1+\frac{1}{n}\left(1+\mu_1+\frac{B_1}{z_0}\right)\right]\ \ \text{if}\ \to-\infty.$$

Notice that (51) is responsible only for the first term of these limits; the second term comes from (48). Thus, by using also the ratios (46), for U_1^{∞} we find

$$\frac{b^{(1)}_{n+1}{}_{n+1}}{b^{(1)}_{n}{}_{n}}\sim 1-\frac{2}{n}\ \ \text{if}\ n\to\infty,\qquad \frac{b^{(1)}_{n-1}{}_{n-1}}{b^{(1)}_{n}{}_{n}}\sim\frac{z_0}{z}\left[1+\frac{1}{n}\left(2-B_2-\frac{B_1}{z_0}\right)\right]\ \ \text{if}\ n\to-\infty.$$

or

$$\left|\frac{b^{(1)}_{n+1}{}_{n+1}}{b^{(1)}_{n}{}_{n}}\right|\sim 1-\frac{2}{n}\ \ \text{if}\ n\to\infty,\qquad \left|\frac{b^{(1)}_{n-1}{}_{n-1}}{b^{(1)}_{n}{}_{n}}\right|\sim\frac{|z_0|}{|z|}\left[1+\frac{1}{|n|}\mathrm{Re}\left(B_2+\frac{B_1}{z_0}-2\right)\right]\ \ \text{if}\ n\to-\infty. \tag{52}$$

Therefore, by the D'Alembert test the series converges for $|z|>|z_0|$; in addition, according to the Raabe test the series converges also at $|z|=|z_0|$ provided that $\mathrm{Re}\left(B_2+B_1/z_0\right)<1$ as stated in (42). Since the above ratios (52) have been obtained by supposing that z is finite, now we consider the behavior of U_1^{∞} as $z\to\infty$. As $\Psi(a,c;y)=y^{-a}$ for large y, we find

$$U_1^{\infty}(z)\sim e^{i\omega z}\;z^{-i\eta-\frac{B_2}{2}}\sum_{n=-\infty}^{\infty}b_n^{(1)}\quad\text{when}\quad z\to\infty, \tag{53}$$

where, on the right-hand side,

$$\left|\frac{b_{n+1}^{(1)}}{b_n^{(1)}}\right|\overset{(46)}{\sim}1+\frac{1}{n}\text{Re}\left(i\eta+\frac{B_2}{2}-2\right)\text{ if } n\to\infty,\qquad \left|\frac{b_{n-1}^{(1)}}{b_n^{(1)}}\right|\overset{(46)}{\sim}\left|\frac{2\omega z_0}{n}\right|\to 0\text{ if } n\to-\infty. \tag{54}$$

Thus, by the Raabe test for $n\to\infty$, the series in the expansion U_1^{∞} converges at $z=\infty$ if $\text{Re}(i\eta+B_2/2-1)<0$ as stated in (41).

2.4.2 Convergence of $\bar{U}_1^{\infty}$

For the solution $\bar{U}_1^{\infty}$ written in (36a) and (37), we have

$$\bar{U}_1^{\infty}(z)=e^{-i\omega z}\sum_{n=-\infty}^{\infty}\bar{b}_n^{(1)}\;\psi_n(y),\quad \psi_n=\Psi\left[-n-\mu_1-i\eta-\frac{B_1}{z_0}-\frac{B_2}{2},-n-\mu_1-\frac{B_1}{z_0};2i\omega z\right],$$

and the ratios (6) read ($y=2i\omega z$)

$$L_1(z)=\left|\frac{\bar{b}_{n+1}^{(1)}\psi_{n+1}(y)}{\bar{b}_n^{(1)}\psi_n(y)}\right|\text{ for } n\to\infty,\qquad L_2(z)=\left|\frac{\bar{b}_{n-1}^{(1)}\psi_{n-1}(y)}{\bar{b}_n^{(1)}\psi_n(y)}\right|\text{ for } n\to-\infty. \tag{55}$$

When $n\to\pm\infty$, relations (21) for $\bar{b}_n^{(1)}$ give

$$2i\omega z_0\left[n+2\mu_1+2+i\eta+\frac{B_1}{z_0}+\frac{B_2}{2}\right]\frac{\bar{b}_{n+1}^{(1)}}{\bar{b}_n^{(1)}}-\left[n+2\mu_1+1+2i\omega z_0+B_2+\frac{2B_1}{z_0}\right]$$
$$+\left[1+\frac{1}{n}\left(\mu_1-1+B_2+\frac{B_1}{z_0}\right)\right]\frac{\bar{b}_{n-1}^{(1)}}{\bar{b}_n^{(1)}}=0,$$

Then, for minimal solutions we find

$$\frac{\bar{b}_{n+1}^{(1)}}{\bar{b}_n^{(1)}}\sim\frac{1}{n}\left[1-\frac{1}{n}\left(\mu_1+3+\frac{B_1}{z_0}\right)\right]\text{ if } n\to\infty,\qquad \frac{\bar{b}_{n-1}^{(1)}}{\bar{b}_n^{(1)}}\sim 2i\omega z_0\left[1+\frac{1}{n}\left(1+i\eta-\frac{B_1}{z_0}-\frac{B_2}{2}\right)\right]\text{ if } n\to-\infty, \tag{56}$$

which correspond respectively to

$$\frac{\bar{b}_{n-1}^{(1)}}{\bar{b}_n^{(1)}}\sim n+\mu_1+2+\frac{B_1}{z_0}\text{ if } n\to\infty\;,\qquad \frac{\bar{b}_{n+1}^{(1)}}{\bar{b}_n^{(1)}}\sim\frac{1}{2i\omega z_0}\left[1-\frac{1}{n}\left(1+i\eta-\frac{B_1}{z_0}-\frac{B_2}{2}\right)\right]\text{ if } n\to-\infty.$$

To obtain the ratios for the functions $\psi_n(y)$, we write the last of equations (39) as

$$\frac{\psi_{n+1}(y)}{\psi_n(y)}-\left(n+\mu_1+1+\frac{B_1}{z_0}+y\right)+\left(n+\mu_1+i\eta+\frac{B_1}{z_0}+\frac{B_2}{2}\right)y\frac{\psi_{n-1}(y)}{\psi_n}=0. \tag{57}$$

If y is bounded ($y/n\to 0$), this equation is satisfied by two expressions for ψ_{n+1}/ψ_n or ψ_{n-1}/ψ_n,

$$\begin{aligned}\frac{\psi_{n+1}}{\psi_n}&\sim n\left[1+\frac{1}{n}\left(\mu_1+1+\frac{B_1}{z_0}\right)\right] &&\Leftrightarrow &&\frac{\psi_{n-1}}{\psi_n}\sim\frac{1}{n}\left[1-\frac{1}{n}\left(\mu_1+\frac{B_1}{z_0}\right)\right],\\ \frac{\psi_{n+1}}{\psi_n}&\sim y\left[1-\frac{1}{n}\left(1-i\eta-\frac{B_2}{2}\right)\right] &&\Leftrightarrow &&\frac{\psi_{n-1}}{\psi_n}\sim\frac{1}{y}\left[1+\frac{1}{n}\left(1-i\eta-\frac{B_2}{2}\right)\right],\end{aligned} \tag{58}$$

where $y=2i\omega z$.

If $1-i\eta-B_2/2=-l$ ($l=0,1,\dots$) we must choose the ratios

$$\frac{\psi_{n+1}}{\psi_n}\sim y\left[1+\frac{l}{n}\right]\text{ if } n\to\infty,\quad \frac{\psi_{n-1}}{\psi_n}\sim\frac{1}{y}\left[1-\frac{l}{n}\right]\text{ if } n\to-\infty:\quad \left(l=i\eta+\frac{B_2}{2}-1\right),$$

which imply that

$$\left.\begin{aligned}&\left|\frac{\bar{b}_{n+1}^{(1)}\psi_{n+1}}{\bar{b}_n^{(1)}\psi_n}\right|\sim\left|\frac{2\omega z}{n}\right| &&\text{if } n\to\infty,\\ &\left|\frac{\bar{b}_{n-1}^{(1)}\psi_{n-1}}{\bar{b}_n^{(1)}\psi_n}\right|\sim\frac{|z_0|}{|z|}\left[1+\frac{1}{|n|}\text{Re}\left(B_2+\frac{B_1}{z_0}-2\right)\right] &&\text{if } n\to-\infty,\end{aligned}\right\}\text{ for }\; i\eta+\frac{B_2}{2}-1=l. \tag{59}$$

The above choices for the ratios between consecutive ψ_n follow from the relation [9]

$$\psi_n(y)=\Psi(a,a+l+1;y)=\frac{l!\;y^{-a}}{\Gamma(a)}\sum_{m=0}^{l}\frac{\Gamma(a+m)\;y^{-m}}{m!\;(l-m)!},\qquad l=0,1,2,\cdots. \tag{60}$$

where $(a)_m$ is the Pochhammer symbol and, for the present case, $a=-n-\mu_1-1-l-B_1/z_0$.

On the other hand, by means of (14), we write

$$\psi_n(y)=\;y^{n+1+\mu_1+\frac{B_1}{z_0}}\,\Psi\left(1-i\eta-\tfrac{B_2}{2},2+n+\mu_1+\tfrac{B_1}{z_0};y\right),$$

which gives

$$\frac{\psi_{n+1}}{\psi_n}=y\,\frac{\Psi\left(1-i\eta-\frac{B_2}{2},3+n+\mu_1+\frac{B_1}{z_0};y\right)}{\Psi\left(1-i\eta-\frac{B_2}{2},2+n+\mu_1+\frac{B_1}{z_0};y\right)},\qquad \frac{\psi_{n-1}}{\psi_n}=\frac{1}{y}\,\frac{\Psi\left(1-i\eta-\frac{B_2}{2},1+n+\mu_1+\frac{B_1}{z_0};y\right)}{\Psi\left(1-i\eta-\frac{B_2}{2},2+n+\mu_1+\frac{B_1}{z_0};y\right)}, \tag{61}$$

Then, if $l\neq i\eta+\frac{B_2}{2}-1$, we can use (61) in conjunction with (51) to select the ratios

$$\frac{\psi_{n+1}}{\psi_n}\sim n\left[1+\frac{1}{n}\left(\mu_1+1+\frac{B_1}{z_0}\right)\right]\;\text{ if } n\to\infty,\qquad \frac{\psi_{n-1}}{\psi_n}\sim\frac{1}{y}\left[1+\frac{1}{n}\left(1-i\eta-\frac{B_2}{2}\right)\right]\;\text{ if } n\to-\infty\;.$$

In effect, when $n\to\infty$, only the second term of (51) is relevant and so we get the first term of the above ratio; when $n\to-\infty$, only the first term of (51) is relevant. Therefore,

$$\left.\begin{array}{ll}\left|\dfrac{\bar{b}^{(1)}_{n+1}\psi_{n+1}}{\bar{b}^{(1)}_{n}\psi_n}\right|\sim 1-\dfrac{2}{n} & \text{if } n\to\infty,\\[2ex] \left|\dfrac{\bar{b}^{(1)}_{n-1}\psi_{n-1}}{\bar{b}^{(1)}_{n}\psi_n}\right|\sim\dfrac{|z_0|}{|z|}\left[1+\dfrac{1}{|n|}\mathrm{Re}\left(B_2+\dfrac{B_1}{z_0}-2\right)\right] & \text{if } n\to-\infty,\end{array}\right\}\;\text{for } i\eta+\frac{B_2}{2}-1\neq l. \tag{62}$$

From (59) and (62) we see that, by the Raabe test, $\bar{U}_1^{\infty}$ converges for any $|z|>|z_0|$; it converges also at $|z|=|z_0|$ if $\mathrm{Re}(B_1+B_1/z_0)<1$ as stated in (42). As the test does not hold for $z\to\infty$, by using $\lim_{y\to\infty}\Psi(a,c;y)=y^{-a}$ for large y, we find

$$\bar{U}_1^{\infty}(z)\sim e^{-i\omega z}z^{i\eta+\mu_1+\frac{B_1}{z_0}+\frac{B_2}{2}}\sum_{n=-\infty}^{\infty}\bar{b}^{(1)}_n\,(2i\omega z)^n\qquad[z\to\infty]$$

which becomes undetermined at $z=\infty$ when $n\to\infty$ due to the first of the relations:

$$\left|\frac{\bar{b}^{(1)}_{n+1}(2i\omega z)^{n+1}}{\bar{b}^{(1)}_{n}(2i\omega z)^{n}}\right|\overset{(56)}{\sim}\left|\frac{2\omega z}{n}\right|\text{if } n\to\infty,\quad \left|\frac{\bar{b}^{(1)}_{n-1}(2i\omega z)^{n-1}}{\bar{b}^{(1)}_{n}(2i\omega z)^{n}}\right|\overset{(56)}{\sim}\left|\frac{z_0}{z}\right|\to 0\text{ if } n\to-\infty\quad[z\to\infty]. \tag{63}$$

For one-sided series with $n\leq 0$, the first relation does not exist and, so, the series converges at $z=\infty$.

2.5 Relations with Leaver-type series solutions

The second version for the two-sided solutions is an extension of the solutions given by Leaver who has considered only one expansion in series of regular confluent hypergeometric functions and one in series of irregular functions [2]. In fact, we can use expansions in series of any of the four functions (13) and add new solutions by means of (16).

- In section 2.5.1 we write the initial set of solutions (65) and show that it is equivalent to the set (24).
- In sections 2.5.2 we state the regions of convergence for the solutions.

The study of convergence is incomplete in Leaver's paper, a fact that now becomes irrelevant. In fact, the regions of convergence are the same as the ones of the first version (equivalence of solutions), except that the restrictions which come from the ratios when $n\to\pm\infty$ in section 2.3, now result from $n\to\mp\infty$. This will imply differences in the regions of the two groups of one-sided series solutions.

Now we denote the sets of solutions by bold letters, that is,

$$\left[\boldsymbol{U}_i(z),\boldsymbol{U}_i^{\infty}(z),\bar{\boldsymbol{U}}_i^{\infty}(z)\right],\qquad[i=1,2,\cdots,16] \tag{64}$$

which take the role of (18). We write only the initial set $(\boldsymbol{U}_1,\boldsymbol{U}_1^{\infty},\bar{\boldsymbol{U}}_1^{\infty})$, but others may be generated by the transformations (16) as indicated in (19).

2.5.1 The initial set of Leaver-type solutions

In the following, the solutions $\boldsymbol{U}_1(z)$ and $\boldsymbol{U}_1^{\infty}(z)$ are the expansions (138) and (140) of Leaver's paper. The solution $\bar{\boldsymbol{U}}_1^{\infty}(z)$ can be interpreted as a linear combination of $\boldsymbol{U}_1(z)$ and $\boldsymbol{U}_1^{\infty}(z)$. Thus,

$$\begin{aligned}
\left[\begin{array}{c}\boldsymbol{U}_1(z)\\ \boldsymbol{U}_1^{\infty}(z)\end{array}\right] &= e^{i\omega z}\sum_{n=-\infty}^{\infty}\boldsymbol{b}_n^{(1)}\left[\begin{array}{c}\tilde{\Phi}\left(\frac{B_2}{2}+i\eta,n+\nu_1+B_2;-2i\omega z\right)\\ \Psi\left(\frac{B_2}{2}+i\eta,n+\nu_1+B_2;-2i\omega z\right)\end{array}\right], \qquad \tfrac{B_2}{2}+i\eta\neq 0,-1,-2,\cdots\\
\bar{\boldsymbol{U}}_1^{\infty}(z) &= e^{-i\omega z}\sum_{n=-\infty}^{\infty}(-1)^n\bar{\boldsymbol{b}}_n^{(1)}\Psi\left(n+\nu_1-i\eta+\tfrac{B_2}{2},n+\nu_1+B_2;2i\omega z\right)
\end{aligned} \tag{65}$$

where $\boldsymbol{b}_n^{(1)}$ and $\bar{\boldsymbol{b}}_n^{(1)}$ satisfy recurrence relations having the form (21) with

$$\begin{aligned}
&\boldsymbol{\alpha}_n^{(1)} = -[n+\nu_1+1]\left[n+\nu_1-i\eta+\tfrac{B_2}{2}\right], \qquad \boldsymbol{\gamma}_n^{(1)} = 2i\omega z_0\left[n+\nu_1+B_2+\tfrac{B_1}{z_0}-1\right],\\
&\boldsymbol{\beta}_n^{(1)} = (n+\nu_1)(n+\nu_1+B_2-1-2i\omega z_0)+B_3-i\omega z_0\left(B_2+\tfrac{B_1}{z_0}\right) \quad \text{for } \boldsymbol{b}_n^{(1)}
\end{aligned} \tag{66a}$$

and

$$\begin{aligned}
&\bar{\boldsymbol{\alpha}}_n^{(1)} = -\left[n+\nu_1+1\right], \qquad \bar{\boldsymbol{\beta}}_n^{(1)} = \beta_n^{(1)}, \qquad \bar{\boldsymbol{\gamma}}_n^{(1)} = 2i\omega z_0\left[n+\nu_1+B_2+\tfrac{B_1}{z_0}-1\right]\left[n+\nu_1-i\eta+\tfrac{B_2}{2}-1\right],\\
&\bar{\boldsymbol{b}}_n^{(1)} = \Gamma\left[n+\nu_1-i\eta+\tfrac{B_2}{2}\right]\boldsymbol{b}_n^{(1)}.
\end{aligned} \tag{66b}$$

From the above connection between the coefficients, we see that

$$\boldsymbol{U}_1(z) = e^{i\omega z}\sum_{n=-\infty}^{\infty}\frac{\bar{\boldsymbol{b}}_n^{(1)}}{\Gamma\left[n+\nu_1+B_2\right]}\;\Phi\left(i\eta+\tfrac{B_2}{2},n+\nu_1+B_2;-2i\omega z\right). \tag{67}$$

This reminds that the coefficients of $U_i(z)$ and $\bar{U}_i^{\infty}(z)$ satisfy the same recurrence relations. On the other side, from $\lim_{z\to\infty}\psi(a,c;y)=y^{-a}$ we find

$$\lim_{z\to\infty}\boldsymbol{U}_1^{\infty}(z) = e^{-i\omega z}z^{i\eta-\frac{B_2}{2}}\boldsymbol{b}_0^{(1)}, \qquad \lim_{z\to\infty}\bar{\boldsymbol{U}}_1^{\infty}(z) = e^{i\omega z}z^{-i\eta-\nu_1-\frac{B_2}{2}}\sum_{n=-\infty}^{\infty}\bar{\boldsymbol{b}}_n^{(1)}(-2i\omega z)^{-n}. \tag{68}$$

Thus, for one-sided series with $n\geq 0$ it is possible (if $\nu_1=0$) to obtain the two asymptotic behaviors (2) from solutions belonging to the same set. Besides this, it is possible another expansion in series of regular confluent hypergeometric functions, namely,

$$\bar{\boldsymbol{U}}_1(z) = e^{i\omega z}\sum_{n=-\infty}^{\infty}\boldsymbol{b}_n^{(1)}(2i\omega z)^{1-n-\nu_1-B_2}\,\tilde{\Phi}\left(1-n-\nu_1+i\eta-\tfrac{B_2}{2},2-n-\nu_1-B_2;-2i\omega z\right). \tag{69}$$

This expansion may be useful for studying solutions for the reduced CHE (121).

By putting $\mu_1=-\nu_1-B_2-(B_1/z_0)$, we verify that the expansions (65) differ from the expansions (24) of section 2.2 because the parameters of the hypergeometric functions present opposite signs in the summation index n. In addition, we see that

$$\left[\alpha_{-n}^{(1)},\beta_{-n}^{(1)},\gamma_{-n}^{(1)}\right] = \left[\boldsymbol{\gamma}_n^{(1)},\boldsymbol{\beta}_n^{(1)},\boldsymbol{\alpha}_n^{(1)}\right], \qquad \left[\bar{\alpha}_{-n}^{(1)},\bar{\beta}_{-n}^{(1)},\bar{\gamma}_{-n}^{(1)}\right] = \left[-\bar{\boldsymbol{\gamma}}_n^{(1)},\bar{\boldsymbol{\beta}}_n^{(1)},-\bar{\boldsymbol{\alpha}}_n^{(1)}\right] \tag{70}$$

if $\mu_1=-\nu_1-B_2-(B_1/z_0)$. Since n runs from minus to plus infinity in two-sided infinite series, both sets are in fact equivalent to each other, having the coefficients rearranged as

$$\left[b_{-n}^{(1)},\;\bar{b}_{-n}^{(1)}\right] \longleftrightarrow \left[\boldsymbol{b}_n^{(1)},\;(-1)^n\bar{\boldsymbol{b}}_n^{(1)}\right] \quad \text{for} \quad \mu_1=-\nu_1-B_2-\tfrac{B_1}{z_0}. \tag{71}$$

As a consequence, when we consider the convergence of the expansions $\boldsymbol{U}_i^{\infty}$ and $\bar{\boldsymbol{U}}_i^{\infty}$, in the conditions (41), (42) and (43) [valid for U_i^{∞} and U_i^{∞}] n must be replaced by $-n$, as in the following.

2.5.2 Convergence of the Leaver-type solutions

We find that

- The expansions $\boldsymbol{U}_i(z)$ in series of regular confluent hypergeometric functions converge for finite values of z, the convergence being undetermined when $z\to\infty$ – see Eq. (75).

- When $n\to-\infty$ the ratio test assures convergence of $\boldsymbol{U}_i^{\infty}(z)$ at $z=\infty$ if the conditions (74) are satisfied. When $n\to\infty$, the test assures that $\boldsymbol{U}_i^{\infty}$ converge for $|z|>|z_0|$ $(i=1,\cdots,4)$ or $|z-z_0|>|z_0|$ $(i=5,\cdots,8)$ but, by the Raabe test, converge also at $|z|=|z_0|$ and $|z-z_0|=|z_0|$ if the parameters of the equation satisfy the conditions given in (72) and (73).
- When $n\to-\infty$ the ratio test does not assure convergence of $\bar{\boldsymbol{U}}_i^{\infty}(z)$ at $z=\infty$ (in fact, the test becomes inconclusive). When $n\to\infty$ the test assures that the $\bar{\boldsymbol{U}}_i^{\infty}$ converge for $|z|>|z_0|$ $(i=1,\cdots,4)$ or $|z-z_0|>|z_0|$ $(i=5,\cdots,8)$ but, by the Raabe test, converge also at $|z|=|z_0|$ and $|z-z_0|=|z_0|$ if the parameters of the equation satisfy the conditions given in (72) and (73).

The regions of convergence for $\boldsymbol{U}_i^{\infty}(z)$ and $\bar{\boldsymbol{U}}_i^{\infty}(z)$ are (using the Raabe test for $n\to\infty$)

$$n\to\infty:\ |z|\geq|z_0|,\ \text{sign "=" if}\begin{cases}\text{Re}\left(B_2+\frac{B_1}{z_0}\right)<1\ \text{in}\ \left(\boldsymbol{U}_1^{\infty},\bar{\boldsymbol{U}}_1^{\infty}\right),\ \left(\boldsymbol{U}_2^{\infty},\bar{\boldsymbol{U}}_2^{\infty}\right),\\ \text{Re}\left(B_2+\frac{B_1}{z_0}\right)>1\ \text{in}\ \left(\boldsymbol{U}_3^{\infty},\bar{\boldsymbol{U}}_3^{\infty}\right),\ \left(\boldsymbol{U}_4^{\infty},\bar{\boldsymbol{U}}_4^{\infty}\right),\end{cases}\tag{72}$$

$$n\to\infty:|z-z_0|\geq|z_0|,\ \text{sign "=" if}\begin{cases}\text{Re}\left(\frac{B_1}{z_0}\right)>-1\ \text{in}\ \left(\boldsymbol{U}_5^{\infty},\bar{\boldsymbol{U}}_5^{\infty}\right),\ \left(\boldsymbol{U}_8^{\infty},\bar{\boldsymbol{U}}_8^{\infty}\right),\\ \text{Re}\left(\frac{B_1}{z_0}\right)<-1\ \text{in}\ \left(\boldsymbol{U}_6^{\infty},\bar{\boldsymbol{U}}_6^{\infty}\right),\ \left(\boldsymbol{U}_7^{\infty},\bar{\boldsymbol{U}}_7^{\infty}\right).\end{cases}\tag{73}$$

The test is inconclusive if $\text{Re}[B_2+(B_1/z_0)]=1$ in (72), and if $\text{Re}[B_1/z_0]=-1$ in (73). In contrast, the restrictions (42) and (43) for the first group come from $n\to-\infty$.

On the other hand, by taking the limit $\lim_{z\to\infty}\boldsymbol{U}_i^{\infty}$ and applying the Raabe test for $n\to-\infty$, we find that the $\boldsymbol{U}_i^{\infty}$ converge at $z=\infty$ if

$$\begin{aligned}n\to-\infty:\quad &\text{Re}\left[i\eta+\tfrac{B_2}{2}-1\right]<0\ \text{for}\ \boldsymbol{U}_1^{\infty},\ \boldsymbol{U}_5^{\infty};\quad \text{Re}\left[i\eta+\frac{B_1}{z_0}+\frac{B_2}{2}\right]<0\ \text{for}\ \boldsymbol{U}_2^{\infty},\ \boldsymbol{U}_6^{\infty};\\ &\text{Re}\left[i\eta-\tfrac{B_2}{2}+1\right]<0\ \text{for}\ \boldsymbol{U}_3^{\infty},\ \boldsymbol{U}_7^{\infty};\quad \text{Re}\left[i\eta-\frac{B_1}{z_0}-\frac{B_2}{2}\right]<0\ \text{for}\ \boldsymbol{U}_4^{\infty},\ \boldsymbol{U}_8^{\infty}.\end{aligned}\tag{74}$$

In contrast, conditions (41) for the first group result from the Raabe test when $n\to+\infty$.

When $z\to\infty$, we find

$$\boldsymbol{U}_1(z)\overset{(50)}{\sim}e^{i\omega z}\ z^{-i\eta-\frac{B_2}{2}}\sum_{n=-\infty}^{\infty}\boldsymbol{b}_n^{(1)}+e^{-i\omega z}\ z^{i\eta+\nu_1-\frac{B_2}{2}}\sum_{n=-\infty}^{\infty}\Gamma_n\,\boldsymbol{b}_n^{(1)}\,(-2i\omega z)^{-n},\qquad \Gamma_n=\Gamma\left(n+\nu_1-i\eta+\tfrac{B_2}{2}\right),$$

where the ratio between the terms of the series becomes undetermined when $n\to-\infty$ due to the following relations (obtained from the recurrence relations):

$$\begin{aligned}n\to\infty:&\quad \frac{\boldsymbol{b}_{n+1}^{(1)}}{\boldsymbol{b}_n^{(1)}}\sim-\frac{2i\omega z_0}{n}\to0 &&\frac{\Gamma_{n+1}\,\boldsymbol{b}_{n+1}^{(1)}\,(-2i\omega z)^{-n-1}}{\Gamma_n\,\boldsymbol{b}_n^{(1)}\,(-2i\omega z)^{-n}}\sim\frac{z_0}{z}\to0,\\ n\to-\infty:&\quad \frac{\boldsymbol{b}_{n-1}^{(1)}}{\boldsymbol{b}_n^{(1)}}\sim1+\frac{1}{n}\left(2-i\eta-\frac{B_2}{2}\right), &&\frac{\Gamma_{n-1}\,\boldsymbol{b}_{n-1}^{(1)}\,(-2i\omega z)^{-n+1}}{\Gamma_n\,\boldsymbol{b}_n^{(1)}\,(-2i\omega z)^{-n}}\sim-\frac{2i\omega z}{n}.\end{aligned}\tag{75}$$

Therefore, the convergence of $\boldsymbol{U}_1(z)$ is undetermined at $z=\infty$, except for one-sided series with $n\geq0$.

3 Expansions in one-sided series of confluent hypergeometric functions

In this section, one-sided series solutions are obtained by truncating the two-sided series. If, for some values of the parameters, the series end on the right-hand and on the left-hand side according (8), we get finite-series solutions; in this case the convergence must be decided by examining each term of the series.

- In section 3.1, the two nonequivalent groups of one-sided series solutions with $n\geq0$ are obtained by setting $\mu_i=\nu_i=0$ in the two-sided series solutions. Some properties of solutions are displayed.
- In sections 3.2 and 3.3, the regions of convergence for one-sided solutions are obtained by restricting n to non-negative values in the regions of the two-sided series.
- In section 3.4, we show that the two groups of one-sided series result also from the truncation of the same group of two-sided series on the left-hand side ($n\geq0$) and on right-hand side ($n\leq0$).

The one-sided solutions of the two groups are again denoted by $\left(U_i, U_i^{\infty}, \bar{U}_i^{\infty}\right)$ and $\left(\boldsymbol{U}_i, \boldsymbol{U}_i^{\infty}, \bar{\boldsymbol{U}}_i^{\infty}\right)$. To avoid misleading interpretations, the expansions U_i and $\boldsymbol{U}_i$ must be rewritten in terms the coefficients $\bar{b}_n^{(i)}$ and $\bar{\boldsymbol{b}}_n^{(i)}$ rather than in terms of $b_n^{(i)}$ and $\boldsymbol{b}_n^{(i)}$. For this purpose we have provided the formulas which connect $\bar{b}_n^{(i)}$ and $\bar{\boldsymbol{b}}_n^{(i)}$ with $b_n^{(i)}$ and $\boldsymbol{b}_n^{(i)}$. By expressing

$$U_i(z) \text{ in terms of } \bar{b}_n^{(i)}, \qquad \boldsymbol{U}_i(z) \text{ in terms of } \bar{\boldsymbol{b}}_n^{(i)}, \tag{76}$$

we suppress from the expansions the term $\Gamma(c-a)$ introduced by the definition (20) for $\tilde{\Phi}(a,c;y)$. In fact, the factor $\Gamma(c-a)$ may lead to incorrect results when we use relations (8).

3.1 The two groups of one-sided series solutions

We require that the series of section 2 begin at $n=0$, by choosing the parameters μ_i and ν_i such that

$$\alpha_{-1}^{(i)} = \bar{\alpha}_{-1}^{(i)} = 0 \text{ in section 2.2 and appendix A}, \qquad \boldsymbol{\alpha}_{-1}^{(i)} = \bar{\boldsymbol{\alpha}}_{-1}^{(i)} = 0 \text{ in section 2.5} \tag{77}$$

according to (8). To satisfy these conditions it is sufficient to take $\mu_i = 0$ in the first case and $\nu_i = 0$ in the second case. This yields the two groups of one-sided solutions with $n \geq 0$, namely,

$$\left[U_i(z), U_i^{\infty}(z), \bar{U}_i^{\infty}(z)\right]\Big|_{\mu_i=0}, \qquad \left[\boldsymbol{U}_i(z), \boldsymbol{U}_i^{\infty}(z), \bar{\boldsymbol{U}}_i^{\infty}(z)\right]\Big|_{\nu_i=0}. \tag{78}$$

The recurrence relations (21) take the forms

$$\begin{aligned} &\alpha_0^{(i)}\, b_1^{(i)} + \beta_0^{(i)}\, b_0^{(i)} = 0, \qquad \alpha_n^{(i)}\, b_{n+1}^{(i)} + \beta_n^{(i)}\, b_n^{(i)} + \gamma_n^{(i)}\, b_{n-1}^{(i)} = 0, \qquad (n \geq 1);\\ &\bar{\alpha}_0^{(i)}\, \bar{b}_1^{(i)} + \bar{\beta}_0^{(i)}\, \bar{b}_0^{(i)} = 0, \qquad \bar{\alpha}_n^{(i)}\, \bar{b}_{n+1}^{(i)} + \bar{\beta}_n^{(i)}\, \bar{b}_n^{(i)} + \bar{\gamma}_n^{(i)}\, \bar{b}_{n-1}^{(i)} = 0, \qquad (n \geq 1) \end{aligned} \tag{79}$$

for the first group, and analogous relations for the second group. These equations can be written in a form similar to (5), namely,

$$\begin{bmatrix} \beta_0^{(i)} & \alpha_0^{(i)} & 0 & & & \\ \gamma_1^{(i)} & \beta_1^{(i)} & \alpha_1^{(i)} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \gamma_n^{(i)} & \beta_n^{(i)} & \alpha_n^{(i)} & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} b_0^{(i)} \\ b_1^{(i)} \\ \vdots \\ b_n^{(i)} \\ \vdots \end{bmatrix} = \mathbf{0}, \tag{80}$$

where the determinant of the tridiagonal matrix must vanish. This constitutes the characteristic equation which is equivalent to the continued fraction

$$\beta_0^{(i)} = \frac{\alpha_0^{(i)}\gamma_1^{(i)}}{\beta_1^{(i)}-}\, \frac{\alpha_1^{(i)}\gamma_2^{(i)}}{\beta_2^{(i)}-}\, \frac{\alpha_2^{(i)}\gamma_3^{(i)}}{\beta_3^{(i)}-} \cdots. \tag{81}$$

Contrary to the case of two-sided series, the solutions of the two groups (78) are not equivalent to each other because now n assumes only non-negative values. We regard two sets which are used in section 3.4. Thus, taking $\mu_1 = 0$ in (24), we have

$$\begin{aligned} &\begin{bmatrix} U_1(z) \\ U_1^{\infty}(z) \end{bmatrix} = e^{i\omega z} \sum_{n=0}^{\infty} b_n^{(1)} \begin{bmatrix} \tilde{\Phi}\left(i\eta + \frac{B_2}{2}, -n - \frac{B_1}{z_0}; -2i\omega z\right) \\ \Psi\left(i\eta + \frac{B_2}{2}, -n - \frac{B_1}{z_0}; -2i\omega z\right) \end{bmatrix}, \qquad i\eta + \tfrac{B_2}{2} \neq 0, -1, -2, \cdots, \\ &\bar{U}_1^{\infty}(z) = e^{-i\omega z} \sum_{n=-\infty}^{\infty} \bar{b}_n^{(1)} \Psi\left(-n - i\eta - \tfrac{B_1}{z_0} - \tfrac{B_2}{2}, -n - \tfrac{B_1}{z_0}; 2i\omega z\right) \end{aligned} \tag{82}$$

where the coefficients $b_n^{(1)}$ and $\bar{b}_n^{(1)}$ satisfy the relations (79) with $\mu_1 = 0$ in (25a) and (25b). Since $\bar{b}_n^{(1)} = (-1)^n$ $\Gamma\left[-n - i\eta - B_1/z_0 - B_2/2\right] b_n^{(1)}$, we have

$$U_1(z) = e^{i\omega z} \sum_{n=0}^{\infty} \frac{(-1)^n \bar{b}_n^{(1)}}{\Gamma[-n-(B_1/z_0)]}\ \Phi\left(i\eta + \tfrac{B_2}{2}, -n - \tfrac{B_1}{z_0}; -2i\omega z\right),$$

as required in (76). On the other side, for $\nu_1=0$ the solutions (65) take the form

$$\begin{aligned}
&\left[\begin{array}{c}\boldsymbol{U}_1(z)\\ \boldsymbol{U}_1^{\infty}(z)\end{array}\right]=e^{i\omega z}\sum_{n=0}^{\infty}\boldsymbol{b}_n^{(1)}\left[\begin{array}{c}\tilde{\Phi}\left(i\eta+\frac{B_2}{2},n+B_2;-2i\omega z\right)\\ \Psi\left(i\eta+\frac{B_2}{2},n+B_2;-2i\omega z\right)\end{array}\right], \qquad i\eta+\tfrac{B_2}{2}\neq 0,-1,-2,\cdots,\\
&\bar{\boldsymbol{U}}_1^{\infty}(z) = e^{-i\omega z}\sum_{n=0}^{\infty}(-1)^n\bar{\boldsymbol{b}}_n^{(1)}\Psi\left(n-i\eta+\tfrac{B_2}{2}n+B_2;2i\omega z\right),
\end{aligned} \tag{83}$$

where $\bar{\boldsymbol{b}}_n^{(1)}=\Gamma\big[n-i\eta+(B_2/2)\big]\boldsymbol{b}_n^{(1)}$. The coefficients of the recurrence relations are obtained by putting $\nu_1=0$ in (66a) and (66b).

In addition to different domains of convergence for the two groups, we find that:

- Solutions of the first group do not admit the Leaver limit $z_0\to 0$, in opposition to solutions of the second group [4].
- The solutions $\boldsymbol{U}_i(z)$ and $\boldsymbol{U}_{i+4}(z)$ of the second group are linearly dependent, that is,

$$\boldsymbol{U}_{i+4}(z)\propto\boldsymbol{U}_i(z), \qquad i=1,2,3,4. \tag{84}$$

The Leaver limit [2] is important because, when $z_0=0$, the CHE (1) reduces to the double-confluent Heun equation (DCHE)

$$z^2\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left(B_3-2\eta\omega z+\omega^2z^2\right)U=0, \tag{85}$$

where $z=0$ and $z=\infty$ are both irregular singularities. Most of the solutions for the DCHE are generated by applying the limit $z_0\to 0$ on solutions of the CHE [2, 4, 7, 17]. The first group does not admit the Leaver limit because $\beta_n^{(i)}=\tilde{\beta}_n^{(i)}$ goes to infinity if $z_0\to\infty$, as we can check.

On the other side, on account of the transformations (16), it is sufficient to discuss only one pair of the relations (84), for example, $(\boldsymbol{U}_1,\boldsymbol{U}_5)$. By using (14) and (20), we write

$$\begin{aligned}
&\boldsymbol{U}_1(z)\overset{(14,20,83)}{=}e^{-i\omega z}\sum_{n=0}^{\infty}\frac{\bar{\boldsymbol{b}}_n^{(1)}}{\Gamma(n+B_2)}\Phi\left[n-i\eta+\tfrac{B_2}{2},n+B_2;2i\omega z\right],\\
&\boldsymbol{U}_5(z)=T_4\boldsymbol{U}_1(z)=e^{-i\omega z}\sum_{n=0}^{\infty}\frac{\bar{\boldsymbol{b}}_n^{(5)}}{\Gamma(n+B_2)}\Phi\left[n-i\eta+\tfrac{B_2}{2},n+B_2;2i\omega(z-z_0)\right],
\end{aligned} \tag{86}$$

To show that these solutions are linearly dependent, we use the addition theorem [9]

$$\Phi(a,c;x+y)=\frac{\Gamma(c)}{\Gamma(a)}\sum_{m=0}^{\infty}\frac{\Gamma(a+m)y^m}{m!\,\Gamma(c+m)}\,\Phi(a+m,c+m;x), \tag{87}$$

together with the Cauchy product

$$\sum_{n=0}^{\infty}f_n\sum_{m=0}^{\infty}g_m^n=\sum_{j=0}^{\infty}h_j, \quad \text{where} \quad h_j=\sum_{k=0}^{j}f_k\,g_{j-k}^k. \tag{88}$$

By taking $x=2i\omega z$ and $y=-2i\omega z_0$, we obtain

$$\begin{aligned}
\boldsymbol{U}_5(z)&\overset{(86,87)}{=}e^{-i\omega z}\sum_{n=0}^{\infty}\frac{\bar{\boldsymbol{b}}_n^{(5)}}{\Gamma\left[n-i\eta+\frac{B_2}{2}\right]}\sum_{m=0}^{\infty}\frac{\Gamma\left[n+m-i\eta+\frac{B_2}{2}\right][-2i\omega z_0]^m}{m!\,\Gamma[n+m+B_2]}\Phi\left[n+m-i\eta+\tfrac{B_2}{2},n+m+B_2;2i\omega z\right]\\
&\overset{(88)}{=}e^{-i\omega z}\sum_{j=0}^{\infty}\frac{1}{\Gamma(j+B_2)}\sum_{k=0}^{j}\boldsymbol{b}_k^{(5)}\frac{[-2i\omega z_0]^{j-k}}{(j-k)!}\Phi\left[j-i\eta+\tfrac{B_2}{2},j+B_2;2i\omega z\right],
\end{aligned}$$

that is,

$$\boldsymbol{U}_5(z)=e^{-i\omega z}\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+B_2)}\left[\sum_{k=0}^{n}\frac{[-2i\omega z_0]^{n-k}}{(n-k)!}\,\boldsymbol{b}_k^{(5)}\right]\Phi\left[n-i\eta+\tfrac{B_2}{2},n+B_2;2i\omega z\right].$$

So, the right-hand side of this equation is identical to the expression (86) for $\boldsymbol{U}_1(z)$ if

$$\bar{\boldsymbol{b}}_n^{(1)}=\sum_{k=0}^{n}\frac{(-2i\omega z_0)^{n-k}}{(n-k)!}\,\bar{\boldsymbol{b}}_k^{(5)}.$$

3.2 Convergence of the one-sided solutions of the first group

The following results come from section 2.4 by considering the ratio test for $n \to \infty$ only.

- The solutions $U_i(z)$ in series of regular confluent hypergeometric functions converge for any finite value of z. The ratio tests are inconclusive at $z = \infty$.
- For the solutions $U_i^{\infty}(z)$ in series of irregular functions the ratio test does not exclude any finite value of z but the functions $\Psi(a, c; y)$ are not appropriate for $y = 0$ and, consequently, the U_i^{∞} are not valid at $z = 0$ (if $i = 1, 2, 3, 4$) and at $z = z_0$ (if $i = 5, 6, 7, 8$). The expansions $U_i^{\infty}(z)$ converge as well at $z = \infty$ under the conditions (89).
- The solutions $\bar{U}_i^{\infty}$ do not converge at $z = \infty$; the $\bar{U}_i^{\infty}$ converge only for finite z, excepting $z = 0$ (if $i = 1, 2, 3, 4$) and $z = z_0$ (if $i = 5, 6, 7, 8$).

According to the Raabe test the U_i^{∞} converge at $z = \infty$ if

$$\begin{aligned} &\mathrm{Re}\left[i\eta + \tfrac{B_2}{2} - 1\right] < 0 \text{ for } U_1^{\infty},\ U_5^{\infty}; \qquad \mathrm{Re}\left[i\eta + \tfrac{B_1}{z_0} + \tfrac{B_2}{2}\right] < 0 \text{ for } U_2^{\infty},\ U_6^{\infty}; \\ &\mathrm{Re}\left[i\eta - \tfrac{B_2}{2} + 1\right] < 0 \text{ for } U_3^{\infty},\ U_7^{\infty}; \qquad \mathrm{Re}\left[i\eta - \tfrac{B_1}{z_0} - \tfrac{B_2}{2}\right] < 0 \text{ for } U_4^{\infty},\ U_8^{\infty}. \end{aligned} \tag{89}$$

Notice that the one-sided expansion $U_i^{\infty}(z)$ may converge for any value of z, excepting $z = 0$ (if $i = 1, 2, 3, 4$) and $z = z_0$ (if $i = 5, 6, 7, 8$). This fact will become useful in section 4.3.

3.3 Convergence of the one-sided solutions of the second group

The following results come from section 3.2 by considering the ratio test for $n \to \infty$ only.

- The expansions $\boldsymbol{U}_i(z)$ in series of regular confluent hypergeometric functions converge for finite values of z.
- Both, the expansions $\boldsymbol{U}_i^{\infty}(z)$ and $\bar{\boldsymbol{U}}_i^{\infty}(z)$, converge for $|z| > |z_0|$ if $i = 1, \cdots, 4$ and for $|z - z_0| > |z_0|$ if $i = 5, \cdots, 8$. By the Raabe test, $\boldsymbol{U}_i^{\infty}$ and $\bar{\boldsymbol{U}}_i^{\infty}$ converge also at $|z| = |z_0|$ and $|z - z_0| = |z_0|$ if the parameters of the equation satisfy the conditions given in (90) and (91).
- The three series expansions $(\boldsymbol{U}_i, \boldsymbol{U}_i^{\infty}, \bar{\boldsymbol{U}}_i^{\infty})$ can converge at $z = \infty$. See, however, comments after Eq. (93).

By the Raabe test the regions of convergence for $\boldsymbol{U}_i^{\infty}(z)$ and $\bar{\boldsymbol{U}}_i^{\infty}(z)$ are

$$|z| \geq |z_0|, \text{ sign “=” if } \begin{cases} \mathrm{Re}\left(B_2 + \frac{B_1}{z_0}\right) < 1 \text{ in } \left(\boldsymbol{U}_1^{\infty}, \bar{\boldsymbol{U}}_1^{\infty}\right),\ \left(\boldsymbol{U}_2^{\infty}, \bar{\boldsymbol{U}}_2^{\infty}\right); \\ \mathrm{Re}\left(B_2 + \frac{B_1}{z_0}\right) > 1 \text{ in } \left(\boldsymbol{U}_3^{\infty}, \bar{\boldsymbol{U}}_3^{\infty}\right),\ \left(\boldsymbol{U}_4^{\infty}, \bar{\boldsymbol{U}}_4^{\infty}\right); \end{cases} \tag{90}$$

$$|z - z_0| \geq |z_0|, \text{ sign “=” if } \begin{cases} \mathrm{Re}\left(\frac{B_1}{z_0}\right) > -1 \text{ in } \left(\boldsymbol{U}_5^{\infty}, \bar{\boldsymbol{U}}_5^{\infty}\right),\ \left(\boldsymbol{U}_8^{\infty}; \bar{\boldsymbol{U}}_8^{\infty}\right); \\ \mathrm{Re}\left(\frac{B_1}{z_0}\right) < -1 \text{ in } \left(\boldsymbol{U}_6^{\infty}, \bar{\boldsymbol{U}}_6^{\infty}\right),\ \left(\boldsymbol{U}_7^{\infty}, \bar{\boldsymbol{U}}_7^{\infty}\right). \end{cases} \tag{91}$$

The test is inconclusive if $\mathrm{Re}[B_2 + (B_1/z_0)] = 1$ in (90), and if $\mathrm{Re}[B_1/z_0] = -1$ in (91).

Although the one-sided expansions $\boldsymbol{U}_i$ in series of regular functions converge for any z, we have to consider the behavior of the solutions when $z \to \infty$. For $\boldsymbol{U}_1(z)$ we find

$$\boldsymbol{U}_1(z) \stackrel{(75)}{\sim} e^{i\omega z}\ z^{-i\eta - \frac{B_2}{2}} \sum_{n=0}^{\infty} \boldsymbol{b}_n^{(1)} + \ e^{-i\omega z}\ z^{i\eta - \frac{B_2}{2}} \Gamma\left[\tfrac{B_2}{2} - i\eta\right] \boldsymbol{b}_0^{(1)}, \qquad z \to \infty, \tag{92}$$

which is a superposition of

$$\boldsymbol{U}_1^{\infty}(z) \sim e^{i\omega z}\ z^{-i\eta - \frac{B_2}{2}} \sum_{n=0}^{\infty} \boldsymbol{b}_n^{(1)}\ , \qquad \bar{\boldsymbol{U}}_1^{\infty}(z) \sim e^{-i\omega z} z^{i\eta - \frac{B_2}{2}}\ \Gamma\left[\tfrac{B_2}{2} - i\eta\right] \boldsymbol{b}_0^{(1)}. \tag{93}$$

Thus, when $z \to \infty$, $\boldsymbol{U}_1$ is a linear combination of the two asymptotic behaviors (2) and, consequently, may be inappropriate for some applications.

3.4 The two possibilities for truncating the two-sided series

By using conditions (8), we show that the two groups of one-sided series solutions can be obtained by truncating the same version of two-sided series, $(U_i, U_i^{\infty}, \bar{U}_i^{\infty})$ or $(\boldsymbol{U}_i, \boldsymbol{U}_i^{\infty}, \bar{\boldsymbol{U}}_i^{\infty})$, on the left ($n \geq 0$) or on right-hand side ($n \leq 0$). We choose the first version.

For $0 \leq n < \infty$, the first set of solutions (24) in notation of Eqs. (44a) and (55) gives

$$\Big[U_1(z), U_1^{\infty}(z)\Big]_{n\geq 0} = e^{i\omega z}\sum_{n=0}^{\infty} b_n^{(1)}\ _n(y), \qquad \Big[\bar{U}_1^{\infty}(z)\Big]_{n\geq 0} = e^{-i\omega z}\sum_{n=0}^{\infty} \bar{b}_n^{(1)}\psi_n(y). \tag{94}$$

Thence, Eq. (35) for U_1 and U_1^{∞} yields

$$\alpha_{-1}\, b_0^{(1)}\ _{-1}(y) + \Big[\alpha_0^{(1)}\, b_1^{(1)} + \beta_0^{(1)}\, b_0^{(1)}\Big]\ _0(y) + \sum_{n=1}^{\infty}\ \Big[\alpha_n^{(1)}\, b_{n+1}^{(1)} + \beta_n^{(1)}\, b_n^{(1)} + \gamma_n^{(1)}\, b_{n-1}^{(1)}\Big]\ _n(y) = 0$$

and Eq. (40) for $\bar{U}_1^{\infty}$ yields

$$\bar{\alpha}_{-1}\, \bar{b}_0^{(1)}\ \psi_{-1}(y) + \Big[\bar{\alpha}_0^{(1)}\, \bar{b}_1^{(1)} + \bar{\beta}_0^{(1)}\, \bar{b}_0^{(1)}\Big]\ \psi_0(y) + \sum_{n=1}^{\infty}\ \Big[\bar{\alpha}_n^{(1)}\, \bar{b}_{n+1}^{(1)} + \bar{\beta}_n^{(1)}\, \bar{b}_n^{(1)} + \bar{\gamma}_n^{(1)}\, \bar{b}_{n-1}^{(1)}\Big]\psi_n(y) = 0,$$

where the coefficients of $_m$ and ψ_m must vanish. In particular, since we are supposing that $b_0^{(1)} \neq 0$ and $\bar{b}_0^{(1)} \neq 0$,

$$\alpha_{-1}^{(1)} \overset{(25a)}{=} -2i\omega z_0 \mu_1 = 0, \qquad \bar{\alpha}_{-1}^{(1)} \overset{(25b)}{=} -2i\omega z_0 \mu_1 \left(\mu_1 + i\eta + \tfrac{B_2}{2} + \tfrac{B_1}{z_0}\right) = 0.$$

By taking $\mu_1 = 0$ we find that the right-hand sides of (94) reduce to the one-sided series solutions (82). For $\bar{U}_1^{\infty}$ it is also possible to take $\mu_1 = -i\eta - B_2/2 - B_1/z_0$ but we do not consider this case.

Now we show that for $-\infty < n \leq 0$, the first set of solutions (24) reduces to the one-sided series solutions (83). By using again the notation of Eqs. (44a) and (55), we have

$$\Big[U_1(z), U_1^{\infty}(z)\Big]_{n\leq 0} = e^{i\omega z}\sum_{n=-\infty}^{0} b_n^{(1)}\ _n(y), \quad \Big[\bar{U}_1^{\infty}(z)\Big]_{n\leq 0} = e^{-i\omega z}\sum_{n=-\infty}^{0} \bar{b}_n^{(1)}\psi_n(y).$$

which is transformed into

$$\Big[U_1(z), U_1^{\infty}(z)\Big]_{n\leq 0} = e^{i\omega z}\sum_{m=0}^{\infty} b_{-m}^{(1)}\ _{-m}(y), \qquad \Big[\bar{U}_1^{\infty}(z)\Big]_{n\leq 0} = e^{-i\omega z}\sum_{m=0}^{\infty} \bar{b}_{-m}^{(1)}\psi_{-m}(y). \tag{95}$$

by replacing n by $-m$ on the right-hand side. In the following we find that, in fact, the right-hand sides of these equations coincide with the solutions (83). First, if $n \leq 0$ Eq. (35) for U_1 and U_1^{∞} becomes

$$\begin{aligned}
&\sum_{n=-\infty}^{0}\ \Big[\gamma_{n+1}^{(1)}\, b_n^{(1)}\ _{n+1}(y) + \beta_n^{(1)}\, b_n^{(1)}\ _n(y) + \alpha_{n-1}^{(1)}\, b_n^{(1)}\ _{n-1}(y)\Big] = \\
&\gamma_1^{(1)} b_0^{(1)}\ _1(y) + \Big[\gamma_0^{(1)} b_{-1}^{(1)} + \beta_0^{(1)} b_0^{(1)}\Big]\ _0(y) + \sum_{n=-\infty}^{-2}\ \gamma_{n+1}^{(1)}\, b_n^{(1)}\ _{n+1}(y)\ + \\
&\sum_{n=-\infty}^{-1}\ \beta_n^{(1)}\, b_n^{(1)}\ _n(y) + \sum_{n=-\infty}^{0}\ \alpha_{n-1}^{(1)}\, b_n^{(1)}\ _{n-1}(y) = 0
\end{aligned}$$

which can be rewritten as

$$\gamma_1^{(1)}\, b_0^{(1)}\ _1(y) + \Big[\gamma_0^{(1)}\, b_{-1}^{(1)} + \beta_0^{(1)}\, b_0^{(1)}\Big]\ _0(y)\ + \sum_{m=-\infty}^{-1}\ \Big[\gamma_m^{(1)}\, b_{m-1}^{(1)} + \beta_m^{(1)}\, b_m^{(1)} + \alpha_m^{(1)}\, b_{m+1}^{(1)}\Big]\ _m(y) = 0. \tag{96}$$

Similarly, if $n \leq 0$ Eq. (40) for $\bar{U}_1^{\infty}$ leads to

$$\bar{\gamma}_1^{(1)}\, \bar{b}_0^{(1)}\ \psi_1(y) + \Big[\bar{\gamma}_0^{(1)}\, \bar{b}_{-1}^{(1)} + \bar{\beta}_0^{(1)}\, \bar{b}_0^{(1)}\Big]\ \psi_0(y) + \sum_{m=-\infty}^{-1}\ \Big[\bar{\gamma}_m^{(1)}\, \bar{b}_{m-1}^{(1)} + \bar{\beta}_m^{(1)}\, \bar{b}_m^{(1)} + \bar{\alpha}_m^{(1)}\, \bar{b}_{m+1}^{(1)}\Big]\psi_m(y) = 0. \tag{97}$$

Eqs. (96) and (97) are satisfied if the coefficients of $_m$ and ψ_m vanish. In particular, since we are supposing that $b_0^{(1)} \neq 0$ and $\bar{b}_0^{(1)} \neq 0$, we require that

$$\gamma_1^{(1)} \overset{(25a)}{=} -\left[1+\mu_1+i\eta+\tfrac{B_2}{2}+\tfrac{B_1}{z_0}\right]\left[\mu_1+B_2+\tfrac{B_1}{z_0}\right]=0, \qquad \bar{\gamma}_1^{(1)} \overset{(25b)}{=} -\left[\mu_1+B_2+\tfrac{B_1}{z_0}\right]=0.$$

Thus, we can satisfy both equations by taking $\mu_1 = -B_2 - B_1/z_0$. In addition, by putting $m=-n$, Eqs. (96) and (97) are rearranged as

$$\begin{aligned}
&\left[\gamma_0^{(1)}\, b_{-1}^{(1)}+\beta_0^{(1)}\, b_0^{(1)}\right]{}_0(y)+\sum_{n=1}^{\infty}\left[\gamma_{-n}^{(1)}\, b_{-n-1}^{(1)}+\beta_{-n}^{(1)}\, b_{-n}^{(1)}+\alpha_{-n}^{(1)}\, b_{-n+1}^{(1)}\right]{}_{-n}(y) \ =0,\\
&\left[\bar{\gamma}_0^{(1)}\, \bar{b}_{-1}^{(1)}+\bar{\beta}_0^{(1)}\, \bar{b}_0^{(1)}\right]\psi_0(y)+\sum_{n=1}^{\infty}\left[\bar{\gamma}_{-n}^{(1)}\, \bar{b}_{-n-1}^{(1)}+\bar{\beta}_{-n}^{(1)}\, \bar{b}_{-n}^{(1)}+\bar{\alpha}_{-n}^{(1)}\, \bar{b}_{-n+1}^{(1)}\right]\psi_{-n}(y)=0.
\end{aligned} \tag{98}$$

Thence, by means of relations (70) and (71) with $\nu_1=0$, Eqs. (95) and (98) reproduce the one-sided series solutions (83), as stated above.

4 Applications for a two-level system

Elsewhere [1] we have seen that the one-sided solutions of the first group may be used to compute the radial part of the wavefunctions for bound states of hydrogen moleculelike ions. In this section we show that such group of solutions is also suitable to solve a time-dependent two-state quantum system given by Ishkhanyan and Gregoryan [3]. For specific values of the parameters, we find finite-series solutions which can be rewritten as power-series solutions; for other values, we find solutions in infinite series of irregular confluent hypergeometric functions. These solutions are bounded for all values of the independent variable z [$z=(1+it)/2$] and vanish when the time t goes to infinity.

4.1 The Lorentzian model

The two-state system is described by the pair of equations [3]

$$i\frac{dA_1(t)}{dt}=e^{-i\delta(t)}\,\Omega(t)\,A_2(t), \qquad i\frac{dA_2(t)}{dt}=e^{i\delta(t)}\,\Omega(t)\,A_1(t), \tag{99}$$

where $\Omega(t)$ (called the Rabi frequency) and $\delta(t)$ (detuning modulation) are real functions of the time t. The decoupled equation for A_2 is

$$\frac{d^2A_2}{dt^2}+\left[-i\frac{d\delta}{dt}-\frac{1}{\Omega}\frac{d\Omega}{dt}\right]\frac{dA_2}{dt}+\Omega^2A_2=0, \qquad [\text{ when } \Omega(t)\neq 0].$$

From the several classes of systems obeying the CHE [3], we select one representing a pulse with Lorentzian shape, given by

$$\Omega(t)=\frac{\Omega_0}{1+t^2}, \qquad \frac{d\delta}{dt}=\Delta_0+\frac{\Delta_1}{1+t^2}, \tag{100}$$

where Ω_0, Δ_0 and Δ_1 are constants. Since $\Omega(t=\pm\infty)=0$, Eqs. (99) allow to impose

$$A_1(t=\pm\infty)=A_2(t=\pm\infty)=0. \tag{101}$$

For (100), the second-order equation for $A_2(t)$ reads

$$\left(1+t^2\right)\frac{d^2A_2}{dt^2}+\left[-i\Delta_0\left(1+t^2\right)+2t-i\Delta_1\right]\frac{dA_2}{dt}+\frac{\Omega_0^2}{1+t^2}\,A_2=0 \tag{102}$$

which, by the substitutions

$$A_2(t)=e^{\Delta_0 z}\left[\tfrac{z}{z-1}\right]^{\alpha}U(z), \quad z=\tfrac{1+it}{2}, \quad \alpha=\tfrac{1}{4}\left(\Delta_1+2R\right), \quad R=\sqrt{\Omega_0^2+\tfrac{\Delta_1^2}{4}}, \tag{103}$$

is transformed into the CHE (1)

$$z(z-1)\tfrac{d^2U}{dz^2}+(-R-1+2z)\tfrac{dU}{dz}+\left[\Delta_0\left(1+\frac{\Delta_1}{2}\right)+2\Delta_0(z-1)-\Delta_0^2z(z-1)\right]U=0.$$

with parameters

$$z_0 = 1, \quad B_1 = -1 - R, \quad B_2 = 2, \qquad B_3 = \Delta_0 \left[1 + \tfrac{1}{2}\Delta_1\right], \quad i\eta = 1, \quad i\omega = \Delta_0. \tag{104}$$

If $\Delta_0 = 0$ the above equation is not a CHE and can be easily integrated.

By (103) the parameter R is a positive real number. For $R = 2, 3, \cdots$, we will find finite-series solutions which are bounded for any value of t and satisfy conditions (101), as stated in Ref. [3]. These terminating series are sometimes called exact isolated solutions (see [18] and references therein). According to a definition given by Kalnins, Miller and Pogosyan [19], in the present case we have a quasiexactly solvable problem whose coefficients of the series satisfy three-term recurrence relations. However, for $R \neq 1, 2, \cdots$, we find infinite-series solutions which are also convergent and bounded for any value of t. More precisely:

- If $R = 2, 3, \cdots$, the one-sided series expansions U_1^{∞} and U_7^{∞} in terms of irregular confluent hypergeometric functions lead to bounded finite-series solutions for Eq. (102) (if $R = 2$ and $\Delta_1 = 0$ there is no finite-series solution). These can also be obtained from power series expansions.
- If $R \neq 1, 2, 3, \cdots$, the one-sided series solutions U_2^{∞} and U_6^{∞} in terms of irregular confluent hypergeometric functions lead to bounded infinite-series solutions for Eq. (102). These cannot be obtained from power series expansions.

The two types of solutions satisfy the conditions (101). We have found no solution appropriate for $R = 1$, that is, for $1 = \Omega_0^2 + (\Delta_1^2/4)$.

4.2 Finite-series solutions for $R = 2, 3, \cdots$

By inserting the expansion U_1^{∞} given in (82) into (103) , for $R = 2, 3, \cdots$ we obtain the following finite-series solutions $A_2^{(1)}(t)$,

$$\begin{aligned} A_2^{(1)}[t(z)] &\overset{(82,103)}{=} e^{\Delta_0 z}\left(\tfrac{z}{z-1}\right)^{\alpha} U_1^{\infty}(z) \\ &\overset{(82,104)}{=} e^{2\Delta_0 z}\left(\tfrac{z}{z-1}\right)^{\alpha} \sum_{n=0}^{R-2} b_n^{(1)} \Psi\left(2, R+1-n; -2\Delta_0 z\right), \quad \alpha = \tfrac{\Delta_1 + 2R}{4}, \end{aligned} \tag{105a}$$

where, in the recurrence relations (79) for $b_n^{(1)}$, we have

$$\begin{aligned} &\alpha_n^{(1)} = -2\Delta_0(n+1), \qquad \beta_n^{(1)} = n\left[n+1-2R\right] + R\left[R-1\right] + \Delta_0\left[2 - R + 2n + \tfrac{1}{2}\Delta_1\right], \\ &\gamma_n^{(1)} = -\left[n+1-R\right]\left[n-R\right]. \end{aligned} \tag{105b}$$

The summation is restricted to $0 \leq n \leq R-2$ because $\gamma_{n=R-1}^{(1)} = 0$ requires that $n \leq R-2$ as consequence of (8). For $\mu_7 = 0$ the U_7^{∞}, written in (A.6), gives a solution $A_2^{(7)}$,

$$A_2^{(7)}[t(z)] = e^{2\Delta_0 z}\left(\tfrac{z}{z-1}\right)^{\alpha - R} \sum_{n=0}^{R-2} b_n^{(7)} \Psi\left(2, R+1-n; -2\Delta_0(z-1)\right), \tag{106a}$$

where, in the relations (79) for $b_n^{(7)}$, we have

$$\begin{aligned} &\alpha_n^{(7)} = 2\Delta_0(n+1), \quad \beta_n^{(7)} = n\left[n+1-2R\right] + R\left[R-1\right] + \Delta_0\left[-2 + R - 2n + \tfrac{1}{2}\Delta_1\right], \\ &\gamma_n^{(7)} = -\left[n+1-R\right]\left[n-R\right]. \end{aligned} \tag{106b}$$

The function $\exp\left(2\Delta_0 z\right) = \exp\left[\Delta_0(1+it)\right]$ is a finite factor. Thence, from $\Psi(a, c; y) \sim y^{-a}$ when $y \to \infty$, it follows that $A_2^{(1)}$ and $A_2^{(7)}$ vanish when $t \to \pm\infty$, as required in (101). Moreover, the finite-series solutions can be written in terms of elementary functions by using the relation (60), that is,

$$\Psi(a, a+l+1; y) = \frac{l!\, y^{-a}}{\Gamma(a)} \sum_{m=0}^{l} \frac{\Gamma(a+m)\, y^{-m}}{m!\, (l-m)!}, \qquad l = 0, 1, 2, \cdots.$$

So, if $R = 2, 3, \cdots$, we find

$$\begin{aligned} A_2^{(1)}[t(z)] &= e^{2\Delta_0 z}\left[\tfrac{z}{z-1}\right]^{\alpha} \tfrac{1}{z^2} \sum_{n=0}^{R-2} (R-n-2)!\, b_n^{(1)} \sum_{m=0}^{R-2-n} \frac{(m+1)(-2\Delta_0 z)^{-m}}{(R-2-n-m)!}, \\ A_2^{(7)}[t(z)] &= e^{2\Delta_0 z}\left[\tfrac{z}{z-1}\right]^{\alpha - R} \tfrac{1}{(z-1)^2} \sum_{n=0}^{R-2} (R-n-2)!\, b_n^{(7)} \sum_{m=0}^{R-2-n} \frac{(m+1)\left[-2\Delta_0(z-1)\right]^{-m}}{(R-2-n-m)!}. \end{aligned} \tag{107}$$

From the above expressions we see that

$$A_2^{(1)} \text{ or } A_2^{(7)} \propto e^{2\Delta_0 z}\left[\frac{z}{z-1}\right]^{\alpha} z^{-R}\times\Big[\text{polynomial of degree } (R-2) \text{ in } z \text{ or } (z-1)\Big].$$

This indicates that the solutions (107) must be equivalent to the ones obtained by using the power series of Appendix B. In effect, we find

$$\begin{aligned} \hat{A}_2^{(2)}[t(z)] &\overset{(103,\text{ B.2})}{=} e^{2\Delta_0 z}\left(\frac{z}{z-1}\right)^{\alpha} z^{-R}\sum_{n=0}^{R-2}\hat{b}_n^{(2)}(z-1)^n, && R=2,3,\cdots \\ \hat{A}_2^{(6)}[t(z)] &\overset{(103,\text{ B.4})}{=} e^{2\Delta_0 z}\left(\frac{z}{z-1}\right)^{\alpha} z^{-R}\sum_{n=0}^{R-2}\hat{b}_n^{(6)}(-z)^n, && R=2,3,\cdots \end{aligned} \tag{108}$$

where, in the recurrence relations (79) for $\hat{b}_n^{(2)}$ and $\hat{b}_n^{(6)}$, we have

$$\begin{aligned} &\hat{\alpha}_n^{(2)} = (n+1)(n+1-R), \qquad \hat{\gamma}_n^{(2)} = 2\Delta_0\,[n+1-R]\,, \\ &\hat{\beta}_n^{(2)} = n\,[n+1+2\Delta_0-2R] + R\,[R-1] + \Delta_0\left[2-R+\tfrac{1}{2}\Delta_1\right]. \end{aligned} \tag{109}$$

and

$$\begin{aligned} &\hat{\alpha}_n^{(6)} = (n+1)(n+1-R), \qquad \hat{\gamma}_n^{(6)} = -2\Delta_0\,[n+1-R]\,, \\ &\hat{\beta}_n^{((6)} = n\,[n+1-2\Delta_0-2R] + R\,[R-1] + \Delta_0\left[R-2+\tfrac{1}{2}\Delta_1\right]. \end{aligned} \tag{110}$$

Thus, from now on we deal with the solutions (108).

In the matrix form (80), the recurrence relations for finite series ($0\leq n\leq R-2$) read

$$\begin{bmatrix} \beta_0 & \alpha_0 & 0 & & \\ \gamma_1 & \beta_1 & \alpha_1 & & \\ 0 & \gamma_2 & \beta_2 & \alpha_2 & \\ & & \ddots & \ddots & \ddots \\ & & & \gamma_{R-2} & \beta_{R-2} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{R-2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad R=2,3,\cdots, \tag{111}$$

where we have omitted the superscripts. The vanishment of the determinant of the above matrix imposes constraints between the constants Δ_0 and Δ_1. Moreover, these relations allow writing $\hat{b}_n^{(2)}$ and $\hat{b}_n^{(6)}$ in terms of $\hat{b}_0^{(2)}$ and $\hat{b}_0^{(6)}$, respectively.

We have no proof that $\hat{A}_2^{(2)}$ and $\hat{A}_2^{(6)}$ are linearly dependent for any integer $R\geq 2$, but we check that this is true for $R=2,3,4$. If $R=2$ both solutions are dependent because

$$\hat{A}_2^{(2)}[t(z)]\Big|_{R=2,\ \Delta_1\neq 0} \propto \hat{A}_2^{(6)}[t(z)]\Big|_{R=2,\ \Delta_1\neq 0} \propto e^{2\Delta_0 z}\left(\frac{z}{z-1}\right)^{\alpha}\frac{1}{z^2}, \quad \alpha = 1+\frac{\Delta_1}{4}, \quad \Delta_0 = -\frac{4}{\Delta_1}.$$

For $\Delta_1=0$ we must take $\hat{b}_0^{(2)}=\hat{b}_0^{(6)}=0$ (trivial solution) in order to fulfill the relation $\hat{\beta}_0^{(2)}\hat{b}_0^{(2)}=\hat{\beta}_0^{(6)}\hat{b}_0^{(6)}=0$, where $\hat{\beta}_0^{(2)}=\hat{\beta}_0^{(6)}=\Delta_0\Delta_1/2+2$. If $\Delta_1\neq 0$ we get a nontrivial solution with $\Delta_0=-4/\Delta_1$.

For $R=3$, the recurrence relations give $\hat{b}_1^{(2)}=-\left(\hat{\beta}_0^{(2)}/\hat{\alpha}_0^{(2)}\right)\hat{b}_0^{(2)}$ and $\hat{b}_1^{(6)}=-\left(\hat{\beta}_0^{(6)}/\hat{\alpha}_0^{(6)}\right)\hat{b}_0^{(6)}$. Hence we find

$$\begin{aligned} \hat{A}_2^{(2)}[t(z)]\big|_{R=3} &= \hat{b}_0^{(2)}\,e^{2\Delta_0 z}\left[\frac{z}{z-1}\right]^{\alpha}\frac{1}{z^3}\left[1-\frac{\hat{\beta}_0^{(2)}}{\hat{\alpha}_0^{(2)}}(z-1)\right], \qquad \alpha=\frac{\Delta_1+6}{4}, \\ \hat{A}_2^{(6)}[t(z)]\big|_{R=3} &= \hat{b}_0^{(6)}\,e^{2\Delta_0 z}\left[\frac{z}{z-1}\right]^{\alpha}\frac{1}{z^3}\left[1+\frac{\hat{\beta}_0^{(6)}}{\hat{\alpha}_0^{(6)}}z\right], \end{aligned} \tag{112}$$

where (for $R=3$)

$$\begin{aligned} &\hat{\alpha}_0^{(2)}=-2, \quad \hat{\beta}_0^{(2)}=6+\Delta_0\left(-1+\tfrac{1}{2}\Delta_1\right), \quad \hat{\beta}_1^{(2)}=2+\Delta_0\left(1+\tfrac{1}{2}\Delta_1\right), \quad \hat{\gamma}_1^{(2)}=-2\Delta_0, \\ &\hat{\alpha}_0^{(6)}=-2, \quad \hat{\beta}_0^{(6)}=6+\Delta_0\left(1+\tfrac{1}{2}\Delta_1\right), \quad \hat{\beta}_1^{(6)}=2+\Delta_0\left(-1+\tfrac{1}{2}\Delta_1\right), \quad \hat{\gamma}_1^{(6)}=2\Delta_0, \end{aligned} \tag{113}$$

whereas the condition on the determinant is

$$\hat{\beta}_0^{(i)}\hat{\beta}_1^{(i)}-\hat{\alpha}_0^{(i)}\hat{\gamma}_1^{(i)}=0, \quad [i=2,6] \quad \Leftrightarrow \quad \left[(\Delta_1)^2-4\right](\Delta_0)^2+16\Delta_1\Delta_0+48=0. \tag{114}$$

The last equation establishes following relations between Δ_0 and Δ_1:

$$\Delta_0 = \tfrac{3}{2} \text{ if } \Delta_1 = -2;\ \Delta_0 = -\tfrac{3}{2} \text{ if } \Delta_1 = 2;\ \Delta_0 = \tfrac{8}{4-(\Delta_1)^2}\left[\Delta_1 \pm \sqrt{3+\tfrac{1}{4}(\Delta_1)^2}\,\right] \text{ if } \Delta_1 \neq \pm 2.$$

To show the linear dependence, we write the Wronskian between the solutions as

$$W(z) = W\left(\hat{A}_2^{(2)}, \hat{A}_2^{(6)}\right) = \hat{A}_2^{(2)}\,\frac{d\hat{A}_2^{(6)}}{dz} - \hat{A}_2^{(6)}\,\frac{d\hat{A}_2^{(2)}}{dz}. \tag{115}$$

Thence, the solutions are linearly dependent since

$$\begin{aligned} W(z)\big|_{R=3} &\overset{(112,115)}{=} \hat{b}_0^{(2)}\hat{b}_0^{(6)}\left[e^{2\Delta_0 z}\left(\frac{z}{z-1}\right)^{\alpha}\frac{1}{z^3}\right]^2\left[\frac{\hat{\alpha}_0^{(2)}\hat{\beta}_0^{(6)}+\hat{\beta}_0^{(2)}\hat{\beta}_0^{(6)}+\hat{\alpha}_0^{(6)}\hat{\beta}_0^{(2)}}{\hat{\alpha}_0^{(2)}\hat{\alpha}_0^{(6)}}\right] \\ &\overset{(113)}{=} \hat{b}_0^{(2)}\hat{b}_0^{(6)}\left[\frac{e^{2\Delta_0 z}}{4}\left(\frac{z}{z-1}\right)^{\alpha}\frac{1}{z^3}\right]^2\left\{\left[(\Delta_1)^2-4\right](\Delta_0)^2+16\Delta_1\Delta_0+48\right\} \overset{(114)}{=} 0. \end{aligned}$$

For $R = 4$, the recurrence relations give

$$\hat{b}_1^{(i)} = -\frac{\hat{\beta}_0^{(i)}}{\hat{\alpha}_0^{(i)}}\,\hat{b}_0^{(i)}, \qquad \hat{b}_2^{(i)} = \left(\frac{\hat{\beta}_0^{(i)}\hat{\beta}_1^{(i)}}{\hat{\alpha}_0^{(i)}\hat{\alpha}_1^{(i)}} - \frac{\hat{\gamma}_1^{(i)}}{\hat{\alpha}_1^{(i)}}\right)\hat{b}_0^{(i)}, \qquad [i=2,6] \tag{116}$$

while Eqs. (109) and (110) give

$$\begin{aligned} &\hat{\alpha}_0^{(2)} = -3, \quad \hat{\alpha}_1^{(2)} = -4, \quad \hat{\beta}_0^{(2)} = 12+\Delta_0\left[-2+\tfrac{1}{2}\Delta_1\right], \quad \hat{\beta}_1^{(2)} = 6+\tfrac{1}{2}\Delta_0\Delta_1, \\ &\hat{\beta}_2^{(2)} = 2+\Delta_0\left[2+\tfrac{1}{2}\Delta_1\right], \quad \hat{\gamma}_1^{(2)} = -4\Delta_0, \quad \hat{\gamma}_2^{(2)} = -2\Delta_0 \end{aligned}$$

and

$$\begin{aligned} &\hat{\alpha}_0^{(6)} = \hat{\alpha}_0^{(2)} = -3, \quad \hat{\alpha}_1^{(6)} = \hat{\alpha}_1^{(2)} = -4 \quad \hat{\beta}_0^{(6)} = 12+\Delta_0\left[2+\tfrac{1}{2}\Delta_1\right], \quad \hat{\beta}_1^{(6)} = \hat{\beta}_1^{(2)}, \\ &\hat{\beta}_2^{(6)} = 2+\Delta_0\left[-2+\tfrac{1}{2}\Delta_1\right], \quad \hat{\gamma}_1^{(6)} = -\hat{\gamma}_1^{(2)}, \quad \hat{\gamma}_2^{(6)} = -\hat{\gamma}_2^{(2)}. \end{aligned}$$

Hence we find $[\alpha = 2 + (\Delta_1/4)]$

$$\begin{aligned} \hat{A}_2^{(2)}[t(z)]\Big|_{R=4} &= \hat{b}_0^{(2)}\,\frac{e^{2\Delta_0 z}}{12z^4}\left[\frac{z}{z-1}\right]^{\alpha}\left[12+4\hat{\beta}_0^{(2)}(z-1)+\left(\hat{\beta}_0^{(2)}\hat{\beta}_1^{(2)}-12\Delta_0\right)(z-1)^2\right], \\ \hat{A}_2^{(6)}[t(z)]\Big|_{R=4} &= \hat{b}_0^{(6)}\,\frac{e^{2\Delta_0 z}}{12z^4}\left[\frac{z}{z-1}\right]^{\alpha}\left[12-4\hat{\beta}_0^{(6)}z+\left(\hat{\beta}_0^{(6)}\hat{\beta}_1^{(2)}+12\Delta_0\right)z^2\right]. \end{aligned} \tag{117}$$

The relations between Δ_0 and Δ_1 are determined from the conditions

$$\hat{\beta}_0^{(i)}\hat{\beta}_1^{(i)}\hat{\beta}_2^{(i)} - \hat{\alpha}_1^{(i)}\hat{\beta}_0^{(i)}\hat{\gamma}_2^{(i)} - \hat{\alpha}_0^{(i)}\hat{\beta}_2^{(i)}\hat{\gamma}_1^{(i)} = 0, \qquad [i=2,6]$$

which, in both cases, are equivalent to

$$\tfrac{1}{2}\Delta_1\left[\tfrac{1}{4}(\Delta_1)^2-1\right](\Delta_0)^3+\left[5\Delta_1-32\right](\Delta_0)^2+54\Delta_1\Delta_0+144=0. \tag{118}$$

This relation permits to show that the Wronskinan of the solutions (117) vanishes and, so, we keep only $A_2^{(2)}(t)$. Besides this, the relation yields two values for Δ_0 in each of the cases: $\Delta_1 = 0$, $\Delta_1 = 2$ and $\Delta_1 = -2$; otherwise, we get three values for Δ_0. The explicit forms of the wavefunctions are generated by introducing in $A_2^{(2)}(t)$ the parameters (Δ_0, Δ_1) which satisfy the characteristic equation (118).

Notice the forward procedure used for computing the series coefficients, that is, by means of the recurrence relations, $\hat{b}_0$ leads to $\hat{b}_1$, $\hat{b}_2$, $\cdots$, as in (116). Infinite-series solutions deal with a backward computation.

4.3 Infinite-series solutions for $R \neq 1, 2, 3, \cdots$

For $R \neq 1, 2, 3, \cdots$ there are many infinite-series solutions which, however, are not bounded for all values of z. So, the solutions (105a) and (106a) become infinite-series expansions which do not converge at $z = \infty$ in virtue of conditions (89). Similarly, the expansions (108) become infinite series which are not convergent at $z = \infty$. Bounded infinite-series solutions for $R \neq 1, 2, 3, \cdots$ result from the one-sided solutions U_2^{∞} and U_6^{∞}, as aforementioned.

Thus, by inserting U_2^∞ and U_6^∞ (with $\mu_2=\mu_6=0$) into Eq. (103) we find

$$\begin{aligned} A_2^{(2)}(t) &\overset{(103,\,\mathrm{A.1})}{=} e^{2\Delta_0 z}\frac{1}{z^R}\left[\frac{z}{z-1}\right]^{\alpha}\sum_{n=0}^{\infty} b_n^{(2)}\Psi\left[2-R,1-R-n;-2\Delta_0 z\right],\\ A_2^{(6)}(t) &\overset{(103,\,\mathrm{A.5})}{=} \frac{e^{2\Delta_0 z}}{z^R}\left[\frac{z}{z-1}\right]^{\alpha}\sum_{n=0}^{\infty} b_n^{(6)}\Psi\left[2-R,1-R-n;-2\Delta_0(z-1)\right], \end{aligned} \tag{119}$$

where, in the recurrence relations (79) for $b_n^{(2)}$,

$$\begin{aligned} &\alpha_n^{(2)}=-2\Delta_0[n+1], \qquad \beta_n^{(2)}=n\left[n+1+2\Delta_0\right]+\Delta_0\left[2-R+\tfrac{1}{2}\Delta_1\right],\\ &\gamma_n^{(2)}=-\left[n+1\right]\left[n-R\right], \end{aligned}$$

and, in the relations for $b_n^{(6)}$,

$$\alpha_n^{(6)}=2\Delta_0[n+1], \quad \beta_n^{(6)}=n\left[n+1-2\Delta_0\right]+\Delta_0\left[R+\tfrac{1}{2}\Delta_1\right], \quad \gamma_n^{(6)}=-\left[n+1\right]\left[n-R\right].$$

These solutions are convergent for any finite z ($|z|\geq 1/2$) and also at $z=\infty$ according to (89). Moreover, they satisfy the condition (101) since $\Psi(a,c;y)\sim y^{-a}$ when $y\to\infty$.

We do not know if the expansions (119) are linearly dependent or independent. Anyway, they supply an analytical starting point for constructing solutions for $R\neq 1,2,3,\cdots$. As in the cases concerning the ordinary spheroidal wave equation [6, 20, 21], this problem demands solutions for the characteristic equation which affords relations between the parameters of the CHE, and the generation of the series coefficients $b_n^{(i)}$. The actual computations regard solutions truncated at a some large n.

The characteristic equation may be solved by the continued fraction method which relies on Eq. (81) and depends on the availability of an initial approximated solution. On the other side, there is the tridiagonal matrix technique [20] which is based on the determinant of the matrix defined in Eq. (80). Falloon proposes to use this method for generating accurate starting estimates for the first method [21].

After solving the characteristic equation it is necessary to construct the series coefficients of the wavefunctions. This includes a backward procedure as a consequence of the fact that the series convergence is established for minimal solutions of the recurrence relations. To clarify this point, firstly we write the relations (79) as

$$\alpha_n^{(i)}\frac{b_{n+1}^{(i)}}{b_n^{(i)}}+\beta_n^{(i)}+\gamma_n^{(i)}\frac{b_{n-1}^{(i)}}{b_n^{(i)}}=0. \tag{120}$$

As an illustration we consider $A_2^{(2)}(t)$. Then, when n is large we find

$$2\Delta_0\left[1+\tfrac{1}{n}\right]\frac{b_{n+1}^{(2)}}{b_n^{(2)}}-\Big[n+1+2\Delta_0\Big]+\Big[n+1-R\Big]\frac{b_{n-1}^{(2)}}{b_n^{(2)}}=0,$$

which is satisfied by

$$\frac{b_{n+1}^{(2)}}{b_n^{(2)}}\sim 1-\frac{R}{n} \quad\Leftrightarrow\quad \frac{b_{n-1}^{(2)}}{b_n^{(2)}}\sim 1+\frac{R}{n} \qquad \text{or} \qquad \frac{b_{n+1}^{(2)}}{b_n^{(2)}}\sim\frac{n}{2\Delta_0} \quad\Leftrightarrow\quad \frac{b_{n-1}^{(2)}}{b_n^{(2)}}\sim\frac{2\Delta_0}{n}.$$

The minimal solution for $b_n^{(2)}$ [10] (which assures the convergence of $A_2^{(2)}$) is

$$\frac{b_{n+1}^{(2)}}{b_n^{(2)}}\sim 1-\frac{R}{n}.$$

So, for some large $n=N$, we set $b_{N+1}^{(2)}/b_N^{(2)}=1$ and write Eq. (120) for $A_2^{(2)}(t)$ as

$$\frac{b_n^{(2)}}{b_{n-1}^{(2)}}=\left[n+1-R\right]\left[n+1+2\Delta_0-2\Delta_0\left(\tfrac{n+1}{n}\right)\frac{b_{n+1}^{(2)}}{b_n^{(2)}}\right]^{-1}.$$

Hence, for $n=1,\cdots,N$ we find

$$\begin{aligned} &\frac{b_1^{(2)}}{b_0^{(2)}}=\frac{2-R}{2+2\Delta_0-4\Delta_0\left[b_2^{(2)}/b_1^{(2)}\right]}, \qquad \frac{b_2^{(2)}}{b_1^{(2)}}=\frac{3-R}{3+2\Delta_0-3\Delta_0\left[b_3^{(2)}/b_2^{(2)}\right]}, \qquad \cdots\\ &\frac{b_N^{(2)}}{b_{N-1}^{(2)}}=\frac{N+1-R}{N+1+2\Delta_0-2\Delta_0\left[\frac{N+1}{N}\right]\left[b_{N+1}^{(2)}/b_N^{(2)}\right]}, \qquad \frac{b_{N+1}^{(2)}}{b_N^{(2)}}=1. \end{aligned}$$

By iterating these results backward we obtain the ratios between successive coefficients: $b_N^{(2)}/b_{N-1}^{(2)},\cdots,b_2^{(2)}/b_1^{(2)},\ b_1^{(2)}/b_0^{(2)}$. Only after this, we can write $b_1^{(2)},b_2^{(2)},\cdots,b_{N+1}^{(2)}$ in terms of $b_0^{(2)}$, as in the case of finite-series solutions.

5 Concluding remarks

We have investigated a group of two-sided series solutions for the CHE, constituted by sets of three expansions in series of regular and irregular confluent hypergeometric functions. Each set is represented by two equivalent expressions, $(U_i, U_i^{\infty}, \bar{U}_i^{\infty})$ and $(\boldsymbol{U}_i, \boldsymbol{U}_i^{\infty}, \bar{\boldsymbol{U}}_i^{\infty})$, which differ from one another basically by sign of the summation index n which appears in the parameters of the hypergeometric functions – compare, for example, (24) with (65). In section 2.2, the two-sided sets $(U_i, U_i^{\infty}, \bar{U}_i^{\infty})$ have been obtained as generalizations of the one-sided solutions that we have found elsewhere [1]. On the other side, the sets $(\boldsymbol{U}_i, \boldsymbol{U}_i^{\infty}, \bar{\boldsymbol{U}}_i^{\infty})$ represent the two-sided solutions found by Leaver in 1986, supplemented by solutions generated through the transformations (16) and having the regions of convergence improved by the Raabe test.

As a matter of fact, in the present case, the ratio tests hold only for finite values of the variable z and, thus, it is necessary to take into account the behavior of the solutions for large values of z. In this manner, we have seen that at $z = \infty$ the solutions U_i^{∞} converge if the conditions (41) are fulfilled, the $\bar{U}_i^{\infty}$ diverge, but we cannot decide on the convergence of the U_i. On the other side, the D'Alembert test implies that the two-sided solutions U_i converge for any finite value of z, while U_i^{∞} and $\bar{U}_i^{\infty}$ converge for $|z| > |z_0|$ (if $i = 1, \cdots, 4$) or $|z - z_0| > |z_0|$ (if $i = 5, \cdots, 8$). However, the Raabe test implies that U_i^{∞} and $\bar{U}_i^{\infty}$ converge also at $|z| = |z_0|$ and $|z - z_0| = |z_0|$ under the conditions (42) and (43).

The two-sided solutions of the first and second expressions, $(U_i, U_i^{\infty}, \bar{U}_i^{\infty})$ and $(\boldsymbol{U}_i, \boldsymbol{U}_i^{\infty}, \bar{\boldsymbol{U}}_i^{\infty})$, depend on the parameters μ_i and ν_i, respectively. If there is an arbitrary constant in the CHE, we can take $\mu_i = \nu_i = 0$ in order to obtain two non-equivalent groups of one-sided series solutions with $n \geq 0$. According to sections 3.2 and 3.3, these groups differ by the convergence properties of their solutions. Besides this, we have seen that the solutions of the first group do not admit the limit $z_0 \to 0$ which gives solutions to the double-confluent Heun equation, whereas some sets of the second group admit the limit. In addition, notice that, under certain conditions, the one-sided solutions reduce to finite-series solutions for which the ratio tests become meaningless.

The results reported in the previous paragraph show that in fact it is useful to deal with two equivalent expressions for the two-sided series. In effect, it is possible to obtain the two groups of one-sided series out of only one of expressions, $(U_i, U_i^{\infty}, \bar{U}_i^{\infty})$ or $(\boldsymbol{U}_i, \boldsymbol{U}_i^{\infty}, \bar{\boldsymbol{U}}_i^{\infty})$, by truncating the selected two-sided series on the left-hand side ($n \geq 0$) and on the right-hand side ($n \leq 0$) as indicated in (9). Nevertheless, in this event, to rewrite the series with $n \leq 0$ in the usual form ($n \geq 0$) we are forced to perform mathematical manipulations as the ones of section 3.4.

In section 4 we have applied one-sided series expansions to solve a time-dependent two-level problem where the wave-functions must vanish when the time t goes to infinity. We have found bounded solutions given by finite series if $R = 2, 3, \cdots$, where R is a parameter of the problem. These solutions can also be obtained from power series solutions for the CHE. In addition, for non-integer values of R we have found analytical infinite-series solutions in terms of irregular confluent functions, convergent and bounded for all admissible values of the independent variable $z = (1 + it)/2$. These solutions can be used as starting point for computing explicit solutions for the two-level system. However, solutions for a characteristic equation and the generation of the series coefficients demand a lot of additional computations (see section 4.3).

A connected issue which also requires further study is the reduced CHE, namely,

$$z(z - z_0)\frac{d^2U}{dz^2} + (B_1 + B_2 z)\frac{dU}{dz} + [B_3 + q(z - z_0)]\,U = 0, \qquad q \neq 0, \tag{121}$$

where z_0, B_i and q are constants. This equation comes from the CHE (1) by means of the limits [22, 23]

$$\omega \to 0, \quad \eta \to \infty \quad \text{such that} \quad 2\eta\omega = -q. \tag{122}$$

Both groups of one-sided solutions of the CHE lead to solutions for the reduced CHE. Therefore, we expect new one-sided solutions for this equation, a fact that may become relevant since Eq. (121) describes several physical systems [23, 24, 25, 26, 27, 28].

A Other two-sided series solutions

In this appendix we apply the transformations (16) on the first set (24) following the order indicated in (19). If $\mu_i = 0$, we obtain one-sided series solutions, as discussed in section 3.

Second set: $i\eta + 1 + \frac{B_2}{2} + \frac{B_1}{z_0} \neq 0, -1, \cdots$ in $U_2(z)$ and $U_2^{\infty}(z)$.

$$\begin{aligned}
&\begin{bmatrix} U_2(z) \\ U_2^{\infty}(z) \end{bmatrix} = e^{i\omega z} z^{1+\frac{B_1}{z_0}} \sum_{n=-\infty}^{\infty} b_n^{(2)} \begin{bmatrix} \tilde{\Phi}\left(i\eta + 1 + \frac{B_2}{2} + \frac{B_1}{z_0}, 2 + \frac{B_1}{z_0} - n - \mu_2; -2i\omega z\right) \\ \Psi\left(i\eta + 1 + \frac{B_2}{2} + \frac{B_1}{z_0}, 2 + \frac{B_1}{z_0} - n - \mu_2; -2i\omega z\right) \end{bmatrix},\\
&\bar{U}_2^{\infty}(z) = e^{-i\omega z} z^{1+\frac{B_1}{z_0}} \sum_{n=-\infty}^{\infty} \bar{b}_n^{(2)} \Psi\left[-n - \mu_2 - i\eta + 1 - \frac{B_2}{2}, 2 - n - \mu_2 + \frac{B_1}{z_0}; 2i\omega z\right],
\end{aligned} \tag{A.1}$$

where in the recurrence relations (21) for $b_n^{(2)}$ we have

$$\alpha_n^{(2)} = -2i\omega z_0(n+\mu_2+1), \qquad \gamma_n^{(2)} = -\left[n+\mu_2+i\eta-1+\tfrac{B_2}{2}\right]\left[n+\mu_2-1+B_2+\tfrac{B_1}{z_0}\right],$$
$$\beta_n^{(2)} = [n+\mu_2]\,[n+\mu_2-1+B_2+2i\omega z_0]+i\omega z_0\left[B_2+\tfrac{B_1}{z_0}\right]+B_3$$

and for $\bar{b}_n^{(2)}$ we have

$$\bar{\alpha}_n^{(2)} = -2i\omega z_0[n+\mu_2+1]\left[n+\mu_2+i\eta+\tfrac{B_2}{2}\right], \qquad \bar{\beta}_n^{(2)} = \beta_n^{(2)},$$
$$\bar{\gamma}_n^{(2)} = -\left[n+\mu_2-1+B_2+\tfrac{B_1}{z_0}\right], \qquad \bar{b}_n^{(2)} = (-1)^n\Gamma[1-n-\mu_2-i\eta-\tfrac{B_2}{2}]b_n^{(2)}.$$

Third set: $2+i\eta-\frac{B_2}{2}\neq 0,-1,\cdots$ in $U_3(z)$ and $U_3^{\infty}(z)$.

$$\begin{aligned}
&\begin{bmatrix} U_3(z) \\ U_3^{\infty}(z) \end{bmatrix} = e^{i\omega z}z^{1+\frac{B_1}{z_0}}\left[z-z_0\right]^{1-B_2-\frac{B_1}{z_0}}\sum_{n=-\infty}^{\infty} b_n^{(3)}\begin{bmatrix} \tilde{\Phi}\left[2+i\eta-\frac{B_2}{2},2+\frac{B_1}{z_0}-n-\mu_3;-2i\omega z\right] \\ \Psi\left[2+i\eta-\frac{B_2}{2},2+\frac{B_1}{z_0}-n-\mu_3;-2i\omega z\right] \end{bmatrix}, \\
&\bar{U}_3^{\infty}(z) = e^{-i\omega z}z^{1+\frac{B_1}{z_0}}\left[z-z_0\right]^{1-B_2-\frac{B_1}{z_0}}\sum_{n=-\infty}^{\infty}\bar{b}_n^{(3)}\Psi\left[-n-\mu_3-i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2},2-n-\mu_3+\tfrac{B_1}{z_0};2i\omega z\right],
\end{aligned} \tag{A.2}$$

where in the relations (21) for $b_n^{(3)}$ we have

$$\alpha_n^{(3)} = -2i\omega z_0[n+\mu_3+1], \qquad \gamma_n^{(3)} = -\left[n+\mu_3+i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}\right]\left[n+\mu_3+1-B_2-\tfrac{B_1}{z_0}\right],$$
$$\beta_n^{(3)} = [n+\mu_3]\left[n+\mu_3+1-B_2-\tfrac{2B_1}{z_0}+2i\omega z_0\right]+\left[i\omega z_0-1-\tfrac{B_1}{z_0}\right]\left[2-B_2-\tfrac{B_1}{z_0}\right]+2-B_2+B_3$$

and for $\bar{b}_n^{(3)}$ we have

$$\bar{\alpha}_n^{(3)} = -2i\omega z_0[n+\mu_3+1]\left[n+1+\mu_3+i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}\right], \qquad \bar{\beta}_n^{(3)} = \beta_n^{(3)},$$
$$\bar{\gamma}_n^{(3)} = -\left[n+\mu_3+1-B_2-\tfrac{B_1}{z_0}\right], \qquad \bar{b}_n^{(3)} = (-1)^n\Gamma\left[-n-\mu_3-i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}\right]b_n^{(3)}.$$

Fourth set: $1+i\eta-\frac{B_2}{2}-\frac{B_1}{z_0}\neq 0,-1,\cdots$ in $U_4(z)$ and $U_4^{\infty}(z)$.

$$\begin{aligned}
&\begin{bmatrix} U_4(z) \\ U_4^{\infty}(z) \end{bmatrix} = e^{i\omega z}(z-z_0)^{1-B_2-\frac{B_1}{z_0}}\sum_{n=-\infty}^{\infty} b_n^{(4)}\begin{bmatrix} \tilde{\Phi}\left(1+i\eta-\frac{B_2}{2}-\frac{B_1}{z_0},-\frac{B_1}{z_0}-n-\mu_4;-2i\omega z\right) \\ \Psi\left(1+i\eta-\frac{B_2}{2}-\frac{B_1}{z_0},-\frac{B_1}{z_0}-n-\mu_4;-2i\omega z\right) \end{bmatrix}, \\
&\bar{U}_4^{\infty}(z) = e^{-i\omega z}(z-z_0)^{1-B_2-\frac{B_1}{z_0}}\sum_{n=-\infty}^{\infty}\bar{b}_n^{(4)}\ \Psi\left[-n-\mu_4-1-i\eta+\tfrac{B_2}{2},-n-\mu_4-\tfrac{B_1}{z_0};2i\omega z\right],
\end{aligned} \tag{A.3}$$

where, in the relations (21) for $b_n^{(4)}$ and $\bar{b}_n^{(4)}$, we have

$$\alpha_n^{(4)} = -2i\omega z_0(n+\mu_4+1), \quad \gamma_n^{(4)} = -\left[n+\mu_4+i\eta+1-\tfrac{B_2}{2}\right]\left[n+\mu_4+1-B_2-\tfrac{B_1}{z_0}\right],$$
$$\beta_n^{(4)} = [n+\mu_4]\,[n+\mu_4+3-B_2+2i\omega z_0)]+i\omega z_0\left[2-B_2-\tfrac{B_1}{z_0}\right]+2-B_2+B_3$$

and for $\bar{b}_n^{(4)} = (-1)^n\Gamma\left[-n-\mu_4-i\eta-1+\frac{B_2}{2}\right]b_n^{(3)}$ we have

$$\bar{\alpha}_n^{(4)} = -2i\omega z_0[n+\mu_4+1]\left[n+\mu_4+2+i\eta-\tfrac{B_2}{2}\right], \qquad \bar{\beta}_n^{(4)} = \beta_n^{(4)},$$
$$\bar{\gamma}_n^{4} = -\left[n+\mu_4+1-B_2-\tfrac{B_1}{z_0}\right], \qquad \bar{b}_n^{(4)} = (-1)^n\Gamma\left[-n-\mu_4-i\eta-1+\tfrac{B_2}{2}\right]b_n^{(4)}.$$

Fifth set: $i\eta+\frac{B_2}{2}\neq 0,-1,\cdots$ in $U_5(z)$ and $U_5^{\infty}(z)$.

$$\begin{aligned}
&\begin{bmatrix} U_5(z) \\ U_5^{\infty}(z) \end{bmatrix} = e^{i\omega z}\sum_{n=-\infty}^{\infty} b_n^{(5)}\begin{bmatrix} \tilde{\Phi}\left[i\eta+\frac{B_2}{2},B_2+\frac{B_1}{z_0}-n-\mu_5;2i\omega(z_0-z)\right] \\ \Psi\left[i\eta+\frac{B_2}{2},B_2+\frac{B_1}{z_0}-n-\mu_5;2i\omega(z_0-z)\right] \end{bmatrix}, \\
&\bar{U}_5^{\infty}(z) = e^{-i\omega z}\sum_{n=-\infty}^{\infty}\bar{b}_n^{(5)}\ \Psi\left[-n-\mu_5-i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2},-n-\mu_5+B_2+\tfrac{B_1}{z_0};2i\omega(z-z_0)\right]
\end{aligned} \tag{A.4}$$

where, in the relations (21) for $b_n^{(5)}$ and $\bar{b}_n^{(5)}$,

$$\alpha_n^{(5)} = 2i\omega z_0(n+\mu_5+1), \qquad \gamma_n^{(5)} = -\left[n+\mu_5+i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}\right]\left[n+\mu_5-1-\tfrac{B_1}{z_0}\right],$$
$$\beta_n^{(5)} = [n+\mu_5]\left[n+\mu_5+1-2i\omega z_0-B_2-\tfrac{2B_1}{z_0}\right]+i\omega B_1+2\eta\omega z_0+B_3+\tfrac{B_1}{z_0}\left[\tfrac{B_1}{z_0}+B_2-1\right]$$

and

$$\bar{\alpha}_n^{(5)} = 2i\omega z_0[n+\mu_5+1]\left[n+\mu_5+1+i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}\right], \qquad \bar{\beta}_n^{(5)} = \beta_n^{(5)},$$
$$\bar{\gamma}_n^{(5)} = -\left[n+\mu_5-1-\tfrac{B_1}{z_0}\right], \qquad \bar{b}_n^{(5)} = (-1)^n\Gamma\left[-n-\mu_4-i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}\right]b_n^{(4)}.$$

Sixth set: $i\eta+1+\frac{B_2}{2}+\frac{B_1}{z_0}\neq 0,-1,\cdots$ in $U_6(z)$ and $U_6^{\infty}(z)$,

$$\begin{bmatrix} U_6(z) \\ U_6^{\infty}(z) \end{bmatrix} = e^{i\omega z}z^{1+\frac{B_1}{z_0}}\sum_{n=-\infty}^{\infty} b_n^{(6)} \begin{bmatrix} \tilde{\Phi}\left[1+i\eta+\frac{B_1}{z_0}+\frac{B_2}{2}, B_2+\frac{B_1}{z_0}-n-\mu_6; 2i\omega(z_0-z)\right] \\ \Psi\left[1+i\eta+\frac{B_1}{z_0}+\frac{B_2}{2}, B_2+\frac{B_1}{z_0}-n-\mu_6; 2i\omega(z_0-z)\right] \end{bmatrix},$$
$$\bar{U}_6^{\infty}(z) = e^{-i\omega z}z^{1+\frac{B_1}{z_0}}\sum_{n=-\infty}^{\infty} \bar{b}_n^{(6)}\Psi\left[-n-\mu_6-1-i\eta+\tfrac{B_2}{2}, B_2+\tfrac{B_1}{z_0}-n-\mu_6; 2i\omega(z-z_0)\right], \tag{A.5}$$

where, in the recurrence relations,

$$\alpha_n^{(6)} = 2i\omega z_0(n+\mu_6+1), \qquad \gamma_n^{(6)} = -\left[n+\mu_6+i\eta+1-\tfrac{B_2}{2}\right]\left[n+\mu_6+1+\tfrac{B_1}{z_0}\right],$$
$$\beta_n^{(6)} = [n+\mu_6][n+\mu_6+3-2i\omega z_0-B_2]-i\omega B_1+2\eta\omega z_0-2i\omega z_0+B_3-B_2+2$$

and

$$\bar{\alpha}_n^{(6)} = 2i\omega z_0[n+\mu_6+1]\left[n+\mu_6+2+i\eta-\tfrac{B_2}{2}\right], \qquad \bar{\beta}_n^{6} = \beta_n^{6},$$
$$\bar{\gamma}_n^{(6)} = -\left[n+\mu_6+1+\tfrac{B_1}{z_0}\right], \qquad \bar{b}_n^{(6)} = (-1)^n\Gamma\left[-n-\mu_4-i\eta-1+\tfrac{B_2}{2}\right]b_n^{(6)}.$$

Seventh set: $2+i\eta-\frac{B_2}{2}\neq 0,-1,\cdots$ in $U_7(z)$ and $U_7^{\infty}(z)$.

$$\begin{bmatrix} U_7(z) \\ U_7^{\infty}(z) \end{bmatrix} = e^{i\omega z}z^{1+\frac{B_1}{z_0}}[z-z_0]^{1-B_2-\frac{B_1}{z_0}}\sum_{n=-\infty}^{\infty} b_n^{(7)} \begin{bmatrix} \tilde{\Phi}\left[2+i\eta-\frac{B_2}{2}, 2-n-\mu_7-B_2-\frac{B_1}{z_0}; 2i\omega(z_0-z)\right] \\ \Psi\left[2+i\eta-\frac{B_2}{2}, 2-n-\mu_7-B_2-\frac{B_1}{z_0}; 2i\omega(z_0-z)\right] \end{bmatrix},$$
$$\bar{U}_7^{\infty}(z) = e^{-i\omega z}z^{1+\frac{B_1}{z_0}}[z-z_0]^{1-B_2-\frac{B_1}{z_0}}\sum_{n=-\infty}^{\infty} \bar{b}_n^{(7)}\;\Psi\left[-n-i\eta-\mu_7-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}, 2-n-\mu_7-B_2-\tfrac{B_1}{z_0}; 2i\omega(z-z_0)\right] \tag{A.6}$$

where, in the relations for $b_n^{(7)}$ and $\bar{b}_n^{(7)}$, we have

$$\alpha_n^{(7)} = 2i\omega z_0(n+\mu_7+1), \qquad \gamma_n^{(7)} = -\left[n+\mu_7+i\eta+\tfrac{B_2}{2}+\tfrac{B_1}{z_0}\right]\left[n+\mu_7+1+\tfrac{B_1}{z_0}\right],$$
$$\beta_n^{(7)} = [n+\mu_7]\left[n+\mu_7+1+B_2+\tfrac{2B_1}{z_0}-2i\omega z_0\right]+2\eta\omega z_0-2i\omega z_0+B_2+B_3+\tfrac{B_1}{z_0}\left[B_2+\tfrac{B_1}{z_0}+1-i\omega z_0\right]$$

and

$$\bar{\alpha}_n^{(7)} = 2i\omega z_0[n+\mu_7+1]\left[n+\mu_7+1+i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}\right], \qquad \bar{\beta}_n^{(7)} = \beta_n^{(7)},$$
$$\bar{\gamma}_n^{(7)} = -\left[n+\mu_7+1+\tfrac{B_1}{z_0}\right], \qquad \bar{b}_n^{(7)} = (-1)^n\Gamma\left[-n-\mu_4-i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}\right]b_n^{(7)}.$$

Eighth set: $i\eta+1-\frac{B_2}{2}-\frac{B_1}{z_0}\neq 0,-1,\cdots$ in $U_8(z)$ and $U_8^{\infty}(z)$.

$$\begin{bmatrix} U_8(z) \\ U_8^{\infty}(z) \end{bmatrix} = e^{i\omega z}[z-z_0]^{1-B_2-\frac{B_1}{z_0}}\sum_{n=-\infty}^{\infty} b_n^{(8)} \begin{bmatrix} \tilde{\Phi}\left[i\eta+1-\frac{B_2}{2}-\frac{B_1}{z_0}, 2-B_2-\frac{B_1}{z_0}-n-\mu_8; 2i\omega(z_0-z)\right] \\ \Psi\left[i\eta+1-\frac{B_2}{2}-\frac{B_1}{z_0}, 2-B_2-\frac{B_1}{z_0}-n-\mu_8; 2i\omega(z_0-z)\right] \end{bmatrix},$$
$$\bar{U}_8^{\infty}(z) = e^{-i\omega z}[z-z_0]^{1-B_2-\frac{B_1}{z_0}}\sum_{n=-\infty}^{\infty} \bar{b}_n^{(8)}\Psi\left[1-n-\mu_8-i\eta-\tfrac{B_2}{2}, 2-n-\mu_8-B_2-\tfrac{B_1}{z_0}; 2i\omega(z-z_0)\right] \tag{A.7}$$

where, in the relations (21) for $b_n^{(8)}$ and $\bar{b}_n^{(8)}$, we have

$$\alpha_n^{(8)} = 2i\omega z_0(n+\mu_8+1), \qquad \gamma_n^{(8)} = -\left[n+\mu_8-1+i\eta+\tfrac{B_2}{2}\right]\left[n+\mu_8-1-\tfrac{B_1}{z_0}\right],$$
$$\beta_n^{(8)} = [n+\mu_8]\,[n+\mu_8-1+B_2-2i\omega z_0]+i\omega B_1+2\eta\omega z_0+B_3$$

and

$$\bar{\alpha}_n^{(8)} = 2i\omega z_0[n+\mu_8+1]\left[n+\mu_8+i\eta+\tfrac{B_2}{2}\right], \qquad \bar{\beta}_n^{(8)} = \beta_n^{(8)},$$
$$\bar{\gamma}_n^{(8)} = -\left[n+\mu_8-1-\tfrac{B_1}{z_0}\right], \qquad \bar{b}_n^{(8)} = (-1)^n\Gamma\left[-n-\mu_4-i\eta+1-\tfrac{B_2}{2}\right]b_n^{(8)}.$$

B Some power series solutions (Baber-Hassé)

The Baber-Hassé's power-series solution [29] have been adapted to the version (1) of the CHE by Leaver [2] who has not considered the transformations (16). We denote the solutions by $U_i^0(z)$; they converge for any finite z provided we take the minimal solutions for the series coefficients.

As initial solution, we take

$$U_1^0(z) = e^{i\omega z}\sum_{n=0}^{\infty}\hat{b}_n^{(1)}(z-z_0)^n,$$

where, in the three-term recurrence relations (79) for $\hat{b}_n^{(1)}$,

$$\hat{\alpha}_n^{(1)} = z_0[n+1]\left[n+B_2+\tfrac{B_1}{z_0}\right], \qquad \hat{\beta}_n^{(1)} = n\,[n+B_2-1+2i\omega z_0]+i\omega z_0\left[B_2+\tfrac{B_1}{z_0}\right]+B_3,$$
$$\hat{\gamma}_n^{(1)} = 2i\omega\left[n-1+i\eta+\tfrac{B_2}{2}\right]. \tag{B.1}$$

Using the transformations (16) as in (19) we get other solutions. The solutions U_2^0 and U_6^0 are used in section 4.2. We find

$$U_2^0(z) = T_1U_1^0(z) = e^{i\omega z}z^{1+\frac{B_1}{z_0}}\sum_{n=0}^{\infty}\hat{b}_n^{(2)}(z-z_0)^n, \tag{B.2}$$

where the coefficients for the recurrence relations are

$$\hat{\alpha}_n^{(2)} = z_0[n+1]\left[n+B_2+\tfrac{B_1}{z_0}\right], \qquad \hat{\gamma}_n^{(2)} = 2i\omega\left[n+i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}\right]$$
$$\hat{\beta}_n^{(2)} = n\left[n+1+2i\omega z_0+B_2+\tfrac{2B_1}{z_0}\right]+\left[1+i\omega z_0+\tfrac{B_1}{z_0}\right]\left[B_2+\tfrac{B_1}{z_0}\right]+B_3. \tag{B.3}$$

For U_6^0,

$$U_6^0(z) = T_1U_5^0(z) = T_1T_4U_1^0(z) = e^{i\omega z}z^{1+\frac{B_1}{z_0}}\sum_{n=0}^{\infty}\hat{b}_n^{(6)}(-z)^n, \tag{B.4}$$

the coefficients for the recurrence relations are

$$\hat{\alpha}_n^{(6)} = z_0[n+1]\left[n+2+\tfrac{B_1}{z_0}\right], \qquad \hat{\gamma}_n^{(6)} = -2i\omega\left[n+i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}\right],$$
$$\hat{\beta}_n^{(6)} = n\left[n+1-2i\omega z_0+B_2+\tfrac{2B_1}{z_0}\right]-i\omega z_0\left[2+2i\eta+\tfrac{B_1}{z_0}\right]+\left[1+\tfrac{B_1}{z_0}\right]\left[B_2+\tfrac{B_1}{z_0}\right]+B_3.$$

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