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Current algebra for a generalized two-site Bose-Hubbard model

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We present a current algebra for a generalized two-site Bose-Hubbard model and use it to get the quantum dynamics of the currents. For different choices of the Hamiltonian parameters we get different currents dynamics. We generalize the Heisenberg equation of motion to write the n-th time derivative of any operator.

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Introduction - Since the first experimental verification of the Bose-Einstein condensation (BEC) [1–3] occurred more then seven decades after its theoretical prediction [4, 5] a great deal of progress has been made in the theoretical and experimental study of this many body physical phenomenon [6–26]. Looking in this direction a laser was used in an experiment to divide a BEC in two parts to study the interference phenomenon between two BECs [27, 28]. These two BECs can be coupled by Josephson tunnelling [29–35] with the atoms tunnelling between the condensates in the same way that a Cooper pair in a superconductor Josephson junction. To study this system a model, known as the *canonical Josephson Hamiltonian*, was proposed by Leggett [8]. Since then many models has been used to study the BECs such as the quantum dynamics of tunnelling of atoms between the two condensates, the entanglement, the quantum phase transitions, the classical analysis, the atom-molecule interconversion and the quantum metrology [36–47]. The algebraic Bethe ansatz method has been used to solve and study some of these models [48–59]. We are considering here a generalized issue of the models [8, 52] by the introducing of the on-well energies and leaving free choice for the interaction parameters. The on-well energies is determined by the internal states of the atoms in the condensates and/or by the kinetic (interaction) energy of the atoms and/or the external potentials. The generalized model is described by the Hamiltonian

$$\hat{H} = \sum_{i,j=1}^{2} K_{ij}\hat{N}_i\hat{N}_j - \sum_{i=1}^{2}(\mu_i - U_i)\hat{N}_i - \sum_{i\neq j}^{2}\Omega_{ij}\hat{a}_i^{\dagger}\hat{a}_j, \quad (1)$$

where, $\hat{a}_i^{\dagger}(\hat{a}_i)$, denote the single-particle creation (annihilation) operators in each well and, $\hat{N}_i = \hat{a}_i^{\dagger}\hat{a}_i$, are the corresponding boson number operators in each condensate. The boson operator total number of particles, $\hat{N} = \hat{N}_1 + \hat{N}_2$, is a conserved quantity, $[\hat{H}, \hat{N}] = 0$. The couplings K_{ij}, with $K_{ij} = K_{ji}$, provides the interaction strength between the bosons and are proportional to the s-wave scattering length, μ_i are the external potential, $\Omega_{ij} = \Omega_{ji}$ is the symmetric amplitude of tunnelling and U_i are the on-well energy per particle.

For the particular choice of the couplings parameters we can get some Hamiltonians, as for example by the choices $K_{ii} = \frac{K}{8}$, $K_{12} = -\frac{K}{8}$, $\Delta\mu = \mu_1 - \mu_2 = 2\mu$, $U_i = 0$, and $\Omega_{12} = \frac{\mathcal{E}_J}{2}$ we get the canonical Josephson Hamiltonian studied in [8]. The case with $K_{12} = K_{21} = 0$, $K_{ii} = U/2$, $U_i = -U/2$, $\epsilon = \mu_1 - \mu_2 = 2\mu$, and $\Omega_{12} = t$ was used to study the interplay between disorder and interaction [43]. For the symmetric case we have $\Delta\mu = 0$ and when we turn on $\Delta\mu$ we break the symmetry. For the symmetric case we also can put $\mu_1 = \mu_2 = \mu$ and change the deep of both wells at the same time. This mean that we also can adjust the on-well energies using the external potential in the symmetric case. In the antisymmetric case $\Delta\mu \neq 0$ we can change the bias of one well and increase the on-well energy. In this case it is called a tilted two-well potential [40, 60].

Symmetries - In this section we will discus the symmetries of the Hamiltonian (1) in the same way that we discus them in [61]. The on-well energies $U_i \neq 0$ don't break the global $U(1)$ gauge invariance neither the discrete $\mathbb{Z}_2$ mirror invariances. The Hamiltonian (1) is invariant under the $\mathbb{Z}_2$ mirror transformation $\hat{a}_j \rightarrow -\hat{a}_j, \hat{a}_j^{\dagger} \rightarrow -\hat{a}_j^{\dagger}$, and under the global $U(1)$ gauge transformation $\hat{a}_j \rightarrow e^{i\alpha}\hat{a}_j$, where α is an arbitrary c-number and $\hat{a}_j^{\dagger} \rightarrow e^{-i\alpha}\hat{a}_j^{\dagger}$, $j = 1, 2$. For $\alpha = \pi$ we get again the $\mathbb{Z}_2$ symmetry. The global $U(1)$ gauge invariance is associated with the conservation of the total number of atoms $\hat{N} = \hat{N}_1 + \hat{N}_2$ and the $\mathbb{Z}_2$ symmetry is associated with the parity of the wave function by the relation

$$\hat{P}\,|\Psi\rangle = (-1)^N|\Psi\rangle, \quad (2)$$

$$|\Psi\rangle = \sum_{n=0}^{N} C_{n,N-n}\frac{(\hat{a}_1^{\dagger})^n}{\sqrt{n!}}\frac{(\hat{a}_2^{\dagger})^{N-n}}{\sqrt{(N-n)!}}|0, 0\rangle, \quad (3)$$

where $\hat{P}$ is the parity operator and $[\hat{H}, \hat{P}] = 0$.

There is also the permutation symmetry of the atoms of the two wells if we have $\Delta\mu = 0$ and $U_1 = U_2$. When we turn on $\Delta\mu$ or put $U_1 \neq U_2$ we break the symmetry. The wave function (3) is symmetric under this permutation

$$\hat{\mathcal{P}}\,|\Psi\rangle = \sum_{n=0}^{N} C_{N-n,n}\frac{(\hat{a}_1^\dagger)^{N-n}}{\sqrt{(N-n)!}}\frac{(\hat{a}_2^\dagger)^n}{\sqrt{n!}}|0,0\rangle = |\Psi\rangle, \quad (4)$$

where $\hat{\mathcal{P}}$ is the permutation operator and $[\hat{H},\hat{\mathcal{P}}] = 0$ if $\Delta\mu = 0$ [51] and $U_1 = U_2$. In the Fig. 1 we represent the two BEC by a two-well potential for the case $\Delta\mu \neq 0$ and $U_1 = U_2$.

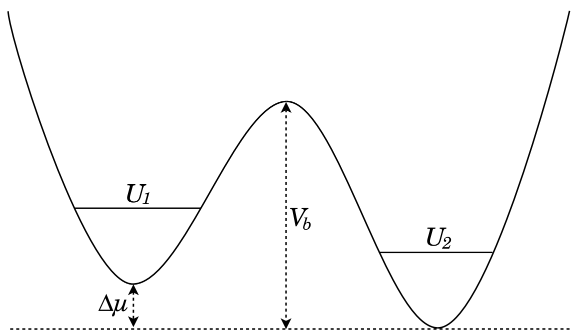

FIG. 1. Two-well potential showing the tunnelling for $U_1 = U_2$ and $\Delta\mu \neq 0$ with the height of the barrier V_b.

The symmetries of the Hamiltonian (1) imply degeneracy. For the conservancy of $\hat{N}$ we have that all wave function of the Hamiltonian (1) are degenerated eigenfunctions of $\hat{N}$ with the same eigenvalue N. For the parity operator $\hat{P}$ all wave function of the Hamiltonian (1) are even or odd depending if N is even or odd. All wave functions are degenerated eigenfunctions of $\hat{P}$ with the same eigenvalue $\lambda = +1$ if N is even or they are degenerated eigenfunctions of $\hat{P}$ with the same eigenvalue $\lambda = -1$ if N is odd. For the permutation operator $\hat{\mathcal{P}}$ all wave function of the Hamiltonian (1) are degenerated eigenfunctions with the same eigenvalue $\lambda = +1$.

Current Algebra - The quantum dynamics of any operator $\hat{O}$ in the Heisenberg picture is determined by the Heisenberg equation of motion

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar}[\hat{H},\hat{O}]. \quad (5)$$

The boson operator total number of particles, $\hat{N} = \hat{N}_1 + \hat{N}_2$, is a conserved quantity, $[\hat{H},\hat{N}] = 0$, and it is commutable compatible operator (CCO) with the boson operators number of particles in each well, $[\hat{N},\hat{N}_1] = [\hat{N},\hat{N}_2] = [\hat{N}_1,\hat{N}_2] = 0$. The boson operators number of particles in each well don't commute with the Hamiltonian and their time evolution is dictated by the Josephson tunnelling current operator,

$$\hat{\mathcal{J}} = \frac{1}{2i}(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1), \quad (6)$$

in coherent opposite phases because of the conservancy of $\hat{N}$, with

$$[\hat{H},\hat{N}_1] = +i\Omega\hat{\mathcal{J}}, \quad [\hat{H},\hat{N}_2] = -i\Omega\hat{\mathcal{J}}, \quad (7)$$

and

$$\frac{d\hat{N}_1}{dt} = -\frac{\Omega}{\hbar}\hat{\mathcal{J}}, \quad (8)$$

$$\frac{d\hat{N}_2}{dt} = +\frac{\Omega}{\hbar}\hat{\mathcal{J}}. \quad (9)$$

From equations (8) and (9) we see that if $\Omega = 0$ we don't have tunnelling.

If we introduce a phase ϕ_{ij} for each term $\hat{a}_i^\dagger\hat{a}_j$, $i,j = 1,2$, we can write the current (6) as

$$\hat{\mathcal{J}} = \frac{1}{2}(e^{i\phi_{12}}\hat{a}_1^\dagger\hat{a}_2 + e^{i\phi_{21}}\hat{a}_2^\dagger\hat{a}_1), \quad (10)$$

with $\phi_{12} = \pi/2$ and $\phi_{21} = 3\pi/2$. So, the phase difference in the current $\hat{\mathcal{J}}$ is $|\Delta\phi| = \pi$. The tunnelling current $\hat{\mathcal{J}}$ together with the imbalance current $\hat{\mathcal{I}}$

$$\hat{\mathcal{I}} = \frac{1}{2}(e^{i\phi_{11}}\hat{N}_1 + e^{i\phi_{22}}\hat{N}_2), \quad (11)$$

with $\phi_{11} = 0$ and $\phi_{22} = \pi$, to get the phase difference in the current $\hat{\mathcal{I}}$ equal to $|\Delta\phi| = \pi$, and the coherent correlation tunnelling current operator $\hat{\mathcal{T}}$

$$\hat{\mathcal{T}} = \frac{1}{2}(e^{i\phi_{12}}\hat{a}_1^\dagger\hat{a}_2 + e^{i\phi_{21}}\hat{a}_2^\dagger\hat{a}_1), \quad (12)$$

with $\phi_{12} = 0$ or 2π and $\phi_{21} = 0$ or 2π, to get the phase difference in the current $\hat{\mathcal{T}}$ equal to $|\Delta\phi| = 0$ or 2π, generates the currents algebra

$$[\hat{\mathcal{T}},\hat{\mathcal{J}}] = +i\hat{\mathcal{I}}, \quad [\hat{\mathcal{T}},\hat{\mathcal{I}}] = -i\hat{\mathcal{J}}, \quad [\hat{\mathcal{J}},\hat{\mathcal{I}}] = +i\hat{\mathcal{T}}. \quad (13)$$

With the identification $\hat{L}_x \equiv \hbar\hat{\mathcal{T}}$, $\hat{L}_y \equiv \hbar\hat{\mathcal{J}}$, and $\hat{L}_z \equiv \hbar\hat{\mathcal{I}}$ we can write it in the standard compact way of the momentum angular

$$[\hat{L}_k,\hat{L}_l] = i\hbar\varepsilon_{klm}\hat{L}_m. \quad (14)$$

where ε_{klm} is the antisymmetric Levi-Civita tensor with $k,l,m = x,y,z$ and $\varepsilon_{xyz} = +1$.

We have two Casimir operators for that currents algebra. One of them is the total number of particles, $\hat{\mathfrak{C}}_1 = \hat{N}$, related to the $U(1)$ symmetry and the another one is related to the momentum angular algebra and the $O(3)$ symmetry, $\hat{\mathfrak{C}}_2 = \hat{\mathcal{T}}^2 + \hat{\mathcal{I}}^2 + \hat{\mathcal{J}}^2$.

We can show that $\hat{\mathfrak{C}}_2$ is just a function of $\hat{\mathfrak{C}}_1$

$$\hat{\mathfrak{C}}_2 = \frac{\hat{\mathfrak{C}}_1}{2}\left(\frac{\hat{\mathfrak{C}}_1}{2} + 1\right). \quad (15)$$

The current algebra (13) is the same for the model [8]. We has been described it in details in [61].

Current Quantum Dynamics - We can rewrite the Hamiltonian (1) using those currents operators

$$\hat{H} = \alpha\hat{\mathcal{I}}^2 + (\beta\hat{\mathfrak{C}}_1 + \gamma)\hat{\mathcal{I}} - 2\Omega\hat{\mathcal{T}} + \frac{\hat{\mathfrak{C}}_1}{2}\left(\frac{\hat{\mathfrak{C}}_1}{2}\rho + \xi\right), \quad (16)$$

where,

$$\begin{aligned}\alpha &= K_{11} - 2K_{12} + K_{22}\\ \beta &= K_{11} - K_{22},\\ \gamma &= U_1 - \mu_1 - U_2 + \mu_2,\\ \rho &= K_{11} + 2K_{12} + K_{22}\\ \xi &= U_1 - \mu_1 + U_2 - \mu_2\\ \Omega &= \Omega_{12} = \Omega_{21}.\end{aligned} \quad (17)$$

We define the Casimir operator $\hat{\mathfrak{Z}} = \beta\hat{\mathfrak{C}}_1 + \gamma$. We can see that the Casimir operators are also conserved quantities, $[\hat{H}, \hat{\mathfrak{C}}_1] = 0$.

The quantum dynamic of those currents are dictated by the momentum angular algebra, their commutation relations with the Hamiltonian and the parameters. We can use the Heisenberg equation of motion (5) to write the second time derivative of any operator $\hat{O}$ in the Heisenberg picture as [61]

$$\frac{d^2\hat{O}}{dt^2} = \left(\frac{i}{\hbar}\right)^2 [\hat{H}, [\hat{H}, \hat{O}]], \quad (18)$$

or as

$$\frac{d^2\hat{O}}{dt^2} = \frac{i}{\hbar}[\hat{H}, \frac{d\hat{O}}{dt}]. \quad (19)$$

It is direct to generalize the Eqs. (18) and (19) for the n-th time derivative of any operator $\hat{O}$ in the Heisenberg picture. So we can write

$$\frac{d^n\hat{O}}{dt^n} = \left(\frac{i}{\hbar}\right)^n \underbrace{[\hat{H}, [\hat{H}, [\hat{H}, \ldots, [\hat{H}, \hat{O}]]],}_{n \text{ commutators}} \quad (20)$$

or as

$$\frac{d^n\hat{O}}{dt^n} = \frac{i}{\hbar}[\hat{H}, \frac{d^{n-1}\hat{O}}{dt^{n-1}}], \quad (21)$$

where we has been defined

$$\frac{d^0\hat{O}}{dt^0} \equiv \hat{O}, \quad (22)$$

and $n \geq 1$. We get the Heisenberg equation of motion (5) for $n = 1$ and the Eqs. (18) and (19) for $n = 2$. Using the Eq. (18) or (19) we found the following equations for the quantum dynamics of the three currents

$$\frac{d^2\hat{\mathcal{I}}}{dt^2} + 4\frac{\Omega^2}{\hbar^2}\hat{\mathcal{I}} = -4\frac{\Omega\alpha}{\hbar^2}\hat{\mathcal{I}}\hat{\mathcal{T}} + 2i\frac{\Omega\alpha}{\hbar^2}\hat{\mathcal{J}} - 2\frac{\Omega}{\hbar^2}\hat{\mathfrak{Z}}\hat{\mathcal{T}}, \quad (23)$$

$$\begin{aligned}\frac{d^2\hat{\mathcal{J}}}{dt^2} + \frac{1}{\hbar^2}\left[\alpha^2 + \hat{\mathfrak{Z}}^2 + 4\Omega^2\right]\hat{\mathcal{J}} &= -4\frac{\alpha^2}{\hbar^2}\hat{\mathcal{I}}^2\hat{\mathcal{J}} - 2i\frac{\alpha^2}{\hbar^2}\hat{\mathcal{I}}\hat{\mathcal{T}} - 2\frac{\alpha}{\hbar^2}\hat{\mathfrak{Z}}\hat{\mathcal{I}}\hat{\mathcal{J}}\\ &- 4\frac{\alpha\Omega}{\hbar^2}\hat{\mathcal{J}}\hat{\mathcal{T}} - 2i\frac{\alpha}{\hbar^2}\hat{\mathfrak{Z}}\hat{\mathcal{T}} - 2i\frac{\alpha\Omega}{\hbar^2}\hat{\mathcal{I}},\end{aligned} \quad (24)$$

$$\begin{aligned}\frac{d^2\hat{\mathcal{T}}}{dt^2} + \frac{1}{\hbar^2}\left(\alpha^2 + \hat{\mathfrak{Z}}^2\right)\hat{\mathcal{T}} &= -4\frac{\alpha^2}{\hbar^2}\hat{\mathcal{I}}\hat{\mathcal{I}}\hat{\mathcal{T}} + 4i\frac{\alpha^2}{\hbar^2}\hat{\mathcal{I}}\hat{\mathcal{J}} - 4\frac{\alpha}{\hbar^2}\hat{\mathfrak{Z}}\hat{\mathcal{I}}\hat{\mathcal{T}}\\ &+ 2i\frac{\alpha}{\hbar^2}\hat{\mathfrak{Z}}\hat{\mathcal{J}} - 4\frac{\Omega\alpha}{\hbar^2}(\hat{\mathcal{I}}^2 - \hat{\mathcal{J}}^2) - 2\frac{\Omega}{\hbar^2}\hat{\mathfrak{Z}}\hat{\mathcal{I}}.\end{aligned} \quad (25)$$

We can see from the Eqs. (23), (24) and (25) that the currents are coupled on the right hand side of these equations. To simplify our analysis we will make some choices of the parameters. Different choices of the parameters of the Hamiltonian gives us different dynamics for the currents. Fortunately the parameters appear in these equations in the linear and quadratic power. So we can consider a perturbation theory in the parameters of the Hamiltonian until the second power terms. If we calculate the n-th time derivative of the current operators we will get the n-th power of the parameters. Here we will need consider only until second order time derivative of the current operators. We can try to use mean field theory (MFT) to decouple the currents to get some insight. In the first approximation, for example, we can use $\langle\hat{L}_k\hat{L}_l\rangle \approx \langle\hat{L}_k\rangle\langle\hat{L}_l\rangle$. But for this approximation we get from the commutation relations (13) that $\langle\hat{\mathcal{I}}\rangle \approx \langle\hat{\mathcal{J}}\rangle \approx \langle\hat{\mathcal{T}}\rangle \approx 0$. Therefore, the currents are correlated by the currents algebra (13) that forbid MFT even in the first approximation. We also can see the correlation between the currents, using the currents algebra (13), writing the Heisenberg uncertainty relations for each couple of currents

$$\langle(\widehat{\Delta\mathcal{T}})^2\rangle\langle(\widehat{\Delta\mathcal{J}})^2\rangle \leq \frac{1}{4}\langle\hat{\mathcal{I}}\rangle^2, \tag{26}$$

$$\langle(\widehat{\Delta\mathcal{T}})^2\rangle\langle(\widehat{\Delta\mathcal{I}})^2\rangle \leq \frac{1}{4}\langle\hat{\mathcal{J}}\rangle^2, \tag{27}$$

$$\langle(\widehat{\Delta\mathcal{J}})^2\rangle\langle(\widehat{\Delta\mathcal{I}})^2\rangle \leq \frac{1}{4}\langle\hat{\mathcal{T}}\rangle^2, \tag{28}$$

where we are introducing the operator $\widehat{\Delta L}_k = \hat{L}_k - \langle\hat{L}_k\rangle$.

Choosing $\alpha = \beta = 0$ we get three linear second order differential equations

$$\frac{d^2\hat{\mathcal{I}}}{dt^2} + 4\frac{\Omega^2}{\hbar^2}\hat{\mathcal{I}} = -2\frac{\Omega\gamma}{\hbar^2}\hat{\mathcal{T}}, \tag{29}$$

$$\frac{d^2\hat{\mathcal{J}}}{dt^2} + \frac{1}{\hbar^2}\left(\gamma^2 + 4\Omega^2\right)\hat{\mathcal{J}} = 0, \tag{30}$$

$$\frac{d^2\hat{\mathcal{T}}}{dt^2} + \frac{\gamma^2}{\hbar^2}\hat{\mathcal{T}} = -2\frac{\Omega\gamma}{\hbar^2}\hat{\mathcal{I}}. \tag{31}$$

We get the dynamics of a simple harmonic oscillator (SHO) with natural angular frequency $\omega = \sqrt{\gamma^2 + 4\Omega^2}/\hbar$ and period of the oscillations $T = \frac{2\pi\hbar}{\sqrt{\gamma^2+4\Omega^2}}$ for the current $\hat{\mathcal{J}}$. The Eqs. (29) and (31) are a system of two linear differential equations of second order. If we diagonalize the matrix of the coefficients of the system of the Eqs. (29) and (31) we get the same angular frequency ω. If we consider the same period of oscillation $T = 40.1$ ms and angular frequency $\omega = 2\pi \times 24.94$ rad·Hz as in [29], with the total number of particles $N = 1150$, we get the angular frequencies $\omega_\Omega = 78.3$ rad·Hz and $\omega_\gamma = 15$ rad·Hz for the parameters of the Hamiltonian. Comparing with the angular frequencies of the trap we found $\omega_x \approx 2\pi\omega_\Omega$, $\omega_y \approx 2\pi \times 0.843\omega_\Omega$ and $\omega_z \approx 2\pi \times 1.150\omega_\Omega$ for the tunnelling amplitude Ω and $\omega_x \approx 2\pi\times 5.2\omega_\gamma$, $\omega_y \approx 2\pi\times 4.4\omega_\gamma$ and $\omega_z \approx 2\pi \times 6.0\omega_\gamma$ for the parameter γ. The height of the barrier is $V_b \approx 2\pi\hbar \times 3.36\omega_\Omega \approx 2\pi\hbar \times 17.53\omega_\gamma$. In the Figs. (2) and (3) we show the numerical solution for the same choice of these parameters. The initial condition for the first derivative for all currents is zero. The currents are normalized by N. For this choice of the parameters the current $\hat{\mathcal{J}}$ is independent and the initial condition determines its amplitude of oscillation. The currents $\hat{\mathcal{I}}$ and $\hat{\mathcal{T}}$ are correlated and the initial condition don't determines their amplitude of oscillation. The currents dynamics are sensitive to the initial condition [29] and they have the same frequency. We have self-trapping

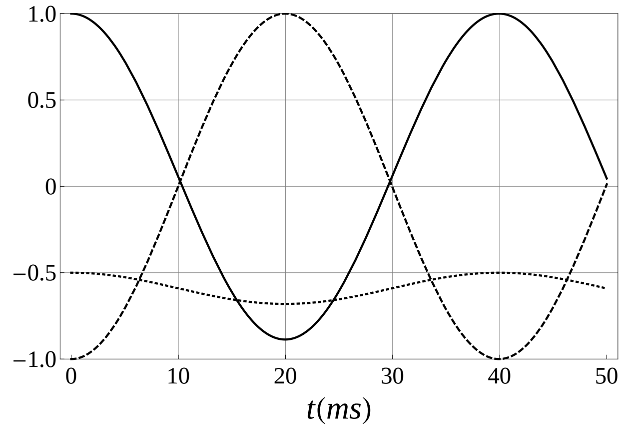

FIG. 2. Current quantum dynamics of the average value of the currents for $\omega_\Omega = 78.3$ rad·Hz and $\omega_\gamma = 15$ rad·Hz. The initial condition for the current $\hat{\mathcal{I}}(t)$ (full line) is $\hat{\mathcal{I}}(0) = 1$. The initial condition for the current $\hat{\mathcal{J}}(t)$ (dashed line) is $\hat{\mathcal{J}}(0) = -1$. The initial condition for the current $\hat{\mathcal{T}}(t)$ (dotted line) is $\hat{\mathcal{T}}(0) = -0.5$.

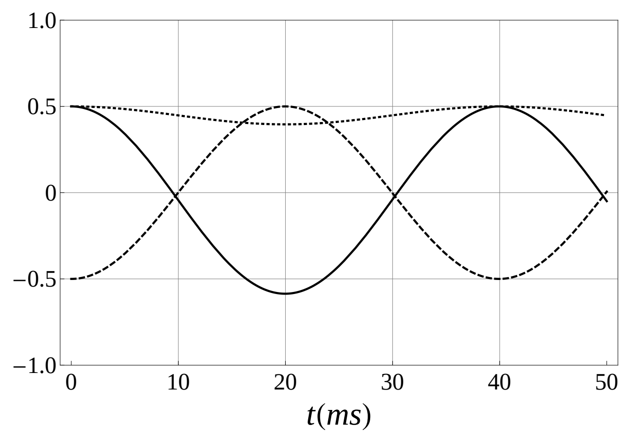

FIG. 3. Current quantum dynamics for $\omega_\Omega = 78.3$ rad·Hz and $\omega_\gamma = 15$ rad·Hz. The initial condition for the current $\hat{\mathcal{I}}(t)$ (full line) is $\hat{\mathcal{I}}(0) = 0.5$. The initial condition for the current $\hat{\mathcal{J}}(t)$ (dashed line) is $\hat{\mathcal{J}}(0) = -0.5$. The initial condition for the current $\hat{\mathcal{T}}(t)$ (dotted line) is $\hat{\mathcal{T}}(0) = 0.5$.

for the current $\hat{\mathcal{T}}$ and Josephson and Rabi dynamics for the currents $\hat{\mathcal{I}}$ and $\hat{\mathcal{J}}$.

In the limit $\alpha = \beta = \gamma = 0$ we get two independent SHO with $\omega_{\mathcal{I}} = \omega_{\mathcal{J}} = 2\frac{\Omega}{\hbar}$ the natural angular frequency. The period of the oscillations is $T = \frac{\pi\hbar}{\Omega}$. In analogy with the classical SHO, the ratio between the elastic constant $\mathcal{K}$ and the mass m is $\frac{\mathcal{K}}{m} = 4\frac{\Omega^2}{\hbar^2}$. The current $\hat{\mathcal{T}}$ is a conserved quantity, $[\hat{\mathcal{H}}, \hat{\mathcal{T}}] = 0$, but this don't means that we don't have tunnelling. We can see from Eqs. (8) and (9) that the quantum dynamic of $\hat{N}_1$, $\hat{N}_2$, and $\hat{\mathcal{I}}$ only depend of the current $\hat{\mathcal{J}}$ and the amplitude of tunnelling Ω.

Summary - We have showed that a current algebra appears when we calculate the quantum dynamics of the tunnelling of the atoms. We generalize the Heisenberg equation of motion to write the n-th time derivative of any operator. Then we calculated the quantum dynamics of these currents and showed that different dynamics appear when we consider different choices of the parameters of the Hamiltonian. The parameters α and ρ determines the non linearity of the interaction, the parameters γ and ξ determines the relation between the on-well energies and the external potentials, the parameter β determines the symmetry of the interaction between the condensates.

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