# A large deviation analysis on the near-equivalence between external and internal reservoirs

João R. Medeiros[a], Sílvio M. Duarte Queirós[a,1,*]

[a] *Centro Brasileiro de Pesquisas Físicas*
*Rua Dr Xavier Sigaud 150, Urca, CEP 22290-180 RJ, Rio de Janeiro, Brazil*

**Abstract**

Within the spirit of van Kampen's "Langevin approach", we discuss the limits of validity of rephrasing the non-equilibrium problem of a particle subject to an external (work) reservoir — a system where the fluctuation-dissipation is not verified — into the simpler case with an internal (heat) reservoir for which the fluctuations and the dissipation arise from the same source. Using a convenient mapping of the thermomechanical parameters we show that, counter-intuitively, such approach is not only valid for steady state time independent quantities, but also for time dependent thermostatistical quantities, namely the injected and dissipated fluxes. We connect this result with the problem of large deviations and conclude that, in this context, we can only distinguish reservoirs by analysing the "fluctuations of accumulated fluctuations". As a by-product, we learn that the best reference approximation to the large deviation functions of a non-Markovian external reservoir system is not the respective internal reservoir limit — as often assumed and suggested by the Langevin approach — but its internal reservoir analogue system obtained from the mapping of the original thermomechanical parameters.

*Keywords:* Non-equilibrium statistical mechanics, External reservoir, Effective temperature, Large deviations, Entropy production

## 1. Introduction

One of the most typical ways of tackling a problem in Physics — and get the solution thereto — is to cast the respective model in a simpler way by redefining the variables/parameters or introducing a phenomenological approach which preserves the backbone of the problem. Concerning the latter, the use of models inspired in the Langevin Equation (LE) is one of the most employed methods [1, 2]; it has a widespread field of applications and has played a relevant role in surveys over the thermostatistical properties of systems far from the thermodynamic limit [3]. Perhaps, the most striking feature of the LE is that in problems of non-equilibrium statistical mechanics, it permits the direct (statistical) characterisation of the position, $x = x(t)$, and velocity, $v = v(t) \equiv dx/dt$, as well as probing the relation between the fluctuation and dissipation in the system, which

*Corresponding author
*Email address:* sdqueiro@gmail.com (Sílvio M. Duarte Queirós)
[1] Associate to National Institute of Science and Technology for Complex Systems, Brazil

is the upshot of the fluctuation-dissipation theorem [4]. The way the two physical effects are connected (or not) dictates the nature of the reservoir. Accordingly [1], a reservoir is:

- *internal* (IntRes) if it allows establishing the dissipation as a property of the reservoir, the fluctuation-dissipation relation is verified and the corresponding theorem as well;
- *external* (ExtRes) if the effects of dissipation and fluctuation that are taking place have different origins and thus the fluctuation-dissipation relation is not verified.

The quintessential IntRes corresponds to the diffusion problem of Brownian motion as treated by Einstein, where the water acts as the reservoir. The random impacts of the water molecules in the pollen grain are the cause of both the dissipation and the fluctuations. Furthermore, since a pollen grain is weightier than water molecules,[2] the noise correlation function falls off very rapidly (typically $10^{-8}$ seconds) and thus the noise, $\eta$, that is responsible for the fluctuation is nicely reproduced by a white noise,

$$\langle \eta\,(t_2)\,\eta\,(t_1)\rangle_c = 2\,\gamma\,T\,\delta\,(t_2 - t_1)\,, \tag{1}$$

where $\gamma$ is the dissipation coefficient and $T$ is the temperature of the bath. For a significantly dense medium, $\eta$ is Gaussian as well. Thence, if we consider a system composed of a particle with mass, $m$, that is subject to a confining potential, $V = V(x)$, we get a dynamics is ruled by,

$$m\frac{d^2x}{dt^2} = -\gamma\,\frac{dx}{dt} - \frac{dV}{dx} + \eta\,(t)\,, \tag{2}$$

Later, Mori and Kubo [5] surveyed the problem of a reservoir the particles of which have got a mass of the same order of magnitude of the focal particle. In that case, the fluctuations cannot be white noise, but they have the same source as dissipation still. In order to square such type problem within the IntRes scenario, Eq. (2) was generalised to,

$$m\frac{d^2x}{dt^2} = -\int_{t_0}^{t} \kappa\,(t-t')\left(\frac{dx}{dt'}\right)dt' - \frac{dV}{dx} + \xi\,(t)\,. \tag{3}$$

For this non-Markovian problem, the fluctuations — which although Gaussian are now represented by $\xi(t)$ — and the dissipation are defined by,

$$\langle \xi\,(t_2)\,\xi\,(t_1)\rangle_c = \frac{\gamma}{\tau}T\exp\left[-\frac{|t_1-t_2|}{\tau}\right], \qquad \kappa\,(t_1-t_2) = \frac{\gamma}{\tau_\kappa}\exp\left[-\frac{|t_1-t_2|}{\tau_\kappa}\right], \tag{4}$$

with $\tau_\kappa = \tau$, which guarantees that both spectra scale equally. In the limit ($\tau \to 0, \tau_\kappa \to 0$), Eq. (4) reads,

$$\langle \xi\,(t_2)\;\xi\,(t_1)\rangle = 2\,\gamma\,T\,\delta\,(t_2-t_1)\,, \qquad \int_{t_0}^{t} \kappa\,(t-t')\left(\frac{dx}{dt'}\right)dt' = \gamma\,\frac{dx}{dt}. \tag{5}$$

[2] A typical pollen grain is $10^4$ times as weighty as a molecule of water.

like Eq. (1) and Eq. (2) is recovered $[\eta(t) \equiv \xi_{\tau\to 0}(t)]$.

It is not hard to grasp that when $\tau \neq \tau_\kappa$, the source of the fluctuations has nothing to do with the origin of the dissipation, resulting in a ExtRes situation. The simplest ExtRes situation corresponds to $\tau \neq 0$ and $\tau_\kappa \to 0$,

$$m\frac{d^2x}{dt^2} = -\gamma\,\frac{dx}{dt} - \frac{dV}{dx} + \xi\,(t)\,, \tag{6}$$

and Eq. (4) keeps on being valid for $\xi(t)$.

The definitive instance of a system described by Eq. (6) is a particle in a frictional medium subject to a coloured random force [6, 7, 8]. This can be set up, *e.g.*, by inserting a charged particle in liquid helium (above superfluid phase though) — or any other situation where thermal noise is negligible in comparison with the fluctuations bore by the external source — and apply a coloured Gaussian electric field to it. Additionally, we can refer to experiments with dye lasers [9] and laser gyroscopes [10] where this type of reservoir emerges.

The description of IntRes and ExtRes cases helps understand that the two types of reservoirs are often distinguishable by the way they affect the energy of the system: while the IntRes is always a heat reservoir, the ExtRes frequently corresponds to a work reservoir, *i.e.*, it changes the energy of the system by performing work on it.

Herein, an important point pertains to calling the quantity $T$ [Eq. (4)] the temperature of the ExtRev system (6). Although its properness might be disputed within a cautious Thermodynamical parlance, we will use that terminology upholding our decision on the fact that $T$ has energy units ($k_B = 1$) and provides us with a typical scale of the fluctuations of velocity and position induced by the ExtRes (*e.g.*, in the form of positive/negative work performed on the system). Such broad understanding of the concept of temperature has been recently employed in the statistics of vortices in superconductors whence an athermal formalism — absolutely analogous to standard Thermodynamics — was derived [11]. Moreover, alternative definitions of temperature concurring with equipartition and fluctuation-dissipation relations were introduced in the field of stochastic dynamics with non-Gaussian reservoirs as well [12].

### *1.1. The hypothesis*

In the remaining of this manuscript, we focus on the IntRes, Eq. (3), and the ExtRes, Eq. (6), systems with $V(x) = \frac{1}{2}\,k\,x^2$. The choice for this potential has an experimental justification; in thermostatistical oriented problems the system is frequently confined by means of an optical tweezer that is very well described by the harmonic potential [20].

To solve the problem, we Fourier-Laplace transform Eqs. (2) and (6),

$$\tilde{f}\,(\mathrm{i}\,q) \equiv \int_0^\infty \mathrm{e}^{-\mathrm{i}\,q\,t}\,f\,(t)\;dt, \tag{7}$$

and for $t_0 = 0$, $x_0 = 0$ and $v_0 = 0$ we get,

$$\tilde{x}\,(\mathrm{i}\,q) = \frac{\tilde{\eta}\,(\mathrm{i}\,q)}{R\,(\mathrm{i}\,q)}, \qquad \tilde{x}\,(\mathrm{i}\,q) = \frac{\tilde{\xi}\,(\mathrm{i}\,q)}{R\,(\mathrm{i}\,q)}, \tag{8}$$

respectively, where,

$$R(q) \equiv m\,q^2 + \gamma\,q + k = m\,(q - \varsigma_+)\,(q - \varsigma_-)\,, \qquad \left(\varsigma_\pm = -\frac{\gamma}{2m} \pm \mathrm{i}\sqrt{4\frac{k}{m} - \frac{\gamma^2}{4m^2}}\right). \quad (9)$$

For both reservoirs, the velocity in the reciprocal space reads,

$$\tilde{v}\,(\mathrm{i}\,q) = \mathrm{i}\,q\,\tilde{x}\,(\mathrm{i}\,q)\,, \quad (10)$$

and the covariance of the fluctuations,

$$\langle \tilde{\eta}\,(\mathrm{i}\,q_1)\;\tilde{\eta}\,(\mathrm{i}\,q_2)\rangle_c = \frac{2}{\mathrm{i}\,q_1 + \mathrm{i}\,q_2}\,\gamma\,T, \quad (11)$$

$$\left\langle \tilde{\xi}\,(\mathrm{i}\,q_1)\;\tilde{\xi}\,(\mathrm{i}\,q_2)\right\rangle_c = \frac{2 + \mathrm{i}\,q_1\,\tau + \mathrm{i}\,q_2\,\tau}{(\mathrm{i}\,q_1 + \mathrm{i}\,q_2)\,(1 + \mathrm{i}\,q_1\,\tau)\,(1 + \mathrm{i}\,q_2\,\tau)}\gamma\,T. \quad (12)$$

The two cases give rise to a steady state and after the transient, we can apply sample and time averages interchangeably. Alternatively to the standard calculation of the statistical moments of a quantity $\mathcal{O}$ by sampling in time,

$$\overline{\mathcal{O}^n} = \lim_{\Xi \to 0} \frac{1}{2\,\Xi} \int_{-\Xi}^{\Xi} [\mathcal{O}\,(t)]^n\;dt, \quad (13)$$

we will resort to the extreme-value theorem [8],

$$\overline{\mathcal{O}^n} = \lim_{z \to 0} z \int_{-\infty}^{\infty} e^{-z\,t}\;[\mathcal{O}\,(t)]^n\;dt, \quad (14)$$

which yields the same results.

Owing to the additive and Gaussian nature of the noise in Eqs. (2) and (6), the steady state distribution is always Boltzmann-like, regardless of the colour of $\eta(t)$ or $\xi(t)$. That independence is further extended to the form of $V(x)$, as long as the potential bears a stationary solution and is differentiable [22].For the harmonic optical tweezer potential, the computation of the cumulants $\langle x^n\,v^m\rangle_c$ by means of Eq. (14) gives the generating function of,

$$p\,(x, v) = \frac{1}{Z}\exp\left[-\mathcal{B}_v\,v^2 - \mathcal{B}_x\,x^2\right], \quad (15)$$

where, for a ExtRes system,

$$\mathcal{B}_v = \frac{1}{2}\frac{m + \tau\,(\gamma + k\,\tau)}{T}, \qquad \mathcal{B}_x = \frac{k}{2}\frac{m + \tau\,(\gamma + k\,\tau)}{T\,(m + \gamma\,\tau)}, \quad (16)$$

and for a IntRes system,

$$\mathcal{B}_v = \frac{1}{2}\frac{m}{T}, \qquad \mathcal{B}_x = \frac{1}{2}\frac{k}{T}. \quad (17)$$

by making $\tau \to 0$. Both steady states are Gaussian and the average energy equals ($\mathcal{E}_{\mathrm{IntRes}} = \lim_{\tau \to 0} \mathcal{E}$),

$$\mathcal{E} \equiv \mathcal{K} + \mathcal{V} = \frac{1}{2}\frac{2m+\tau\,\gamma}{[m+\tau\,(\gamma+k\,\tau)]}T, \tag{18}$$

Concretely, although Eqs. (2) and (6) represent different dynamics, we recognise the steady state distribution of the ExtRes system as Gaussian; the same distribution as the well-known stationary solution to the Fokker-Planck Equation of a particle subject to an IntRes.[3]

To characterise a system from a thermostatistical perspective, one is not only interested in the steady state distributions, but in the heat and work fluxes as well. However, for ExtRes systems the formulae thereof are difficult to compute. Bearing in mind the functional equivalence of the steady state solutions, Eq. (15), for both kinds of reservoirs one is tempted to ask the following question: *In order to circumvent the natural analytical difficulties of an ExtRes system is it possible to represent it in the form of an IntRes system, for which calculations are simpler, and therefrom reverse the thermostatistical results to the original case*?

The answer to that question is the aim of the present work, which was fuelled by other papers on external noise [13], including surveys over the thermodynamical consequences of the nature of a reservoir [14, 15], namely the approachability to absolute zero. Moreover, our endeavour fits what van Kampen coined as "the Langevin approach" [1] to a system, where the goal is to recast a system in a standard Langevin Equation reproducing drift and diffusion properties. This approach was also conjectured for multiplicative noise cases or to rephrase non-stationary into stationary models [16].

## 2. Hypothesis testing

On the one hand, let us assume that the ExtRes system with parameters $m$, $T$, $\gamma$, $\tau$ and $k$, Eq. (6), has its statistics replicated by IntRes system, Eq. (2), with proxy thermomechanical parameters.[4] From the steady state averages values of the total, kinetic and potential energies, $\mathcal{E}$, $\mathcal{K}$ and $\mathcal{V}$,

$$\begin{cases} \langle v^2 \rangle = & \frac{T}{m+\tau(\gamma+k\,\tau)} & = & \frac{T^*}{m^*} \\ \langle x^2 \rangle = & \frac{T\,(m+\tau\,\gamma)}{k\,[m+\tau(\gamma+k\,\tau)]} & = & \frac{T^*}{k^*} \\ \mathcal{E} = & \frac{2m+\tau\,\gamma}{2[m+\tau(\gamma+k\,\tau)]}T & = & T^* \end{cases}, \tag{19}$$

we get the following proxy-parameters,

$$m^* \leftarrow m + \frac{\gamma\,\tau}{2}, \qquad T^* \leftarrow \frac{m^*}{m+\tau\,(\gamma+k\,\tau)}T, \qquad k^* \leftarrow \frac{m^*}{m+\tau\gamma}k, \qquad \gamma^* \leftarrow \gamma. \tag{20}$$

The relations in Eq. (20) state that the proxy IntRes analogue system must be colder and the particle heavier. Note that these quantitative relations are not universal, [5] but

[3] That difference in the dynamics can be appraised in the long-term velocity covariance for the two cases that we present in Appendix B.

[4] Hereinafter, we will refer to this system as the IntRes proxy system and its parameters and quantities marked with an asterisk.

[5] And there is no reason whatsoever to be like that.

the qualitative relation between real and mapped parameters are bound to go like that for every confining potential case. From Eqs. (2), (15) and (16), we confirm that the Gaussianity of the steady state is preserved, the moments of the steady state (kinetic and potential) energies are unaltered and hence the mapping (20) is deemed significant from the time independent point of view.

On the other hand, the energy, namely that of the steady state, $\mathcal{E}$, ensues from the superposition of the total injected, $J_{\text{inj}}(\Xi)$, and dissipated, $J_{\text{dis}}(\Xi)$, fluxes (up to time $\Xi$),

$$\mathcal{E} \equiv \lim_{\Xi \gg \frac{m}{\gamma}} J_{\text{inj}}(\Xi) + J_{\text{dis}}(\Xi) . \tag{21}$$

For both IntRes and ExtRes cases, $J_{\text{dis}}(\Xi)$ flows in the form of heat whereas $J_{\text{inj}}(\Xi)$ is the total work made by the ExtRes on the system when it corresponds to a work reservoir. After the transient, we have up to time $\Xi$ (see Appendix C),

$$\mathcal{E} \equiv \lim_{\Xi \gg \frac{m}{\gamma}} \int_0^{\Xi} \langle v(t)\ \xi(t) \rangle\ dt - \gamma \int_0^{\Xi} \left\langle [v(t)]^2 \right\rangle\ dt. \tag{22}$$

For the ExtRes system, we obtain,

$$\lim_{\Xi \gg \frac{m}{\gamma}} \langle J_{\text{inj}}(\Xi) \rangle = -\frac{T\,\gamma\,\tau\,(m - k\,\tau^2)}{[m + \tau\,(\gamma + k\,\tau)]^2} + \frac{\gamma\,T}{m + \tau\,(\gamma + k\,\tau)} \Xi, \tag{23}$$

and,

$$\lim_{\Xi \gg \frac{m}{\gamma}} \langle J_{\text{dis}}(\Xi) \rangle = \frac{T\,[m\,(2m + 5\gamma\,\tau + 2k\,m\,\tau^2) + \tau^2\,(\gamma^2 - k\,\gamma\,\tau)]}{2\,[m + \tau\,(\gamma + k\,\tau)]^2} - \frac{\gamma\,T}{m + \tau\,(\gamma + k\,\tau)} \Xi. \tag{24}$$

The time independent energy $\mathcal{E}$ [Eq. (21)] matches the steady state value given in Eq. (18).

For an IntRes system, namely one that is playing the role of proxy for an ExtRev situation, we replace the coloured $\xi(t)$ noise by a white noise $\eta^*(t)$ and from Eqs. (21) and (22) we have,

$$\mathcal{E} = \frac{\gamma\,T^*}{m^*}\Xi + \left( T^* - \frac{\gamma\,T^*}{m^*}\Xi \right). \tag{25}$$

As also previously suggested in [17], Equations (8) and (10) clearly show that the position depends on the nature of the fluctuations and so do the velocity and the time dependent fluxes, or its assymptotic limit,

$$\mathcal{J}(\Xi) \equiv \Xi \lim_{\Xi \to \infty} \frac{1}{\Xi} J(\Xi) . \tag{26}$$

But, what is the actual extent of this dependence?

At first glance, due to the different dynamical features imposed by $\eta^*$ and $\xi$, one expects a utterly different statistics of the time dependent fluxes for IntRes and ExtRes systems, even for a proxy system, that must reflect its internal character. In other words, in applying the mapping relations to the fluxes in the ExtRes case and taking into consideration the time dependencies we have just mentioned, the best guess is a relation like,

$$\langle \mathcal{J}(\Xi) \rangle = \langle \mathcal{J}(\Xi) \rangle^* + \varphi\,(m^*, \gamma^*, k^*, T^*)\ \tau\,\Xi. \tag{27}$$

However, when we apply the mapping relations provided by Eq. (20) in Eqs. (23) and (24) to assess the factual difference between the fluxes of an ExtRes and its proxy IntRes — *i.e.*, $\varphi$ — we simply get,

$$\varphi = 0, \tag{28}$$

and thus,

$$\langle \mathcal{J}\,(\Xi)\rangle = \langle \mathcal{J}\,(\Xi)\rangle^{*} \qquad \text{(after scaling)}\,. \tag{29}$$

Moreover the value of $\mathcal{E}$ is preserved. According to what we have said in the previous paragraph this equivalence is against all odds and counter-intuitive.

From Eqs. (21) and (22), we identify the fluxes as quantities proportional to the accumulated fluctuations, *i.e.*, they behave in the form of a large deviation. Putting together this feature with the mapping equivalence Eq. (29), we understand that, under the conditions herein presented, the average value of the large deviation of the fluctuations is in practice insensitive to the nature of the reservoir. In Fig. 1 (left-hand panel), we show the evolution of $J_{\mathrm{inj}}\,(\Xi)$ and $J_{\mathrm{dis}}\,(\Xi)$ for the ExtRes system [Eq. (6)], which agrees with the behaviour of its proxy IntRes problem defined by Eqs. (2) and (20).

At this point, we have found a reason for further checking the true impact of the nature of the reservoir in the fluxes that guarantee the steady state. That is particularly important for systems that are far from the thermodynamical limit and for which the fluctuations of its thermostatistical quantities are crucial to a proper characterisation. With that goal in mind, we can look at the large deviation function (LDF) of the fluxes, $L\,(\mathcal{J})$, wherefrom the respective cumulants can be obtained. In accordance, should the mapping between reservoirs be fully valid, the LDF of the proxy IntRes case will read [18, 19],

$$L^{*}\,(\mathcal{J}) = \frac{1}{Z_L}\exp\left[-\frac{\left(\mathcal{J}-\frac{\gamma\,T^{*}}{m^{*}}\Xi\right)^{2}}{4\,T^{*}\,\mathcal{J}}\right]\Theta\,[\mathcal{J}]\,, \tag{30}$$

for either (injected/dissipated) fluxes.

More than concerned about the form of $L\,(\mathcal{J})$, we set our sight at the cumulants. This means that instead of employing standard methods to find the LDF [21], we have opted to successively compute the cumulants in a way that: *i)* dispenses with the hard task of defining the propagator, *ii)* provides exact results, *iii)* allows obtaining results when one has to deal with non-Gaussian (shot-noise) reservoirs [19]. That being said, we get for the proxy IntRes case the second order cumulant,

$$\left\langle \mathcal{J}^{2}\,(\Xi)\right\rangle_{c}^{*} = 2\frac{\gamma\,T^{*2}}{m^{*}}\Xi = \frac{\gamma\,T^{2}\,\left(2\,m+\gamma\,\tau\right)}{\left(m+\tau\,(\gamma+k\,\tau)\right)^{2}}\Xi, \tag{31}$$

whereas for the original ExtRes problem (see Appendix C),

$$\left\langle \mathcal{J}^{2}\,(\Xi)\right\rangle_{c} = \frac{\gamma\,T^{2}\,\left(2\,m+\gamma\,\tau\right)}{\left(m+\tau\,(\gamma+k\,\tau)\right)^{2}}\Xi + \frac{\gamma^{2}\,T^{2}\,\left(3\,m+\tau\,(\gamma-k\,\tau)\right)}{\left(m+\tau\,(\gamma+k\,\tau)\right)^{3}}\tau\,\Xi \tag{32}$$

which are slightly different and unmappable. Qualitatively, the form of the relation between the variance of the fluxes Eqs. (31) and (32),

$$\left\langle \mathcal{J}^{\,2}\,(\Xi)\right\rangle = \left\langle \mathcal{J}^{\,2}\,(\Xi)\right\rangle^{*} + \varphi_{2}\,(m^{*},\gamma^{*},k^{*},T^{*})\;\tau\,\Xi \tag{33}$$

is actually the relation we were already expecting for the average fluxes, a conjecture that our calculations surprisingly proved wrong.

One might attempt to recast Eq. (20) in order to accommodate the fluctuations of $\mathcal{J}(\Xi)$ — the fluctuations of the accumulated fluctuations — but all of them end up hitting the buffers because they yield a wrong value of $\mathcal{E}$. From these efforts, we learn the key issue amounts to the fact that the parameter $\gamma$, which establishes the dissipation in the system, is always invariant under mapping whatever the conditions we try to impose to obtain it.

Besides the hypothesis we have probed, the mapping hypothesis — or "Langevin approach" [1] — furnishes a by-product for the treatment of LDFs of coloured ExtRev systems. Specifically, as previously stated, solutions to time-dependent probabilistc problems within a ExtRev context are hard to obtain and generally achieved by approximative methods [7]. In these circumstances, the white noise limit of a coloured ExtRes system[6] is systematically assumed as the preferable zeroth order approximation to which higher-order corrections are added. Our analysis points to a better proposal though: the proxy IntRes system. For instance, in adopting an Edgeworth expansion to the LDF of the fluxes,[7]

$$L_{\mathrm{ExtRes}}(\mathcal{J}) = \exp\left[\sum_{n=1}^{\infty} \Delta_{\langle J^n\rangle_c} \frac{1}{n!}\frac{\partial^n}{\partial \mathcal{J}^n}\right] L_0(\mathcal{J}), \tag{34}$$

the best(neatest) reference distribution, $L_0(\mathcal{J})$, is provided by Eq. (30) instead of its "unstarred" version directly derived from Eq. (2); best because it already includes ExtRes colour effects.

The dashed line in Fig. 1 (right-hand panel) represents the calculation of Eq. (34) up to $n = 2$ (equivalent results for $n = 3$ are presented in Appendix C). Note that the approximation involving the second order cumulant is in fact first order since the reference distribution naturally embodies the colour effects of the reservoir on the average.

## 3. Concluding Remarks

Stimulated by previous works on systems in contact with external reservoirs — *i.e.*, reservoirs for which dissipation and fluctuations have different origins and do not abide by the fluctuation-dissipation relation — we have assessed the hypothesis of having the long-term thermostatistical behaviour of a coloured ExtRes system represented by a proxy IntRes system, after a convenient scaling of its thermomechanical parameters [Eq. (20) for our case study]. That hypothesis fits van Kampen's "Langevin approach"to a stochastic system. *At first*, because of the Boltzmann weight nature of the IntRes and ExtRes steady state distributions, the *mathematical* mapping is deemed plausible for time independent quantities. Moreover, when we analysed key time dependent quantities like the average injected and dissipated fluxes[8] — that are the large deviation of powers and for which

[6] This is equivalent to a proper IntRes system governed by Eq. (2).

[7] $\Delta_{\langle J^n\rangle_c}$ is the difference between the $n^{th}$-order cumulant of the actual $L_{\mathrm{ExtRes}}(J)$ and the reference $L_0(J)$ LDFs.

[8] That guarantee the existence of the steady state.

the time dependent nature of the fluctuations [Eq. (4)] comes explicitly into play — we verified that the establishment of a proxy IntRes system could yet be possible.

Only when we look at the fluctuations of the fluxes — which are in practice the fluctuations of the accumulated fluctuations — the fluctuation-dissipation relation speaks louder and the functional equivalence scenario breaks down. Although these facts could be understood as a signature of a certain insensitiveness of the large deviation treatment to the thermostatistical properties of a system, what the present result conveys is that we can always rephrase the first order thermal properties of a non-Markovian Gaussian system similarly to what we can do with a Markovian non-Gaussian IntRes system like a massive particle in contact to an Anderson thermostat [12]. [9]

Complementarily, the present analysis gave us arguments to claim that in a standard approximative treatment of an ExtRes system, the best zeroth-order is not its white-noise limit, but its proxy IntRes since it already includes colour effects in the form of the proxy temperature, $T^*$ and mass, $m^*$.

Still taking into consideration our mapping, we can look at the we look at the entropy production/exchange that is related to the dissipated and injected fluxes. If the system attains a steady state, then the relation between the total entropy, $S$, the entropy production, $\Pi$, and entropy exchange, $\Psi$, is,

$$\frac{dS}{dt} = \Pi - \Psi = 0 \qquad\qquad (t \gg m/\gamma) \, .$$

where $\Pi$ and $\Psi$ are associated with $\mathcal{J}_{\mathrm{inj}}$ and $\mathcal{J}_{\mathrm{dis}}$, respectively. In other words, by assuming our mapping hypothesis and an effective temperature $T^*$ for our system, we can easily verify that $\Pi = \Psi = \gamma/m^*$, which is exactly the same form obtained for thermal (internal) reservoirs, and assert that $T^*$ is an actually worth energy scale that retrieves the standard form of long-term entropy production/exchange.

As a final side note, yet in a LDF context, if one reminds that the Jarzynski equality [23], *for long trajectories*, is nothing but the LDF of the power input to the system by some external force, our analysis provides a different interpretation as to why that equality fails for external reservoirs [24]. Putting it differently, one cannot circumvent the umbilical link between the dissipation and the fluctuations of an IntRes system so that the generating function of the work made by the driving a system from steady state A to B is always equal to the ratio between the partition functions of those states, whatever the reservoir.

## Acknowledgement

We thank W. A. M. Morgado for several discussions on this subject matter. Financial support from CAPES (JMR) is acknowledged as well as FAPERJ [APQ1-110.635/2014] and CNPq [308737/2013-0] (SMDQ).

[9] It is worth reminding that LDF of the fluxes is able to preserve the information on the initial conditions of the system, namely the initial temperature [18], ergo it is a thermostatically sensitive function.

## Appendix A. Poles location in complex plane

For the calculations expressed in the main text (MT) we used the poles structure shown in Figure A.2. To avoid issues related to the Cauchy's principal value we shifted the moments $q$ in Eqs. (8),(9) and (10) written in the MT by $\mathrm{i}\,\varepsilon$, which is then made going to zero at the end of all calculations. Hence, those formulae become,

$$\tilde{x}\,(\mathrm{i}\,q+\varepsilon)=\frac{\tilde{\eta}\,(\mathrm{i}\,q+\varepsilon)}{R\,(\mathrm{i}\,q+\varepsilon)},\qquad \tilde{x}\,(\mathrm{i}\,q+\varepsilon)=\frac{\tilde{\xi}\,(\mathrm{i}\,q+\varepsilon)}{R\,(\mathrm{i}\,q+\varepsilon)} \tag{A.1}$$

and,

$$\begin{cases} \langle\tilde{\eta}\,(\mathrm{i}\,q_1+\varepsilon)\;\tilde{\eta}\,(\mathrm{i}\,q_2+\varepsilon)\rangle &= \frac{1}{\mathrm{i}\,q_1+\mathrm{i}\,q_2+2\,\varepsilon}2\,\gamma\,T \\ \left\langle\tilde{\xi}\,(\mathrm{i}\,q_1+\varepsilon)\;\tilde{\xi}\,(\mathrm{i}\,q_2+\varepsilon)\right\rangle &= \frac{2+\mathrm{i}\,q_1\,\tau+\mathrm{i}\,q_2\,\tau+2\,\varepsilon\,\tau}{(\mathrm{i}\,q_1+\mathrm{i}\,q_2+2\,\varepsilon)[1+(\mathrm{i}\,q_1+\varepsilon)\,\tau][1+(\mathrm{i}\,q_2+\varepsilon)\,\tau]}\gamma\,T \end{cases}. \tag{A.2}$$

## Appendix B. Calculation of velocity-velocity correlations

The long-term covariance of the velocity is defined as,

$$C_v(s)\equiv\lim_{\Xi\to\infty}\frac{1}{\Xi}\int_0^{\Xi}\langle v\,(t)\;v\,(t+s)\rangle\;dt. \tag{B.1}$$

Using the Fourier-Laplace representation, the external reservoir system velocity covariance, $C_v\,(s)$, reads,

$$C_v\,(s)=\lim_{\Xi\to\infty}\frac{1}{t}\int_0^{\Xi}\int dt\,\mathrm{e}^{(\mathrm{i}\,q_2+\varepsilon)\,s}\prod_{j=1}^{2}\frac{dq_j}{2\pi}\frac{(\mathrm{i}\,q_j+\varepsilon)\,e^{(\mathrm{i}\,q_1+\varepsilon)\,t}}{m\,R\,(\mathrm{i}\,q_j+\varepsilon)}\left\langle\prod_{l=1}^{2}\tilde{\xi}\,(\mathrm{i}\,q_l+\varepsilon)\right\rangle. \tag{B.2}$$

Using the poles structure depicted in Figure A.2 and after some algebra we get,

$$C_v=\mathcal{T}\left\{e^{-\frac{\gamma}{2m}|s|}\left[\left(m+k\,\tau^2\right)\cos\left(s\frac{\Omega}{2}\right)-\frac{\gamma\,\left(m-k\,\tau^2\right)}{m^2\,\Omega}\sin\left(s\frac{\Omega}{2}\right)\right]-e^{-\frac{|s|}{\tau}}\gamma\,\tau\right\},$$

where,

$$\mathcal{T}=\frac{T}{\prod_{l=1}^{2}\left(m+(-1)^l\,\gamma\,\tau+k\,\tau^2\right)},\qquad \Omega\equiv\sqrt{4\frac{k}{m}-\frac{\gamma^2}{m^2}}.$$

The same happens for the internal reservoir system, where the equation is basically the same as Eq. (B.2), *i.e.*,

$$C_v\,(s)=\lim_{\Xi\to\infty}\frac{1}{\Xi}\int_0^{\Xi}\int dt\,\mathrm{e}^{(\mathrm{i}\,q_2+\varepsilon)\,s}\prod_{j=1}^{2}\frac{dq_j}{2\pi}\frac{(\mathrm{i}\,q_j+\varepsilon)\,\mathrm{e}^{(\mathrm{i}\,q_1+\varepsilon)\,t}}{m\,R\,(\mathrm{i}\,q_j+\varepsilon)}\left\langle\prod_{l=1}^{2}\tilde{\eta}\,(\mathrm{i}\,q_l+\varepsilon)\right\rangle. \tag{B.3}$$

Because the noise is actually different the results of the integration is different and yields,

$$C_{v,\,\tau=0}=e^{-\frac{\gamma}{2m}|s|}T\,\left[\frac{1}{m}\cos\left(s\frac{\Omega}{2}\right)-\frac{\gamma}{m\,\Omega}\sin\left(s\frac{\Omega}{2}\right)\right]. \tag{B.4}$$

## Appendix C. Calculation of the moments of the injected/dissipated fluxes

Hereinafter, we only present the calculations for the external reservoir system. For the specific calculations of the internal reservoir system we refer the reader to Ref. [19].

### Appendix C.1. Averages

The average injected flux, $\langle \mathcal{J}_{\rm inj}(\Xi)\rangle$, was obtained from the computation of,

$$\langle \mathcal{J}_{\rm inj}(\Xi)\rangle = \Xi \lim_{\Xi\to\infty}\frac{1}{\Xi}\int_0^{\Xi}\int\prod_{j=1}^{2}\frac{dq_j}{2\pi}\mathrm{e}^{(\mathrm{i}\,q_j+\varepsilon)\,t}\frac{\mathrm{i}\,q_1+\varepsilon}{m\,R(\mathrm{i}\,q_1+\varepsilon)}\left\langle\prod_{l=1}^{2}\tilde{\xi}(\mathrm{i}\,q_l+\varepsilon)\right\rangle dt. \quad (C.1)$$

In this case, we had to pay attention to the fact that the integral in $q_2$ does not suit the direct application of Jordan's lemma. Because of that, the value of the integration over the upper arch, which is finite, must be subtracted from result obtained from the calculation of the residue. Taking this detail into consideration we got Eq. (23).

The average dissipated flux was obtained from,

$$\langle \mathcal{J}_{\rm dis}(\Xi)\rangle = -\gamma\,\Xi \lim_{\Xi\to\infty}\frac{1}{\Xi}\int_0^{\Xi}\int\prod_{j=1}^{2}\frac{dq_j}{2\pi}\frac{\mathrm{i}\,q_j+\varepsilon}{m\,R(\mathrm{i}\,q_j+\varepsilon)}\mathrm{e}^{(\mathrm{i}\,q_j+\varepsilon)\,t}\left\langle\prod_{l=1}^{2}\tilde{\xi}(\mathrm{i}\,q_l+\varepsilon)\right\rangle dt. \quad (C.2)$$

In the dissipative case there is no problem with Jordan's lemma and Eq. (24) is obtained after some algebra.

### Appendix C.2. Second order cumulant

The variances of both cases are obtained calculating,

$$\begin{aligned}\left\langle \mathcal{J}^2_{\rm inj}(\Xi)\right\rangle_c &= \Xi \lim_{\Xi\to\infty}\frac{1}{\Xi}\int_0^{\Xi}\int\prod_{j=1}^{2}dt_j\frac{dq_{2j-1}}{2\pi}\frac{dq_{2j}}{2\pi}\frac{(\mathrm{i}\,q_{2\,j}+\varepsilon)}{m^2R(\mathrm{i}\,q_{2\,j}+\varepsilon)}\\ &\quad\times\mathrm{e}^{(\mathrm{i}\,q_{2j-1}+\mathrm{i}\,q_{2j}+2\,\varepsilon)\,t_j}\left\langle\prod_{l=1}^{4}\tilde{\xi}(\mathrm{i}\,q_l+\varepsilon)\right\rangle-\langle\mathcal{J}(\Xi)\rangle^2, \end{aligned} \quad (C.3)$$

and

$$\begin{aligned}\left\langle \mathcal{J}^2_{\rm dis}(\Xi)\right\rangle_c &= \Xi \lim_{\Xi\to\infty}\frac{1}{\Xi}\int_0^{\Xi}\int\prod_{j=1}^{2}dt_j\frac{dq_{2j-1}}{2\pi}\frac{dq_{2j}}{2\pi}\frac{(\mathrm{i}\,q_{2\,j-1}+\varepsilon)(\mathrm{i}\,q_{2\,j}+\varepsilon)}{m^4R(\mathrm{i}\,q_{2\,j-1}+\varepsilon)\;R(\mathrm{i}\,q_{2\,j}+\varepsilon)}\\ &\quad\times\mathrm{e}^{(\mathrm{i}\,q_{2j-1}+\mathrm{i}\,q_{2j}+2\,\varepsilon)\,t_j}\left\langle\prod_{l=1}^{4}\tilde{\xi}(\mathrm{i}\,q_l+\varepsilon)\right\rangle-\langle\mathcal{J}(\Xi)\rangle^2, \end{aligned} \quad (C.4)$$

respectively.

Since the external coloured noise is Gaussian, we can dynamically define it as [22],

$$\xi(t)\propto\int_{-\infty}^{t}\exp\left[-\frac{1}{\tau}(t-t')\right]dW_{t'}, \quad (C.5)$$

where $W_t$ is the standard Wiener process. Thus, it is not difficult to verify that one can apply the Esserlis-Wick theorem to break the $n^{th}$-order moments of $dW$ into sums over all combinations of products of pairs $\langle dW\, dW' \rangle$. It must be emphasised that for the variance of the total injected flux the split into sub-terms must also take into consideration whether $\tilde{\xi}\left(\mathrm{i}\, q_l + \varepsilon\right)$ comes from the velocity or else directly represents the effect of the external reservoir on the system.

The successive computation of all the terms finally yield Eq. (32),

$$\left\langle \mathcal{J}^2 \left(\Xi\right) \right\rangle_c = \frac{\gamma\, T^2 \, \left(2\, m + \gamma\, \tau\right)}{\left[m + \tau\left(\gamma + k\,\tau\right)\right]^2}\Xi + \frac{\gamma^2\, T^2 \, \left[3\, m + \tau\left(\gamma - k\,\tau\right)\right]}{\left[m + \tau\left(\gamma + k\,\tau\right)\right]^3}\tau\,\Xi. \tag{C.6}$$

*Appendix C.3. Third order cumulant*

In order to bring our analysis farther afield and to strengthen our assertion on the superiority of the proxy IntRes approach in comparison to the white noise limit approach, we computed the third order cumulant. Since the injected and dissipated LDFs are equal, we centered our efforts in the (easier) injected case,

$$\begin{aligned}\left\langle \mathcal{J}_{\mathrm{inj}}^3 \left(\Xi\right) \right\rangle_c \;=\;& \Xi \lim_{\Xi\to\infty} \frac{1}{\Xi} \int_0^{\Xi} \int \prod_{j=1}^{3} dt_j \frac{dq_{2j-1}}{2\pi}\frac{dq_{2j}}{2\pi} \frac{\left(\mathrm{i}\, q_{2\,j} + \varepsilon\right)\, e^{\left(\mathrm{i}\, q_{2j-1} + \mathrm{i}\, q_{2j} + 2\,\varepsilon\right)\, t_j}}{m^2 R\left(\mathrm{i}\, q_{2\,j} + \varepsilon\right)} \left\langle \prod_{l=1}^{6} \tilde{\xi}\left(\mathrm{i}\, q_l + \varepsilon\right) \right\rangle \\ & -3 \left\langle \mathcal{J}_{\mathrm{inj}}^2 \left(\Xi\right) \right\rangle_c \left\langle \mathcal{J}\left(\Xi\right)\right\rangle - \left\langle \mathcal{J}\left(\Xi\right)\right\rangle^3 . \end{aligned} \tag{C.7}$$

Following Ref. [19], we understood that only the closed diagrams $\mathsf{G}$ and $\mathsf{H}$, emerging from the first term on the right-hand side, have a non-zero contribution to the cumulant. The former is defined by the pairs of $\tilde{\xi}$s coming from different times $t_j$ and associated with different quantities (velocity and fluctuations), *e.g.*, $\left\langle \tilde{\xi}\left(\mathrm{i}\, q_1 + \varepsilon\right)\, \tilde{\xi}\left(\mathrm{i}\, q_4 + \varepsilon\right) \right\rangle \left\langle \tilde{\xi}\left(\mathrm{i}\, q_3 + \varepsilon\right)\, \tilde{\xi}\left(\mathrm{i}\, q_6 + \varepsilon\right) \right\rangle \Big\langle \tilde{\xi}$
which zeroes out in the IntRes system case. The latter comes from pairs of $\tilde{\xi}$s from different times $t_j$, but the pairings are different: one pair of $\tilde{\xi}$s from different quantities and the other two pairs composed of $\tilde{\xi}$s from the same quantity, *i.e.*, noise-noise and velocity-velocity, *e.g.*,
$\left\langle \tilde{\xi}\left(\mathrm{i}\, q_1 + \varepsilon\right)\, \tilde{\xi}\left(\mathrm{i}\, q_4 + \varepsilon\right) \right\rangle \left\langle \tilde{\xi}\left(\mathrm{i}\, q_3 + \varepsilon\right)\, \tilde{\xi}\left(\mathrm{i}\, q_5 + \varepsilon\right) \right\rangle \left\langle \tilde{\xi}\left(\mathrm{i}\, q_2 + \varepsilon\right)\, \tilde{\xi}\left(\mathrm{i}\, q_6 + \varepsilon\right) \right\rangle$. The end result is,

$$\begin{aligned}\left\langle \mathcal{J}^{\,3} \left(\Xi\right) \right\rangle_c \;=\;& 3\frac{\gamma\, T^3 \, \left(2\, m + \gamma\, \tau\right)^2}{\left[m + \tau\left(\gamma + k\,\tau\right)\right]^3}\Xi + 3\frac{\gamma^2\, T^3 \, \left(8\, m^2 + m\,\tau\left(7\,\gamma - 4\,k\,\tau\right) + \gamma\,\tau^2\left(2\gamma - k\,\tau\right)\right)}{\left[m + \tau\left(\gamma + k\,\tau\right)\right]^4}\tau\,\Xi \\ & + 3\frac{\gamma^3\, T^3 \, \left(8\, m^2 + m\,\tau\left(5\,\gamma - 8\,k\,\tau\right) + \gamma\,\tau^2\left(\gamma - 3\,k\,\tau\right)\right)}{\left[m + \tau\left(\gamma + k\,\tau\right)\right]^5}\tau^2\,\Xi. \end{aligned} \tag{C.8}$$

The first term on the right-hand side is exactly the internal reservoir proxy result before the mapping relation. Applying Eq. (20),

$$\left\langle \mathcal{J}^3 \left(t\right) \right\rangle_c^* = 12\frac{\gamma}{m^*} T^{*\,3}\,\Xi. \tag{C.9}$$

that clearly isolate the cumulant corrections,

$$\left\langle \mathcal{J}^{\,3} \left(\Xi\right) \right\rangle_c = \left\langle \mathcal{J}^3 \left(t\right) \right\rangle_c^* + \varphi_{31}\left(m^*, \gamma^*, k^*, T^*\right)\,\tau\,\Xi + \varphi_{32}\left(m^*, \gamma^*, k^*, T^*\right)\,\tau^2\,\Xi. \tag{C.10}$$

## Appendix D. Edgeworth expansion

We write the Large Deviation Function of the injected and dissipated fluxes $J_{\mathrm{inj(dis)}}$ for the external reservoir system using the Edgeworth expansion. In the MT have shown that the mapping hypothesis to the internal reservoir proxy fails at the second order cumulant of the fluxes or as we mentioned in the MT the fluctuations of the accumualted fluctuations. Having chosen the reference distribution given by Eq. (30) one uses the Edgeworth expansion to approximate the real "coloured" external reservoir system $L(\mathcal{J})$,

$$L(\mathcal{J}) = \exp\left[\sum_{n=1}^{\infty}(\langle \mathcal{J}^{*n}\rangle_c - \langle \mathcal{J}^n\rangle_c)\frac{\hat{D}^n}{n!}\right] L^*(\mathcal{J}) \tag{D.1}$$

where $\hat{D}$ is the differential operator of order $n$, $\langle \mathcal{J}^n\rangle$ and $\langle \mathcal{J}^{*n}\rangle$ are the $n-^{th}$ cumulant of the actual and reference (zero-th order) LDFs, respectively.

The first cumulant of both external and internal reservoir cases coincide as was previously pointed out, so our first correction to the LDF is the difference in the second cumulants

$$\langle \mathcal{J}^2(\Xi)\rangle_c^* = \frac{\gamma T^2(2m+\gamma\tau)}{(m+\tau(\gamma+k\tau))^2}\Xi, \tag{D.2}$$

and

$$\langle \mathcal{J}^2(\Xi)\rangle_c = \frac{\gamma T^2\left(2m+\gamma\tau\right)}{\left[m+\tau\left(\gamma+k\tau\right)\right]^2}\Xi + \frac{\gamma^2 T^2\left[2m+\tau\left(\gamma-k\tau\right)\right]}{\left[m+\tau\left(\gamma+k\tau\right)\right]^3}\tau\Xi$$

We limit ourselves to this one first contribution truncating the sum in the exponent to the second order, if we expand the exponential up to first order we get our Edgeworth approximation,

$$\begin{aligned} L(\mathcal{J}) &\approx \exp\left[\frac{\gamma^2T^2(2m+\tau(\gamma-k\tau))}{(m+\tau(\gamma+k\tau))^3}\tau\Xi\frac{\hat{D}^2}{2!}+\dots\right]L^*(\mathcal{J}) \\ &\approx \left(1+\frac{\gamma^2T^2\left[2m+\tau\left(\gamma-k\tau\right)\right]}{\left[m+\tau\left(\gamma+k\tau\right)\right]^3}\tau\Xi\frac{\hat{D}^2}{2}\right)L^*(\mathcal{J}), \end{aligned} \tag{D.3}$$

which yields,

$$\begin{aligned} L(\mathcal{J}) \approx\; & \frac{1}{Z_l}\, e^{-\frac{(\Xi T\gamma-\mathcal{J}(m+\tau(\gamma+k\tau)))^2}{2\mathcal{J}T(2m+\gamma\tau)(m+\tau(\gamma+k\tau))}}\left[1+\frac{1}{(4\mathcal{J}^4(2m+\gamma\tau)^2(m+\tau(\gamma+k\tau))^5)}\right. \\ & \times(\Xi\gamma^2\tau(3m+\tau(\gamma+k\tau))(\Xi^4T^4\gamma^4-4\mathcal{J}\Xi^2T^3\gamma^2(2m+\gamma\tau)(m+\tau(\gamma+k\tau))- \\ & \left.-2\mathcal{J}^2\Xi^2T^2\gamma^2(m+\tau(\gamma+k\tau))^2+\mathcal{J}^4(m+\tau(\gamma+k\tau))^4))\right]. \end{aligned} \tag{D.4}$$

With respect to the third order, we have the difference of the cumulants given by Eq. (C.8) minus Eq. (C.9).


[1] N.G. van Kampen, Stochastic Processes in Physics and Chemestry, Elsevier, Amsterdam, 2007.

[2] C.W. Gardiner, Handbook of Stochastic Methods for Physics, Chemestry and Natural Sciences, Springer-Verlag, Berlin, 1997.

[3] T. Chou, K. Mallick, R.K.P. Zia, Rep. Prog. Phys. 74 (2011) 116601; U. Seifert, Rep. Prog. Phys. 75 (2012) 126001; R. Klages, W. Just, C. Jarzynski (Eds.), Nonequilibrium Statistical Physics of Small Systems: Fluctuation Relations and Beyond, Weinheim, Wiley-VCH Verlag, 2013.

[4] H. Nyquist, Phys. Rev. 32 (1928) 110; L. Onsager, Phys. Rev. 37 (1931) 405; R. Kubo, Rep. Prog. Phys. 29 1966 255; U. Marini, B. Marconi, A. Puglisi, L. Rondoni, A. Vulpiani, Phys. Rep. 461 (2008) 111.

[5] H. Mori, Prog. Theo. Phys. 33 (1965) 423; R. Kubo (Ed.), 1965 Tokyo Summer Lectures in Theoretical Physics, New York: Benjamin, 1966.

[6] M.I. Dykman, Phys. Rev. A 42 (1990) 2020.

[7] Several examples can be found in: F. Moss, P.V.E. McClintock (Eds.), Noise in Nonlinear Dynamical Systems (Cambridge, Cambridge University Press, 2009).

[8] D.O. Soares-Pinto, W.A.M. Morgado, Physica A 365 (2006) 289.

[9] P. Lett, R. Short, L. Mandel, Phys. Rev. Lett. 52 (1984) 341; S. Zhu, A.W. Yu, R. Roy, Phys. Rev. A 34 (1986) 4333; R.F. Fox and R. Roy, Phys. Rev. A 35 (1987) 1838; W. Yu, G.P. Agrawal, R. Roy, Opt. Lett. 12 (1987) 806.

[10] K. Vogel, H. Risken, W. Schleich, M. James, F. Moss, P.V.E. McClintock, Phys. Rev. A 35 (1987) 463; K. Vogel, Th. Leiber, H. Risken, P. Hanggi, W. Schleich, Phys. Rev. A 35 (1987) 4882.

[11] F.D. Nobre, A.M.C. Souza, E.M.F Curado, Phy. Rev. E 86 (2012) 061113; E.M.F. Curado, A.M.C. Souza, F.D. Nobre, R.F.S Andrade, Phy. Rev. E 89 (2014) 022117.

[12] W.A.M. Morgado, S.M. Duarte Queirós, D.O. Soares-Pinto, J. Stat. Mech. (2011) P06010; W.A.M. Morgado, T. Guerreiro, Physica A 391 (2012) 3816.

[13] M.I. Dykman, E. Mori, J. Ross, P.M. Hunt, J. Chem. Phys. 100 (1994) 5735; A. Fiasconaro, B. Spagnolo, Phys. Rev. E 80 (2009) 041110; A.C. Tzemos, D.P.K. Ghikas, Phys. Lett. A 377 (2013) 2307; A. Dechant, E. Lutz, Connecting Active and Passive Microrheology in Living Cells (2013) Preprint arXiv:1307.6466.

[14] A. Dechant, E. Lutz, D.A. Kessler, E. Barkai, Phys. Rev. X 4 (2014) 011022.

[15] E. Torrontegui, R. Kosloff, Phys. Rev. E 88 (2013) 032103.

[16] A.B. Kolomeisky, J. Phys. Chem. Lett. 2 (2011) 1502.

[17] J. Mehl, B. Lander, C. Bechinger, V. Blickle, U. Seifert Phys. Rev. Lett. 108 (2012) 220601

[18] J. Farago, J. Stat. Phys. 107 (2002) 781; J.S. Lee, C. Kwon, H. Park, Phys. Rev. E 87 (2013) 020104(R)

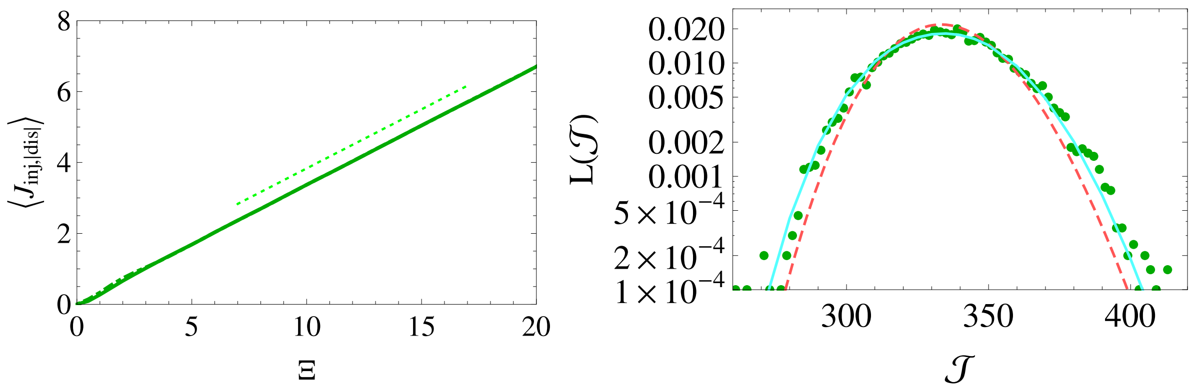


Figure 1: Numerical simulation with $m = \gamma = k = T = \tau = 1$. *Left-hand panel*: Total injected/dissipated fluxes *vs* time for the external reservoir system. The instantaneous fluxes, which are slopes of the lines after the transient, equal 1/3, the same as the dashed (light green) line. According to Eq. (20) the same behaviour is found for an internal reservoir system with $m^* = 3/2$, $T^* = 1/2$ and $k^* = 3/4$. *Right-hand panel*: Empirical LDF of the total injected flux (points) and mapped internal reservoir proxy system LDF Eq. (30) (full line) *vs* total injected flux at time $\Xi = 1000$. Although the average values (close to the peaks) concur, the widths (and also the shape) of the curves are different. In the mapped version (red dashed line), the variance goes as $\Xi/3$ [Eq. (31)], whereas the calculations show it grows as $4\,\Xi/9$ [Eq. (32)]. The result is plainly improved using Eq. (34) as depicted by the full cyan line.

Figure A.2: Location in the complex plane of the poles involved in the calculations.