Integral relations for solutions of the confluent Heun equation

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Abstract

Abstract: Firstly, we construct kernels for integral relations among solutions of the confluent Heun equation (CHE). Additional kernels are systematically generated by applying substitutions of variables. Secondly, we establish integral relations between known solutions of the CHE that are power series and solutions that are series of special functions. Thirdly, by using one of the integral relations as an integral transformation we obtain a new series solution of the ordinary spheroidal wave equation (a particular CHE). ¿From this solution we construct new series solutions of the general CHE, and show that these are suitable for solving the radial part of the two-center problem in quantum mechanics.

1. Introductory remarks

Recently we have found that the transformations of variables which preserve the form of the general Heun equation correspond to transformations which preserve the form of the equation for kernels of integral relations among solutions of the Heun equation [1]. In fact, by using the known transformations of the Heun equation [2, 3] we have found prescriptions for transforming kernels and, in this manner, we have generated several new kernels for the equation.

The above correspondence can be extended to the confluent equations of the Heun family, that is, to the (single) confluent, double-confluent, biconfluent and triconfluent Heun equations [4, 5], as well as to the reduced forms of such equations [6, 7]. In the present study we investigate only the confluent Heun equation (CHE). Specifically:

- we deal with the construction and transformations of integral kernels for the CHE;
- from some of these kernels, we establish integral relations between known solutions of the CHE;
- from another kernel, we obtain new solutions in series of confluent hypergeometric functions for the CHE; we show that these solutions are suitable to solve the radial part of the two-centre problem in quantum mechanics [8].

We write the CHE as [9]

$$z(z-z_0)\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left[B_3-2\omega\eta(z-z_0)+\omega^2z(z-z_0)\right]U=0, \qquad \text{[CHE]} \tag{1}$$

where z_0, B_i, η and ω are constants. This CHE is called *generalized spheroidal wave equation* by Leaver [9], but sometimes the last expression refers to a particular case of the CHE [5, 10]. Excepting the Mathieu equation, the CHE is the most studied of the confluent Heun equations and includes the (ordinary) spheroidal equation [5] as a particular case. Its recent occurrence in several classes of quantum two-state systems [11], certainly will require new solutions for Eq. (1). On the other side, the reduced confluent Heun equation (RCHE) is written as

$$z(z-z_0)\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left[B_3+q(z-z_0)\right]U=0, \qquad \text{[RCHE]} \tag{2}$$

where z_0, B_i and q ($q\neq 0$) are constants. The RCHE describe the angular part of the Schrödinger equation for an electron in the field of a point electric dipole [12, 13]. It appears as well in the study of two-level systems [14], polymer dynamics [15] and theory of gravitation [16]. The form (2) for the RCHE results from the CHE (1) by means of the limits

$$\omega\to 0,\quad \eta\to\infty \quad \text{such that} \quad 2\eta\omega=-q, \qquad \text{[Whittaker-Ince limit]}. \tag{3}$$

In both equations, $z=0$ and $z=z_0$ are regular singular points with exponents $(0,1+B_1/z_0)$ and $(0,1-B_2-B_1/z_0)$, respectively, that is, from ascending power series solutions we find

$$\lim_{z\to 0}U(z)\sim 1,\qquad \lim_{z\to 0}U(z)\sim z^{1+\frac{B_1}{z_0}};\qquad \lim_{z\to z_0}U(z)\sim 1,\qquad \lim_{z\to z_0}U(z)\sim (z-z_0)^{1-B_2-\frac{B_1}{z_0}}:\quad \text{[CHE, RCHE]}. \tag{4}$$

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In contrast, at the irregular singular point $z = \infty$ the behaviour of the solutions is different for each equation since

$$\lim_{z\to\infty} U(z) \sim e^{\pm i\omega z}\ z^{\mp i\eta-(B_2/2)}\ \text{ for the CHE (1)}, \qquad \lim_{z\to\infty} U(z) \sim e^{\pm 2i\sqrt{qz}}\ z^{(1/4)-(B_2/2)}\ \text{for the RCHE (2)}, \tag{5}$$

as follow from the normal and the subnormal Thomé solutions [17] for the CHE and RCHE, respectively.

By the concepts of Ref. [7], the s-rank of the singularity at $z = \infty$ is 2 for the CHE, and 3/2 for the RCHE. However, more important is the fact that the solutions exhibit the above behaviour predicted by the normal or subnormal Thomé solutions, and fact that the Whittaker-Ince limit (3) may generate solutions to the RCHE. In effect, most of the known solutions for the RCHE [18, 19, 20] has been obtained from solutions of the CHE by means of the limit (3). On account of this, and for the sake of brevity, we restrict ourselves to integral relations among solutions of the CHE. We also do not consider kernels for double-confluent Heun equations which result by taking the Leaver limit $z_0 \to 0$ of Eqs. (1) and (2).

Integral relations, in principle, enable us to transform known solutions into solutions with different properties. However, apart from the Mathieu equation, only in rare cases this task has been accomplished successfully. One case is constituted by the solutions in series of associated Legendre functions for the Lamé equation, obtained by Erdélyi in 1948 as a transformation of solutions in Fourier-Jacobi series [21]; however, as far we are aware, his solutions have not been extended for the general Heun equation (of which Lamé equation is a particular case). Another example is an expansion in series of irregular confluent hypergeometric functions for the CHE, which was obtained by Leaver in 1985 as an integral transformation of a power-series solution [9]: the transformation was constructed for $\eta = \pm i(B_2/2 - 1)$ and, then, the result was extended for arbitrary η.

To establish integral relations for solutions the CHE it is necessary to get appropriate integral kernels. To this end, in section 2 we proceed as in case of the general Heun equation [1]. In other words, firstly we insert into the integral connecting two solutions a weight function $w(z,t)$ which allows to write the CHE and the equation for its kernels in terms of differential operators functionally identical (respecting z and t). In this manner, by examining each variable substitution which leaves invariant the form of the CHE (one variable, z) we find prescriptions for the variables transformations which preserve the form of the equation for the kernels (two variables, z and t). By using these substitutions, we may systematically convert a given (initial) kernel into new kernels. As initial kernels we use the ones obtained as limits of kernels of the general Heun equation [1], adapting them to the form (1) for the CHE.

In section 3 we find integral relations which transform the Jaffé power-series solutions [22] into known expansions in series of irregular confluent hypergeometric functions, including the aforementioned solution given by Leaver. Similarly, the power-series solutions of Baber and Hassé [23] are transformed into known expansions in series of regular confluent hypergeometric functions. We also consider non-integral transformations (involving only substitutions of variables) which do not modify the type of series: they transform, for example, a power-series solution into another power-series solution, and an expansion in series of special functions into another expansion in series special functions.

In contrast, by applying an integral transformation on an asymptotic (Thomé) solution, in section 4 we obtain a new solution in series of irregular confluent hypergeometric functions for the spheroidal equation. That solution is extended to any CHE (not just the spheroidal equation). Then, by substitutions of variables (non-integral transformations) we obtain a group of solutions for the CHE; we show that these solutions afford bounded and convergent solutions to the radial part of the Schrödinger equation for an electron in the field of two Coulomb centres [8] (the two-centre problem). In section 5 we present concluding remarks and mention open issues, while in appendix A we give some formulas concerning special functions, and in appendix B we discuss asymptotic solutions for the CHE.

2. Kernels for the confluent Heun equation

In this section we regard kernels for the CHE (1). In particular,

- in section 2.1 we get the correspondences among substitutions of variables which preserve the form of the CHE and the substitutions which preserve the equation of the kernels of the CHE;
- in section 2.2 we construct a group of kernels with an arbitrary constant of separation λ, given by products of two confluent hypergeometric functions and elementary functions;
- in section 2.3 we find another group of kernels with an arbitrary constant of separation λ, given by products of confluent hypergeometric functions and Gauss hypergeometric functions (and elementary functions);
- taking suitable values for λ we get kernels given products of elementary functions and one special function; thus, in sections 2.4 and 2.5 we find products of elementary and confluent hypergeometric functions and, in section 2.6, products of elementary and Gauss hypergeometric functions.

Later on, in section 4, we will need kernels for the ordinary spheroidal wave equation [5]

$$\tfrac{d}{dx}\left[\left(1-x^2\right)\tfrac{dX(x)}{dx}\right] + \left[\gamma^2(1-x^2) + \bar{\lambda} - \tfrac{\mu^2}{1-\ x^2}\right] X(x) = 0. \tag{6}$$

Such kernels are obtained from the ones of the CHE through the substitutions

$$x = 1 - 2z, \qquad X(x) = z^{\mu/2}\,(z-1)^{\mu/2}U(z), \tag{7}$$

which give

$$z(z-1)\tfrac{d^2U}{dz^2} + \left[-(\mu+1) + 2(\mu+1)z\right]\tfrac{dU}{dz} + \left[\mu(\mu+1) - \bar{\lambda} + 4\gamma^2 z(z-1)\right]U = 0, \tag{8a}$$

that is, the CHE (1) with

$$z_0 = 1, \qquad B_2 = -2B_1 = 2(\mu+1), \qquad B_3 = \mu(\mu+1) - \bar{\lambda}, \qquad \eta = 0, \qquad \omega^2 = 4\gamma^2. \tag{8b}$$

Thus, the spheroidal equation (6) will be treated as a CHE with $z_0 = 1$, $\eta = 0$ and $B_2 = -2B_1$, namely,

$$z(z-1)\frac{d^2U}{dz^2} + (B_1 - 2B_1 z)\frac{dU}{dz} + \left[B_3 + \omega^2 z(z-1)\right]U = 0. \tag{9}$$

2.1. Transformations of the CHE and its kernels

Defining the operator L_z by

$$L_z = z[z-z_0]\tfrac{\partial^2}{\partial z^2} + [B_1 + B_2 z]\tfrac{\partial}{\partial z} + \left[\omega^2 z(z-z_0) - 2\omega\eta z\right] \tag{10}$$

and interpreting this as an ordinary differential operator, the CHE (1) reads

$$[L_z + B_3 + 2\eta\omega z_0]U(z) = 0. \tag{11}$$

The adjoint operator $\bar{L}_z$ corresponding to L_z is [24]

$$\bar{L}_z = z(z-z_0)\frac{\partial^2}{\partial z^2} + [-2z_0 - B_1 + (4-B_2)z]\frac{\partial}{\partial z} + \left[\omega^2 z(z-z_0) - 2\omega\eta z + 2 - B_2\right]. \tag{12}$$

On the other side, if $U(z)$ is a known solution of equation CHE, we seek new solutions $\mathcal{U}(z)$ having the form

$$\mathcal{U}(z) = \textstyle\int_{t_1}^{t_2} K(z,t)U(t)dt = \int_{t_1}^{t_2} w(z,t)G(z,t)U(t)dt = \int_{t_1}^{t_2} t^{-1-\frac{B_1}{z_0}}(t-z_0)^{B_2+\frac{B_1}{z_0}-1}G(z,t)U(t)dt, \tag{13}$$
$$w(z,t) = t^{-1-\frac{B_1}{z_0}}(t-z_0)^{B_2+\frac{B_1}{z_0}-1},$$

where the kernel $K(z,t)$ or $G(z,t)$ is determined from a partial differential equation. The general theory is usually established for the function $K(z,t)$ [24], but to study the transformations of kernels we will deal with $G(z,t)$. If the integration endpoints t_1 and t_2 are independent of z, by applying L_z to integral (13) we find

$$L_z\mathcal{U}(z) = \int_{t_1}^{t_2}\left[L_z K(z,t)\right]U(t)dt = \int_{t_1}^{t_2} U(t)\left[L_z - \bar{L}_t\right]K(z,t)dt + \int_{t_1}^{t_2} U(t)\bar{L}_t K(z,t)dt, \tag{14}$$

$\bar{L}_t$ being obtained from $\bar{L}_z$ by replacing z with t. Now we demand that

$$\left[L_z - \bar{L}_t\right]K(z,t) = 0 \quad \Leftrightarrow \quad \left[L_z - L_t\right]G(z,t) = 0. \tag{15}$$

Thence, by using the Lagrange identity

$$U(t)\bar{L}_t K(z,t) - K(z,t)L_t U(t) = \frac{\partial}{\partial t}P(z,t),$$

where the bilinear concomitant $P(z,t)$ is given by

$$\begin{aligned} P(z,t) &= t(t-z_0)\left[U(t)\frac{\partial K(z,t)}{\partial t} - K(z,t)\frac{dU(t)}{dt}\right] - \left[(B_2-2)t + B_1 + z_0\right]U(t)K(z,t) \\ &= t^{-\frac{B_1}{z_0}}(t-z_0)^{B_2+\frac{B_1}{z_0}}\left[U(t)\tfrac{\partial G(z,t)}{\partial t} - G(z,t)\tfrac{dU(t)}{dt}\right], \end{aligned} \tag{16}$$

Eq. (14) reduces to

$$L_z\mathcal{U}(z) = \int_{t_1}^{t_2}\left[K(z,t)L_t U(t) + \frac{\partial P(z,t)}{\partial t}\right]dt = -(B_3 + 2\eta\omega z_0)\int_{t_1}^{t_2} K(z,t)U(t)dt + \int_{t_1}^{t_2}\frac{\partial P(z,t)}{\partial t}dt,$$

where in the last step we have used equation (11). Using equation (13) as well, this yields

$$[L_z + B_3 + 2\eta\omega z_0]\mathcal{U}(z) = P(z,t_2) - P(z,t_1). \tag{17}$$

Therefore, $\mathcal{U}(z)$ is also a solution of the CHE if: (i) the kernel satisfies Eq. (15), (ii) the integral (13) exists and (iii) the right-hand side of Eq. (17) vanishes.

Now let us examine the transformations of the solutions $U(z)$ and kernels $G(z,t)$. If $U(z) = U(B_1,B_2,B_3;z_0,\omega,\eta;z)$ denotes one solution of the CHE, the following transformations [18, 25, 26] – T_1, T_2, T_3 and T_4 – leave invariant the form of the CHE:

$$\begin{aligned} &T_1U(z) = z^{1+\frac{B_1}{z_0}}U(C_1,C_2,C_3;z_0,\omega,\eta;z), && T_2U(z) = (z-z_0)^{1-B_2-\frac{B_1}{z_0}}\,U(B_1,D_2,D_3;z_0,\omega,\eta;z),\\ &T_3U(z) = U(B_1,B_2,B_3;z_0,-\omega,-\eta;z), && T_4U(z) = U(-B_1-B_2z_0,B_2,B_3+2\eta\omega z_0;z_0,-\omega,\eta;z_0-z), \end{aligned} \tag{18}$$

where

$$\begin{aligned} &C_1 = -B_1 - 2z_0, \qquad C_2 = 2 + B_2 + \tfrac{2B_1}{z_0}, \qquad C_3 = B_3 + \left[1+\tfrac{B_1}{z_0}\right]\left[B_2+\tfrac{B_1}{z_0}\right],\\ &D_2 = 2 - B_2 - \tfrac{2B_1}{z_0}, \qquad D_3 = B_3 + \tfrac{B_1}{z_0}\left(\tfrac{B_1}{z_0} + B_2 - 1\right). \end{aligned} \tag{19}$$

By composition of these transformations, from an initial solution we may generate a group containing up to 16 solutions. To get the corresponding transformations for the kernels, we notice that the operators L_z and L_t which appear in the CHE (11) and $[L_z - L_t]G(z,t) = 0$ have the same functional form. Hence, if $G(z,t) = G(B_1,B_2;z_0,\omega,\eta;z,t)$ is a solution of the Eq. (15), we find that the transformations R_1, R_2, R_3 and R_4, given by

$$\begin{aligned} &R_1G(z,t) = (zt)^{1+\frac{B_1}{z_0}}\,G(C_1,C_2;z_0,\omega,\eta;z,t), && R_2G(z,t) = [(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}}\,G(B_1,D_2;z_0,\omega,\eta;z,t),\\ &R_3G(z,t) = G(B_1,B_2;z_0,-\omega,-\eta;z,t), && R_4G(z,t) = G(-B_1-B_2z_0,B_2;z_0,-\omega,\eta;z_0-z,z_0-t) \end{aligned} \tag{20}$$

do not change the form of the kernel equation (15). These transformations may generate a group containing up to 16 kernels when applied to an initial kernel.

For another version of the CHE we have obtained initial kernels as limits of kernels for the general Heun equation [1]. For the version (1), in the following we reobtain these kernels by solving Eq. (15) and use the transformations (20) to generate groups of kernels closed under such transformations.

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2.2. First group of kernels: products of confluent hypergeometric functions

For a particular problem, kernels given by products of confluent hypergeometric functions have already appeared in the literature [27]. In the first place we show that Eq. $[L_z - L_t]G(z,t) = 0$ is satisfied by 16 of such products, denoted by $G_1^{(i,j)}$ and defined as $(i,j = 1,2,3,4)$

$$G_1^{(i,j)}(z,t) = e^{-i\omega(z+t)}\varphi^i(\xi)\times\bar{\varphi}^j(\zeta), \tag{21}$$

where $\varphi^i(\xi)$ and $\bar{\varphi}^j(\zeta)$ are the confluent hypergeometric functions (A.2), having the following arguments and parameters:

$$\varphi^i(\xi):\quad \xi = -\tfrac{2i\omega}{z_0}(z-z_0)(t-z_0), \qquad a = \tfrac{B_2}{2} - i\eta - \lambda, \qquad c = B_2 + \tfrac{B_1}{z_0}; \tag{22a}$$

$$\bar{\varphi}^j(\zeta):\quad \zeta = \tfrac{2i\omega}{z_0}zt, \quad a = \lambda, \quad c = -\tfrac{B_1}{z_0}, \tag{22b}$$

where λ is an arbitrary constant of separation. In the second place, by the transformation R_3 we may get another set of kernels, $G_2^{(i,j)}$, given by

$$G_2^{(i,j)}(z,t) = R_3G_1^{(i,j)}(z,t) = G_1^{(i,j)}(z,t)\big|_{(\eta,\omega)\mapsto(-\eta,-\omega)}, \tag{23}$$

The transformations R_1, R_2 and R_4 are superfluous in this case.

To obtain the kernels (21), first we write

$$G(z,t) = e^{-i\omega(z+t)}f(z,t), \tag{24}$$

in Eq. (15). This leads to

$$z(z-z_0)\frac{\partial^2 f}{\partial z^2}+\left[B_1+(B_2+2i\omega z_0)z-2i\omega z^2\right]\frac{\partial f}{\partial z}$$
$$-t(t-z_0)\frac{\partial^2 f}{\partial t^2}-\left[B_1+(B_2+2i\omega z_0)t-2i\omega t^2\right]\frac{\partial f}{\partial t}-2i\omega\left(\frac{B_2}{2}-i\eta\right)(z-t)f=0. \tag{25}$$

Then, by the substitutions

$$\xi=-\tfrac{2i\omega(z-z_0)(t-z_0)}{z_0},\qquad \zeta=\tfrac{2i\omega zt}{z_0},\quad f=X(\xi)Y(\zeta) \tag{26}$$

we find the confluent hypergeometric equations

$$\xi\tfrac{d^2X}{d\xi^2}+\left[B_2+\tfrac{B_1}{z_0}-\xi\right]\tfrac{dX}{d\xi}-\left[\tfrac{B_2}{2}-i\eta-\lambda\right]X=0,\qquad \zeta\tfrac{d^2Y}{d\zeta^2}+\left[-\tfrac{B_1}{z_0}-\zeta\right]\tfrac{dY}{d\zeta}-\lambda Y=0, \tag{27}$$

where λ is the constant of separation. The solutions for the above equations are: $X(\xi)=\varphi^i(\xi)$ with $a=(B_2/2)-i\eta-\lambda$ and $c=B_2+(B_1/z_0)$; and $Y(\zeta)=\bar{\varphi}^j(\zeta)$ with $a=\lambda$ and $c=-B_1/z_0$. Inserting these solutions into (24) and (26) we find the kernels (21). Thence, the kernels given by regular confluent hypergeometric functions are

$$G_1^{(1,1)}(z,t)=e^{-i\omega(z+t)}\Phi\left[\tfrac{B_2}{2}-i\eta-\lambda,B_2+\tfrac{B_1}{z_0};-\tfrac{2i\omega}{z_0}(z-z_0)(t-z_0)\right]\Phi\left[\lambda,-\tfrac{B_1}{z_0};\tfrac{2i\omega}{z_0}zt\right], \tag{28}$$

$$G_1^{(1,2)}(z,t)=e^{-i\omega(z+t)+\frac{2i\omega}{z_0}zt}[zt]^{1+\frac{B_1}{z_0}}\Phi\left[\frac{B_2}{2}-i\eta-\lambda,B_2+\frac{B_1}{z_0};-\frac{2i\omega}{z_0}(z-z_0)(t-z_0)\right]\Phi\left[1-\lambda,2+\frac{B_1}{z_0};-\frac{2i\omega}{z_0}zt\right], \tag{29}$$

$$\begin{aligned}G_1^{(2,1)}(z,t)\;&=\;e^{-i\omega(z+t)-\frac{2i\omega}{z_0}(z-z_0)(t-z_0)}\left[(z-z_0)(t-z_0)\right]^{1-B_2-\frac{B_1}{z_0}}\\ &\times\;\Phi\left[1+i\eta+\lambda-\frac{B_2}{2},2-B_2-\frac{B_1}{z_0};\frac{2i\omega}{z_0}(z-z_0)(t-z_0)\right]\Phi\left[\lambda,-\frac{B_1}{z_0};\frac{2i\omega}{z_0}zt\right],\end{aligned} \tag{30}$$

$$\begin{aligned}G_1^{(2,2)}(z,t)\;&=\;e^{i\omega(z+t)}[zt]^{1+\frac{B_1}{z_0}}\left[(z-z_0)(t-z_0)\right]^{1-B_2-\frac{B_1}{z_0}}\\ &\times\;\Phi\left[1+i\eta+\lambda-\tfrac{B_2}{2},2-B_2-\tfrac{B_1}{z_0};\tfrac{2i\omega}{z_0}(z-z_0)(t-z_0)\right]\Phi\left[1-\lambda,2+\tfrac{B_1}{z_0};-\tfrac{2i\omega}{z_0}zt\right].\end{aligned} \tag{31}$$

The remaining kernels are obtained by replacing one or both functions Φ by Ψ. In this manner we obtain the 16 kernels. The transformations R_1, R_2 and R_4 are superfluous because they simply rearrange these kernels. For instance,

$$\begin{aligned}R_1G_1^{(1,1)}\;&=\;e^{-i\omega(z+t)+\frac{2i\omega}{z_0}zt}[zt]^{1+\frac{B_1}{z_0}}\Phi\left[2+\tfrac{B_1}{z_0}-\lambda_1,2+\tfrac{B_1}{z_0};-\tfrac{2i\omega}{z_0}zt\right]\\ &\times\;\Phi\left[\tfrac{B_1}{z_0}+\tfrac{B_2}{2}+1-i\eta-\lambda_1,B_2+\tfrac{B_1}{z_0};-\tfrac{2i\omega}{z_0}(z-z_0)(t-z_0)\right],\end{aligned}$$

where we have transformed λ into λ_1. By putting $\lambda_1=\lambda+1+(B_1/z_0)$, we see that $R_1G_1^{(1,1)}=G_1^{(1,2)}$.

2.3. Second group: product of hypergeometric and confluent hypergeometric functions

Now we find a group of kernels $G^{(i,j)}$ given by products of the four confluent hypergeometric φ^i given in (A.2) with the six Gauss hypergeometric functions F^j written in (A.5), (A.6) and (A.7). These kernels take the form

$$G^{(i,j)}(z,t)=e^{-i\omega(z+t)}\left(z+t-z_0\right)^{-\lambda}\varphi^i(\xi)\times F^j(\zeta),\qquad [i=1,\cdots,4;\qquad j=1,\cdots,6] \tag{32}$$

where λ is a constant of separation, whereas the arguments and parameters for the hypergeometric functions are

$$\varphi^i(\xi):\ \xi=2i\omega(z+t-z_0),\qquad a=\frac{B_2}{2}-i\eta-\lambda,\qquad c=B_2-2\lambda; \tag{33a}$$

$$F^j(\zeta):\ \zeta=\frac{zt}{z_0(z+t-z_0)},\qquad a=\lambda,\quad b=B_2-1-\lambda,\qquad c=-\tfrac{B_1}{z_0}. \tag{33b}$$

We can show that the transformations R_i do not generate new kernels.

The above kernels are constructed by inserting

$$G(z,t)=e^{-i\omega(z+t)}f(z,t)=e^{-i\omega(z+t)}g(\xi,\zeta) \tag{34}$$

into $[L_z - L_t]G(z,t) = 0$, where ξ and ζ are defined in Eqs. (33a) and (33b) . Thus we find

$$\xi\left[\xi\frac{\partial^2 g}{\partial\xi^2} + (B_2-\xi)\frac{\partial g}{\partial\xi} - \left(\frac{B_2}{2} - i\eta\right)g\right] + \zeta(1-\zeta)\frac{\partial^2 g}{\partial\zeta^2} + \left(-\frac{B_1}{z_0} - B_2\zeta\right)\frac{\partial g}{\partial\zeta} = 0.$$

The separation of variables $g(\xi,\zeta) = X(\xi)Y(\zeta)$ leads to

$$\xi\frac{d^2X}{d\xi^2} + [B_2-\xi]\frac{dX}{d\xi} - \left[\frac{B_2}{2} - i\eta - \frac{\bar{\lambda}}{\xi}\right]X = 0, \qquad \zeta(1-\zeta)\frac{d^2Y}{d\zeta^2} - \left[\frac{B_1}{z_0} + B_2\zeta\right]\frac{dY}{d\zeta} - \bar{\lambda}Y = 0, \tag{35}$$

where $\bar{\lambda}$ is a constant of separation. Putting $\bar{\lambda} = \lambda(B_2 - 1 - \lambda)$, we find that $Y(\zeta)$ is given by hypergeometric functions $Y(\zeta) = F^j(\zeta)$ as in Eqs. (32) and (33b), while $X(\xi)$ obeys the equation

$$\xi\frac{d^2X}{d\xi^2} + [B_2-\xi]\frac{dX}{d\xi} - \left[\frac{B_2}{2} - i\eta - \frac{\lambda(B_2-1-\lambda)}{\xi}\right]X = 0.$$

The substitution $X(\xi) = \xi^{-\lambda}\bar{X}(x)$ gives the confluent hypergeometric equation

$$\xi\frac{d^2\bar{X}}{d\xi^2} + \left[B_2 - 2\lambda - \xi\right]\frac{d\bar{X}}{d\xi} - \left[\frac{B_2}{2} - i\eta - \lambda\right]\bar{X} = 0, \tag{36}$$

whose solutions are $\bar{X}(\xi) = \varphi^i(\xi)$. In this manner, by inserting the previous solutions for $\bar{X}(\xi)$ and $Y(\zeta)$ into

$$G(z,t) = e^{-i\omega(z+t)}(z+t-z_0)^{-\lambda}\,\bar{X}(\xi)\,Y(\zeta) \tag{37}$$

we obtain kernels having the form (32).

The kernels $G^{(1,j)}$ and $G^{(2,j)}$ in terms of regular confluent hypergeometric functions are

$$G^{(1,j)}(z,t) = e^{-i\omega(z+t)}\left[z+t-z_0\right]^{-\lambda}F^j(\zeta)\,\Phi\left[\frac{B_2}{2} - i\eta - \lambda, B_2 - 2\lambda; 2i\omega(z+t-z_0)\right], \tag{38}$$

$$G^{(2,j)}(z,t) = e^{i\omega(z+t)}\left[z+t-z_0\right]^{1-B_2+\lambda}F^j(\zeta)\,\Phi\left[1 + i\eta + \lambda - \frac{B_2}{2}, 2 + 2\lambda - B_2; -2i\omega(z+t-z_0)\right], \tag{39}$$

whereas $G^{(3,j)}$ and $G^{(4,j)}$ in terms of irregular functions are obtained by substituting $\Psi(a,c;u)$ for $\Phi(a,c;u)$, that is,

$$G^{(3,j)}(z,t) = G^{(1,j)}(z,t)\big|_{\Phi\mapsto\Psi}, \qquad G^{(4,j)}(z,t) = G^{(2,j)}(z,t)\big|_{\Phi\mapsto\Psi}. \tag{40}$$

The functions $F^j(\zeta)$ are given by

$$F^1(\zeta) = F\left[\lambda, B_2 - 1 - \lambda; -\frac{B_1}{z_0}; \frac{zt}{z_0(z+t-z_0)}\right], \tag{41}$$

$$F^2(\zeta) = \left[\frac{zt}{z_0(z+t-z_0)}\right]^{1+\frac{B_1}{z_0}}F\left[\lambda + 1 + \frac{B_1}{z_0}, B_2 + \frac{B_1}{z_0} - \lambda; 2 + \frac{B_1}{z_0}; \frac{zt}{z_0(z+t-z_0)}\right], \tag{42}$$

$$F^3(\zeta) = F\left[\lambda, B_2 - 1 - \lambda; B_2 + \frac{B_1}{z_0}; \frac{(z-z_0)(t-z_0)}{z_0(z_0-z-t)}\right], \tag{43}$$

$$F^4(\zeta) = \left[\frac{(z-z_0)(t-z_0)}{z_0(z+t-z_0)}\right]^{1-B_2-\frac{B_1}{z_0}}F\left[-\lambda - \frac{B_1}{z_0}, \lambda + 1 - B_2 - \frac{B_1}{z_0}; 2 - B_2 - \frac{B_1}{z_0}; \frac{(z-z_0)(t-z_0)}{z_0(z_0-z-t)}\right], \tag{44}$$

$$F^5(\zeta) = \left[\frac{z_0(z+t-z_0)}{zt}\right]^{\lambda}F\left[\lambda, \lambda + 1 + \frac{B_1}{z_0}; 2 + 2\lambda - B_2; \frac{z_0(z+t-z_0)}{zt}\right], \tag{45}$$

$$F^6(\zeta) = \left[\frac{z_0(z+t-z_0)}{zt}\right]^{B_2-1-\lambda}F\left[B_2 + \frac{B_1}{z_0} - \lambda, B_2 - 1 - \lambda; B_2 - 2\lambda; \frac{z_0(z+t-z_0)}{zt}\right]. \tag{46}$$

By using the explicit form for the kernels and the fact that the separation constant is arbitrary, it is possible to show that the transformations R_i simply rearrange the previous kernels. For instance, we get

$$R_3G^{(1,j)}(z,t) = e^{i\omega(z+t)}\left[z+t-z_0\right]^{-\lambda_3}\Phi\left[\frac{B_2}{2} + i\eta - \lambda_3, B_2 - 2\lambda_3; -2i\omega(z+t-z_0)\right]H^j(\zeta),$$

where $H^j(\zeta)$ is obtained by substituting λ_3 for λ in $F^j(\zeta)$. Thence, putting $\lambda_3 = B_2 - \lambda - 1$ and taking into account that $F(a,b;c;u) = F(b,a;c;u)$, we find that $H^5(\zeta) = F^6(\zeta)$, $H^6(\zeta) = F^5(\zeta)$ and $H^j(\zeta) = F^j(\zeta)$ if $j = 1,2,3,4$. For this reason, $R_3G^{(1,j)}$ is equivalent to $G^{(2,j)}$.

2.4. Third group: confluent hypergeometric functions

An initial set has the form

$$G_1^{(i)}(z,t)=e^{-i\omega(z+t)}\varphi^i(\xi),\qquad [i=1,2,3,4] \tag{47}$$

where the $\varphi^i(\xi)$ denote the four solutions (A.2) for the confluent hypergeometric equation with the following argument and parameters:

$$\xi=-\tfrac{2i\omega}{z_0}(z-z_0)(t-z_0),\qquad a=\tfrac{B_2}{2}-i\eta,\qquad c=B_2+\tfrac{B_1}{z_0}. \tag{48}$$

The set (47) is obtained by putting $\lambda=0$ and Y constant in (27). Besides this, from (47) we form four sets by using the rules R_2 and R_4, namely,

$$G_1^{(i)}(z,t),\qquad G_2^{(i)}(z,t)=R_1G_1^{(i)}(z,t),\qquad G_3^{(i)}(z,t)=R_4G_2^{(i)}(z,t),\qquad G_4^{(i)}(z,t)=R_2G_3^{(i)}(z,t). \tag{49}$$

The four pairs in terms of regular confluent hypergeometric functions $\Phi(a,c;u)$ read

$$\begin{aligned}
&G_1^{(1)}(z,t)=e^{-i\omega(z+t)}\Phi\left[\tfrac{B_2}{2}-i\eta,B_2+\tfrac{B_1}{z_0};-\tfrac{2i\omega}{z_0}(z-z_0)(t-z_0)\right],\\
&G_1^{(2)}(z,t)=e^{i\omega(z+t)-\frac{2i\omega zt}{z_0}}\left[(z-z_0)(t-z_0)\right]^{1-B_2-\frac{B_1}{z_0}}\Phi\left[1+i\eta-\tfrac{B_2}{2},2-B_2-\tfrac{B_1}{z_0};\tfrac{2i\omega}{z_0}(z-z_0)(t-z_0)\right];
\end{aligned} \tag{50}$$

$$\begin{aligned}
&G_2^{(1)}(z,t)=e^{-i\omega(z+t)}[zt]^{1+\frac{B_1}{z_0}}\Phi\left[1-i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2},B_2+\tfrac{B_1}{z_0};-\tfrac{2i\omega}{z_0}(z-z_0)(t-z_0)\right],\\
&G_2^{(2)}(z,t)=e^{i\omega(z+t)-\frac{2i\omega zt}{z_0}}[zt]^{1+\frac{B_1}{z_0}}\left[(z-z_0)(t-z_0)\right]^{1-B_2-\frac{B_1}{z_0}}\Phi\left[i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2},2-B_2-\tfrac{B_1}{z_0};\tfrac{2i\omega}{z_0}(z-z_0)(t-z_0)\right];
\end{aligned} \tag{51}$$

$$\begin{aligned}
&G_3^{(1)}(z,t)=e^{-i\omega(z+t)}\left[(z-z_0)(t-z_0)\right]^{1-B_2-\frac{B_1}{z_0}}\Phi\left(1-i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2},-\tfrac{B_1}{z_0};\tfrac{2i\omega zt}{z_0}\right),\\
&G_3^{(2)}(z,t)=e^{-i\omega(z+t)+\frac{2i\omega zt}{z_0}}(zt)^{1+\frac{B_1}{z_0}}\left[(z-z_0)(t-z_0)\right]^{1-B_2-\frac{B_1}{z_0}}\Phi\left(i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2},2+\tfrac{B_1}{z_0};-\tfrac{2i\omega zt}{z_0}\right);
\end{aligned} \tag{52}$$

$$\begin{aligned}
&G_4^{(1)}(z,t)=e^{-i\omega(z+t)}\Phi\left(\tfrac{B_2}{2}-i\eta,-\tfrac{B_1}{z_0};\tfrac{2i\omega zt}{z_0}\right),\\
&G_4^{(2)}(z,t)=e^{-i\omega(z+t)+\frac{2i\omega zt}{z_0}}(zt)^{1+\frac{B_1}{z_0}}\Phi\left(1+i\eta-\tfrac{B_2}{2},2+\tfrac{B_1}{z_0};-\tfrac{2i\omega zt}{z_0}\right).
\end{aligned} \tag{53}$$

We get the pairs in terms of irregular confluent hypergeometric functions by replacing $\Phi(a,c;u)$ by $\Psi(a,c;u)$:

$$G_i^{(3)}(z,t)=G_i^{(1)}(z,t)\Big|_{\Phi\to\Psi},\qquad G_i^{(4)}(z,t)=G_i^{(2)}(z,t)\Big|_{\Phi\to\Psi},\qquad [i=1,2,3,4]. \tag{54}$$

Thus, by using also R_3, we find that this group is constituted by 32 kernels. Notice that R_1, R_2 and R_4 generate only four sets of kernels instead of 16 sets because in some cases these transformations rearrange the kernels of a given set in a different order: we can test this by computing, for example, $R_2G_1^{(i)}$ or $R_2G_2^{(i)}$. Notice that kernels whose arguments of the hypergeometric functions are $\pm(2i\omega zt/z_0)$ have been known since long [28].

For the spheroidal equation ($\eta=0,\ z_0=1,\ B_2=-2B_1$), sixteen of the previous kernels reduce to four kernels in terms of elementary functions, namely,

$$G_1^{(\pm)}(z,t)=e^{\pm i\omega(z+t)\mp 2i\omega zt},\qquad G_2^{(\pm)}(z,t)=e^{\pm i\omega(z+t)\mp 2i\omega zt}[zt(z-1)(t-1)]^{1+B_1}. \tag{55}$$

For instance,

$$\begin{aligned}
&G_1^{(1)}(z,t)\propto G_1^{(4)}(z,t)\propto G_1^{(+)}(z,t), && G_4^{(1)}(z,t)\propto G_4^{(4)}(z,t)\propto G_1^{(-)}(z,t),\\
&G_2^{(2)}(z,t)\propto G_2^{(4)}(z,t)\propto G_2^{(+)}(z,t), && G_3^{(2)}(z,t)\propto G_3^{(4)}(z,t)\propto G_2^{(-)}(z,t).
\end{aligned}$$

The kernel $G_2^{(-)}(z,t)$ will be used in section 4.1.

2.5. Fourth group: confluent hypergeometric functions again

To obtain new kernels given by confluent hypergeometric functions we take

$$G_1^{(i)}(z,t) = G^{(i,1)}(z,t)\big|_{\lambda=0},$$

where the $G^{(i,1)}$ denote the kernels (32) with $j=1$. Since the above choice for λ eliminates the Gauss hypergeometric function $[F(0,b;c;\zeta)=1]$ we find

$$G_1^{(i)}(z,t) = e^{-i\omega(z+t)}\varphi^i(\xi), \qquad [i=1,2,3,4] \tag{56a}$$

where $\varphi^i(\xi)$ denote the solutions (A.2) for the confluent hypergeometric equation with

$$\xi = 2i\omega(z+t-z_0), \qquad a=\frac{B_2}{2}-i\eta, \qquad c=B_2 \qquad \text{[see Eq. (33a)]}. \tag{56b}$$

Other choices for λ also lead to kernels in terms of confluent hypergeometric functions. However, such kernels are obtained from the initial set (56a) by using the transformations R_i. In this manner we find four sets, namely,

$$G_1^{(i)}(z,t), \quad G_2^{(i)}(z,t) = R_1 G_1^{(i)}(z,t), \quad G_3^{(i)}(z,t) = R_2 G_2^{(i)}(z,t), \quad G_4^{(i)}(z,t) = R_1 G_3^{(i)}(z,t), \tag{57}$$

since R_4 does not generate new kernels. The kernels given by regular confluent hypergeometric functions are

$$\begin{aligned} G_1^{(1)}(z,t) &= e^{-i\omega(z+t)}\Phi\left[\tfrac{B_2}{2}-i\eta, B_2; 2i\omega(z+t-z_0)\right], \\ G_1^{(2)}(z,t) &= e^{i\omega(z+t)}[z+t-z_0]^{1-B_2}\Phi\left[1+i\eta-\tfrac{B_2}{2}, 2-B_2; -2i\omega(z+t-z_0)\right]; \end{aligned} \tag{58}$$

$$\begin{aligned} G_2^{(1)}(z,t) &= e^{-i\omega(z+t)}[zt]^{1+\frac{B_1}{z_0}}\Phi\left[1-i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}, 2+B_2+\tfrac{2B_1}{z_0}; 2i\omega(z+t-z_0)\right], \\ G_2^{(2)}(z,t) &= e^{i\omega(z+t)}[zt]^{1+\frac{B_1}{z_0}}[z+t-z_0]^{-1-B_2-\frac{2B_1}{z_0}}\Phi\left[i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}, -B_2-\tfrac{2B_1}{z_0}; -2i\omega(z+t-z_0)\right]; \end{aligned} \tag{59}$$

$$\begin{aligned} G_3^{(1)}(z,t) &= e^{-i\omega(z+t)}[zt]^{1+\frac{B_1}{z_0}}[(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}} \quad \Phi\left[2-i\eta-\tfrac{B_2}{2}, 4-B_2; 2i\omega(z+t-z_0)\right], \\ G_3^{(2)}(z,t) &= e^{i\omega(z+t)}[zt]^{1+\frac{B_1}{z_0}}[(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}}[z+t-z_0]^{B_2-3}\Phi\left[i\eta-1+\tfrac{B_2}{2}, B_2-2; -2i\omega(z+t-z_0)\right]; \end{aligned} \tag{60}$$

$$\begin{aligned} G_4^{(1)}(z,t) &= e^{-i\omega(z+t)}[(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}}\Phi\left[1-i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}, 2-B_2-\tfrac{2B_1}{z_0}; 2i\omega(z+t-z_0)\right], \\ G_4^{(2)}(z,t) &= e^{i\omega(z+t)}[(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}}[z+t-z_0]^{B_2+\frac{2B_1}{z_0}-1}\Phi\left[i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}, B_2+\tfrac{2B_1}{z_0}; -2i\omega(z+t-z_0)\right]. \end{aligned} \tag{61}$$

Replacing $\Phi(a,c;u)$ by $\Psi(a,c;u)$ and using the transformation R_3, once more we get a group with 32 kernels. Some particular cases of these kernels are already known [4]. Furthermore, if $\eta=0$ this group can be expressed in terms of Bessel functions by means of (A.12).

2.6. Fifth group: hypergeometric functions

To get kernels given by hypergeometric functions we take $G_1^{(i)}(z,t) = G^{(1,i)}(z,t)|_{\lambda=(B_2/2)-i\eta}$, where $G^{(1,i)}$ are the kernels given in (38). In fact, for this choice for λ we obtain $\Phi(0,c;\xi)=1$ and, thence,

$$G_1^{(i)}(z,t) = e^{-i\omega(z+t)}[z+t-z_0]^{i\eta-\frac{B_2}{2}}F^i(\zeta), \qquad \zeta = zt/[z_0(z+t-z_0)], \tag{62}$$

where the hypergeometric functions $F^i(\zeta)$ are obtained by putting $\lambda=(B_2/2)-i\eta$ in Eqs. (41-46). Explicitly

$$G_1^{(1)}(z,t) = e^{-i\omega(z+t)}[z+t-z_0]^{i\eta-\frac{B_2}{2}} \quad F\left[\tfrac{B_2}{2}-i\eta, \tfrac{B_2}{2}+i\eta-1; -\tfrac{B_1}{z_0}; \tfrac{zt}{z_0(z+t-z_0)}\right], \tag{63}$$

$$G_1^{(2)}(z,t) = e^{-i\omega(z+t)}[z+t-z_0]^{i\eta-1-\frac{B_1}{z_0}-\frac{B_2}{2}}[zt]^{1+\frac{B_1}{z_0}} \quad F\left[1-i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}, i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}; 2+\tfrac{B_1}{z_0}; \tfrac{zt}{z_0(z+t-z_0)}\right], \tag{64}$$

$$G_1^{(3)}(x,t) = e^{-i\omega(z+t)}\left[z+t-z_0\right]^{i\eta-\frac{B_2}{2}} F\left[\tfrac{B_2}{2}-i\eta, \tfrac{B_2}{2}+i\eta-1; B_2+\tfrac{B_1}{z_0}; \tfrac{(z-z_0)(t-z_0)}{z_0(z_0-z-t)}\right], \tag{65}$$

$$\begin{aligned} G_1^{(4)}(x,t) &= e^{-i\omega(z+t)}\left[z+t-z_0\right]^{i\eta-1+\frac{B_1}{z_0}+\frac{B_2}{2}}\left[(z-z_0)(t-z_0)\right]^{1-B_2-\frac{B_1}{z_0}} \\ &\times\ F\left[i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}, 1-i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}; 2-B_2-\tfrac{B_1}{z_0}; \tfrac{(z-z_0)(t-z_0)}{z_0(z_0-z-t)}\right], \end{aligned} \tag{66}$$

$$G_1^{(5)}(x,t) = e^{-i\omega(z+t)}\left[zt\right]^{i\eta-\frac{B_2}{2}} F\left[\tfrac{B_2}{2}-i\eta, 1-i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}; 2-2i\eta; \tfrac{z_0(z+t-z_0)}{zt}\right], \tag{67}$$

$$G_1^{(6)}(x,t) = e^{-i\omega(z+t)}\left[z+t-z_0\right]^{2i\eta-1}\left[zt\right]^{1-i\eta-\frac{B_2}{2}}\quad F\left[i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}, i\eta-1+\tfrac{B_2}{2}; 2i\eta; \tfrac{z_0(z+t-z_0)}{zt}\right]. \tag{68}$$

The transformations R_1, R_2 and R_4 at most rearrange the preceding kernels. For example,

$$R_1 G_1^{(1)}(z,t) = G_1^{(2)}(z,t), \qquad R_2 G_1^{(1)}(z,t) = G_1^{(1)}(z,t), \qquad R_4 G_1^{(1)}(z,t) = G_1^{(3)}(z,t).$$

However, we find six additional kernels $G_2^{(j)}(z,t)$ by using the transformations R_3 as

$$G_2^{(i)}(z,t) = R_3 G_1^{(i)}(z,t) \tag{69}$$

So, $G_2^{(i)}$ is obtained by replacing (η,ω) by $(-\eta,-\omega)$ in $G_1^{(i)}$.

3. Integral relations between known solutions

In this section we use some kernels to obtain integral relations among solutions of the CHE. We find that:

- the Jaffé solutions in power series are tranformed into Leaver's expansions in series of irregular confluent hypergeometric series;
- the Baber-Hassé solutions in power series are transformed into solutions given by series of regular confluent hypergeometric functions.

Relations for solutions generated by transformations of the CHEs may be obtained by transforming also the kernel, since each transformation of a solution corresponds to a transformation of a kernel.

3.1. Jaffé's solutions in power series and Leaver's solutions

By $U_1^J(z)$ and $U_1^L(z)$ we denote respectively the Jaffé [22] and the Leaver [9] solutions for the CHE, namely,

$$U_1^J(z) = e^{i\omega z} z^{-i\eta-\frac{B_2}{2}} \sum_{n=0}^{\infty} a_n^1 \ \left(\tfrac{z-z_0}{z}\right)^n \ , \tag{70a}$$

$$U_1^L(z) = e^{i\omega z}\sum_{n=0}^{\infty} a_n^1\ \Gamma\left(n+B_2+\tfrac{B_1}{z_0}\right)\Psi\left(n+i\eta+\tfrac{B_2}{2}, -\tfrac{B_1}{z_0}; -2i\omega z\right), \tag{70b}$$

where the recurrence relations for the a_n^1 are $(a_{-1}^1 = 0)$

$$\begin{aligned} (n+1)\left[n+B_2+\tfrac{B_1}{z_0}\right]a_{n+1}^1 + \Big[-2n\Big(n+B_2+\tfrac{B_1}{z_0}+i\eta-i\omega z_0\Big)+B_3+\Big(B_2+\tfrac{B_1}{z_0}\Big)\left(i\omega z_0-i\eta-\tfrac{B_2}{2}\right)\Big]a_n^1 \\ +\left[n-1+i\eta+\tfrac{B_2}{2}\right]\left[n+\tfrac{B_2}{2}+\tfrac{B_1}{z_0}+i\eta\right]a_{n-1}^1 = 0. \end{aligned}$$

The convergence of solutions (70a) and (70b) is discussed in the Leaver paper [9], where it is used the minimal solution for the coefficients a_n^1. In fact, three-term recurrence relations as the above ones admit two independent solutions, say, f_n and g_n. If $\lim_{n\to\infty}(f_n/g_n) = 0$, f_n is called minimal solution [29, 30]. In addition, it is necessary to suppose that there is an arbitrary parameter in the CHE. The series converges only for special values of that parameter, determined from a transcendental (characteristic) equation which results from the recurrence relations [9].

By supposing that $U_1^J(z)$ converge for $|z| \geq |z_0|$ and by using Eq. (13), we find the relation

$$U_1^L(z) = C_1\int_{z_0}^{\infty} t^{-1-\frac{B_1}{z_0}}\left[t-z_0\right]^{B_2+\frac{B_1}{z_0}-1} G_4^{(4)}(z,t)U_1^J(t)dt, \qquad \text{Re}\left[n+B_2+\frac{B_1}{z_0}\right] > 0, \qquad \text{Re}[i\omega z] < 0, \tag{71}$$

where C_1 is a constant and $G(z,t)=G_4^{(4)}(z,t)$ is the kernel indicated in (54). In fact, by setting $y=t/z_0$, we find that the right-hand side of (71) is equivalent to

$$e^{-i\omega z}z^{1+\frac{B_1}{z_0}}\sum_{n=0}^{\infty}a_n^1\int_{z_0}^{\infty}dy\Big[e^{2i\omega zy}\,(y-1)^{n+B_2+\frac{B_1}{z_0}-1}\,y^{-n-i\eta-\frac{B_2}{2}}\;\Psi\left(1+i\eta-\tfrac{B_2}{2},2+\tfrac{B_1}{z_0};-2i\omega zy\right)\Big]\;.$$

Then, by using the integral [31]

$$\int_1^{\infty}e^{-ay}(y-1)^{\mu-1}y^{\alpha+k-\mu-\frac{1}{2}}\Psi\left(\frac{1}{2}+\alpha-k,2\alpha+1;ay\right)dy=\Gamma(\mu)e^{-a}\Psi\left(\frac{1}{2}+\alpha+\mu-k,2\alpha+1;a\right),\qquad(72)$$
$$[\mathrm{Re}\;\mu>0,\;\mathrm{Re}\;a>0],$$

we obtain the relation (71).

For the bilinear concomitant (16) we find

$$\begin{aligned}P_1(z,t)\;&=\;z^{1+\frac{B_1}{z_0}}e^{i\omega z(\frac{2t}{z_0}-1)}\;t^{-i\eta-\frac{B_2}{2}}\;(t-z_0)^{B_2+\frac{B_1}{z_0}}\sum_{n=0}^{\infty}a_n\left(\frac{t-z_0}{t}\right)^n\\ &\times\quad\left\{\left[2i\omega\left(\tfrac{z}{z_0}-1\right)t-\tfrac{nz_0}{t-z_0}+\tfrac{1}{t}\left(i\eta+1+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}\right)\right]\Psi+t\tfrac{\partial\Psi}{\partial t}\right\},\end{aligned}$$

where

$$\Psi=\Psi\left(1+i\eta-\tfrac{B_2}{2},2+\tfrac{B_1}{z_0};-\tfrac{2i\omega zt}{z_0}\right),\qquad\tfrac{\partial\Psi}{\partial t}=\tfrac{2i\omega z}{z_0}\left(1+i\eta-\tfrac{B_2}{2}\right)\Psi\left(2+i\eta-\tfrac{B_2}{2},3+\tfrac{B_1}{z_0};-\tfrac{2i\omega zt}{z_0}\right).$$

Since $\Psi(a,b;y)=y^{-a}$ when $|y|\to\infty$ and $\mathrm{Re}(i\omega z)<0$, the exponential factor assures that $P_1(z,t)$ vanishes when $t/z_0\to\infty$. On the other hand, the condition $\mathrm{Re}[B_2+\frac{B_1}{z_0}]>0$ assures that $P_1(z,t)$ vanishes also for $t=z_0$ since $(t-z_0)^{B_2+B_1/z_0}\to 0$.

In this manner, we have extended the results of Leaver [9] who has considered only relations between solutions with $i\eta=\pm(B_2/2-1)$. Notice also that the conditions given in (71) are necessary only to assure the integral relation between the solutions. In fact the Leaver solutions can be derived directly from the differential equation without imposing those conditions [9].

For the present case the transformation T_1 is ineffective and, so, from $\left(U_1^J,U_1^L\right)$ we can obtain only 8 pairs of solutions by composition of the transformations (18); to each pair corresponds a kernel generated by the transformations (20). For example, taking $U_2^J(z)=T_2U_1^J(z)$ and $U_2^L(z)=T_2U_1^L(z)$, we find

$$U_2^J(z)=e^{i\omega z}(z-z_0)^{1-B_2-\frac{B_2}{2}}z^{-i\eta-1+\frac{B_1}{z_0}+\frac{B_2}{2}}\sum_{n=0}^{\infty}a_n^2\;\left(\tfrac{z-z_0}{z}\right)^n\;,\qquad(73a)$$

$$U_2^L(z)=e^{i\omega z}(z-z_0)^{1-B_2-\frac{B_2}{2}}\sum_{n=0}^{\infty}\;a_n^2\Gamma\left(n+2-B_2-\tfrac{B_1}{z_0}\right)\Psi\left(n+i\eta+1-\tfrac{B_2}{2}-\tfrac{B_1}{z_0},-\tfrac{B_1}{z_0};-2i\omega z\right),\qquad(73b)$$

where the recurrence relations for a_n^2 are ($a_{-1}^2=0$)

$$\begin{aligned}&(n+1)\left[n+2-B_2-\tfrac{B_1}{z_0}\right]a_{n+1}^2+\Big[-2n\Big(n+2+i\eta-i\omega z_0-B_2-\tfrac{B_1}{z_0}\Big)+B_3+\Big(2-B_2+\tfrac{B_1}{z_0}\Big)(i\omega z_0-i\eta)+\tfrac{B_1}{z_0}\\ &\qquad+\left(1-\tfrac{B_2}{2}\right)\left(1+\tfrac{B_1}{z_0}-\tfrac{B_2}{2}\right)\Big]a_n^2+\left[n+1+i\eta-\tfrac{B_2}{2}\right]\left[n+i\eta-\tfrac{B_2}{2}-\tfrac{B_1}{z_0}\right]a_{n-1}^2=0.\end{aligned}$$

Using Eq. (13), we find that

$$U_2^L(z)=C_2\int_{z_0}^{\infty}dt\,t^{-1-\frac{B_1}{z_0}}[t-z_0]^{B_2+\frac{B_1}{z_0}-1}U_2^J(t)R_2G_4^{(4)}(z,t),\qquad\mathrm{Re}\left[n+2-B_2-\frac{B_1}{z_0}\right]>0,\quad\mathrm{Re}[i\omega z]<0,$$

where C_2 is a constant, $G_4^{(4)}(z,t)$ is the kernel indicated in (54), and the transformation R_2 is given in (20); then,

$$R_2G_4^{(4)}=e^{-i\omega(z+t)+\frac{2i\omega zt}{z_0}}\;[(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}}\;(zt)^{1+\frac{B_1}{z_0}}\Psi\left(i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2},2+\tfrac{B_1}{z_0};-\tfrac{2i\omega zt}{z_0}\right).$$

We have supposed that the Jaffé solutions converge for $|z|\geq|z_0|$, but we must be careful about the point $z=\infty$, since [9]

$$\lim_{z\to\infty}U_1^J(z)=e^{i\omega z}z^{-i\eta-\frac{B_2}{2}}\sum_{n=0}^{\infty}a_n^1\quad\text{with}\quad\frac{a_{n+1}^1}{a_n^1}=1-\frac{\sqrt{-2i\omega z_0}}{\sqrt{n}}+\frac{i(\eta-\omega z_0)-(3/4)}{n}\qquad(74)$$

where the ratio a_{n+1}^1/a_n^1 holds for the minimal solution of the recurrence relations when $n\to\infty$. Thus, the D'Alambert test is inconclusive as to the convergence of $\sum a_n^1$. For the radial part of the two-center problem we could use the Raabe test for convergence, as in Eq. (106).

3.2. Solutions in power series and solutions in series of confluent hypergeometric functions

We find another pair of solutions for the CHE which are again connected by the integral (13). By one side we have the Baber-Hassé expansion [18, 23]

$$U_1^{\text{baber}}(z)=e^{i\omega z}\sum_{n=0}^{\infty}a_n^1(z-z_0)^n, \tag{75a}$$

where the coefficients satisfy the relations ($a_{-1}^1=0$)

$$z_0\left(n+B_2+\tfrac{B_1}{z_0}\right)(n+1)\,a_{n+1}^1+\beta_n^1 a_n^1+2i\omega\left(n+i\eta+\tfrac{B_2}{2}-1\right)a_{n-1}^1=0, \tag{75b}$$

with $\beta_n^1=n(n+B_2-1+2i\omega z_0)+B_3+i\omega z_0\left[B_2+B_1/z_0\right]$. The minimal solutions for a_n^1 yield solutions convergent for any finite value of z. On the other side, if $(B_2/2)-i\eta$ is not zero or negative integer we have the solution [18]

$$U_1(z)=e^{-i\omega z}\sum_{n=0}^{\infty}b_n^1\;\Phi\left(\frac{B_2}{2}-i\eta,n+B_2;2i\omega z\right), \tag{76a}$$

where the recurrence relations for b_n^1 are obtained from the previous ones by taking

$$b_n^1=\tfrac{C(-z_0)^n\Gamma(n+B_2+B_1/z_0)}{\Gamma(n+B_2)}a_n^1,\qquad C=\text{constant}.$$

This yields

$$-(n+B_2)(n+1)b_{n+1}^1+\beta_n^1 b_n^1-2i\omega z_0\frac{\left(n+B_2+\frac{B_1}{z_0}-1\right)\left(n+i\eta+\frac{B_2}{2}-1\right)}{n+B_2-1}b_{n-1}^1=0. \tag{76b}$$

Now, if we insert $U_1^{\text{baber}}(t)$ and the kernel $G_4^{(1)}(z,t)$ given in (53) into Eq. (13), we find the solution $U_1(z)$, that is,

$$U_1=K\int_{t_1}^{t_2}t^{-1-\frac{B_1}{z_0}}\left[t-z_0\right]^{B_2+\frac{B_1}{z_0}-1}G_4^{(1)}(z,t)U_1^{\text{baber}}(t)dt,\qquad \text{Re}\left[n+B_2+\frac{B_1}{z_0}\right]>0,\quad \text{Re}[-\frac{B_1}{z_0}]>0, \tag{77}$$

where K is a constant. In effect, by taking $t_1=0$ and $t_2=z_0$, the above integral is proportional to

$$e^{-i\omega z}\sum_{n=0}^{\infty}(-z_0)^n a_n^{(1)}\int_0^1 d\left(\tfrac{t}{z_0}\right)\left[\left(\tfrac{t}{z_0}\right)^{-1-\frac{B_1}{z_0}}\left(1-\tfrac{t}{z_0}\right)^{n+B_2-1+\frac{B_1}{z_0}}\Phi\left(\tfrac{B_2}{2}-i\eta,-\tfrac{B_1}{z_0};2i\omega z\tfrac{t}{z_0}\right)\right]$$

Then, by using the relation [31]

$$\int_0^1\left[x^{\lambda-1}(1-x)^{2\mu-\lambda}\Phi\left(\tfrac{1}{2}+\mu-\nu,\lambda;yx\right)\right]dx=\tfrac{\Gamma(\lambda)\Gamma(1+2\mu-\lambda)}{\Gamma(1+2\mu)}\Phi\left(\tfrac{1}{2}+\mu-\nu,1+2\mu;y\right), \tag{78}$$
$$\left[\text{Re}(\lambda)>0,\quad \text{Re}(1+2\mu-\lambda)>0\right],$$

we find the solution $U_1(z)$ given in (76a) provided that $\text{Re}\left[n+B_2+(B_1/z_0)\right]>0$ and $\text{Re}[-B_1/z_0]>0$. On the other side, from $d\Phi(a,b;\xi)/d\xi=(a/b)\Phi(a+1,b+1;\xi)$ for $\xi=2i\omega zt/z_0$, $a=B_2/2-i\eta$ and $b=-B_1/z_0$, we find that the bilinear concomitant (16) is given by

$$\begin{aligned}P_1(z,t)&=-e^{-i\omega z}t^{-\frac{B_1}{z_0}}(t-z_0)^{B_2+\frac{B_1}{z_0}}\\&\times\left\{\Phi(a,b;\xi)\sum_{n=1}^{\infty}na_n^1(t-z_0)^{n-1}+2i\omega\Big[\Phi(a,b;\xi)-\frac{az}{bz_0}\Phi(a+1,b+1;\xi)\Big]\sum_{n=0}^{\infty}a_n^1(t-z_0)^n\right\}.\end{aligned}$$

Therefore, $P_1(z,t=0)=P_1(z,t=z_0)=0$ due to the conditions $\text{Re}(-B_1/z_0)>0$ and $\text{Re}(B_2+B_1/z_0)>0$.

Observe that from the pair $\left(U_1^{\text{baber}},U_1\right)$ we can obtain 16 pairs of solutions by using the four transformations (18) and composition of them: to each pair corresponds a kernel which is obtained by using the transformations (20).

4. New solutions for the confluent equation

In section 4.1, by an integral transformation we find a new solution in series of irregular confluent hypergeometric functions for the ordinary spheroidal equation. Then, in section 4.2 we extend that solution to the general case (no restriction on the parameters of the CHE). In this manner, we obtain an initial solution, $\mathcal{U}_1(z)$, which allows to generate a group of solutions $\mathcal{U}_i(z)$ for the CHE by by means of transformations (18). Finally, in section 4.3, we show that the new solutions are suitable for the radial part of the two-center problem of the quantum mechanics.

Initially we make some comments on the recurrence relations and the ratio test for convergence. As in the preceding section, the three-term recurrence relations for the series coefficients b_n^i of $\mathcal{U}_i(z)$ have the form

$$\alpha_0^i\, b_1^i + \beta_0^i\, b_0^i = 0, \qquad \alpha_n^i\, b_{n+1}^i + \beta_n^i\, b_n^i + \gamma_n^i\, b_{n-1}^i = 0 \quad (n \geq 1) \tag{79}$$

where α_n^i, β_n^i and γ_n^i depend on the parameters of the differential equation and on the summation index n. By omitting the superscripts, these relations take the form

$$\left[\begin{array}{cccccc|ccc} \beta_0 & \alpha_0 & 0 & & & & & & \\ \gamma_1 & \beta_1 & \alpha_1 & & & & & & \\ 0 & \gamma_2 & \beta_2 & \alpha_2 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & \gamma_{\rm N} & \beta_{\rm N} & & \alpha_{\rm N} & & \\ \hline & & & & \gamma_{\rm N+1} & & \beta_{\rm N+1} & \alpha_{\rm N+1} & \\ & & & & & & \ddots & \ddots & \ddots \end{array}\right] \left[\begin{array}{c} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{\rm N} \\ \hline b_{\rm N+1} \\ \vdots \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \end{array}\right], \tag{80}$$

where we have split the matrix into blocks. This system of homogeneous linear equations has nontrivial solutions for b_n only if the determinant of the above tridiagonal matrix vanishes: this demands some arbitrary parameter in the matrix elements and, as a consequence, in the differential equation. The condition on the determinant can also be expressed by an (characteristic) equation given by the continued fraction [9]

$$\beta_0 = \frac{\alpha_0\gamma_1}{\beta_1 -}\;\frac{\alpha_1\gamma_2}{\beta_2 -}\;\frac{\alpha_2\gamma_3}{\beta_3 -}\cdots . \tag{81}$$

The solution of the characteristic equation and the computation of the series coefficients are important aspects concerning applications of solutions of the CHE [32, 33]. However, we do not consider these questions connected with numerical study of solutions. We only mention that the problem is simplified if $\gamma_{n={\rm N}+1}^i = 0$ for some ${\rm N} \geq 0$; then, the series terminates at $n = {\rm N}$ leading to a finite-series solution with $0 \leq n \leq N$ (see page 146 of [34]) which is called polynomial or quasi-polynomial solution. In this case, only the left upper block of the matrix is relevant.

On the other side, the convergence of a series like $\sum_{n=0}^{\infty} f_n(z)$ is obtained by computing the limit

$$L(z) = \lim_{n\to\infty}\left|\frac{f_{n+1}(z)}{f_n(z)}\right|. \tag{82a}$$

By the D'Alembert ratio test the series converges in the region where $L(z) < 1$ and diverges where $L_1(z) > 1$. If $L(z) = 1$, the D'Alembert test is inconclusive; however, by the Raabe test [35, 36], if

$$L(z) = 1 + \frac{A}{n} + O\left(\frac{1}{n^2}\right) \tag{82b}$$

(where A is a constant) the series converges if $A < -1$ and diverges if $A > -1$; the test is inconclusive if $A = -1$.

4.1. An integral transformation for the spheroidal equation

For the spheroidal equation in the form (9) we will find a solution $\mathcal{U}_1(z)$ given by

$$\mathcal{U}_1(z) = e^{i\omega z} z^{1+B_1}(z-1)^{1+B_1}\sum_{n=0}^{\infty} b_n^1 \Psi(2+B_1, 2+B_1-n; -2i\omega z), \quad [B_1 \neq -2, -3, \cdots] \tag{83a}$$

where the coefficients b_n^1 satisfy the relations $(b_{-1}^1 = 0)$

$$-2i\omega(n+1)b_{n+1}^1 + \big[n\,(n+1+2i\omega) + i\omega(2+B_1) - B_1\,(1+B_1) + B_3\big]b_n^1 - n\,(n+B_1+1)\,b_{n-1}^1 = 0. \tag{83b}$$

$\mathcal{U}_1(z)$ is not valid if $B_1 = -2, -3, \cdots$, because in these cases the function $\Psi(a, c; y)$ becomes a polynomial of fixed degree and, accordingly, (83a) is not a series expansion. This follows from the relation [37]

$$\Psi(-l, \alpha+1; y) = (-1)^l\, l!\, L_l^{\alpha}(y), \qquad [l = 0, 1, \cdots] \tag{84}$$

where the $L_l^{(\alpha)}(y)$ denote Laguerre polynomials of degree l. Besides this, the above expansion in general does not hold at $z=0$ because in most cases $\Psi(a,c;y)$ goes to infinity at $z=0$ [5]. The convergence of Ψ for $z\neq 0$ will be discussed later on.

We get the expansion (83a) by applying an integral transformation to the asymptotic expansion $\mathcal{W}_2(z)$ given in Eq. (B.4). First, for the spheroidal equation, by writing $W(z)=\mathcal{W}_2(z)$ and $a_n^2=b_n^1$, we find

$$W(z)=e^{i\omega z}\,(z-1)^{1+B_1}\sum_{n=0}^{\infty}b_n^1\;z^{-n-1},\qquad \text{[Eq. (B.4) for the spheroidal equation]} \tag{85}$$

where the coefficients b_n^1 satisfy (83b). In the second place, the solution $\mathcal{U}_1$ is obtained by inserting $U(t)=W(t)$ and $G(z,t)=G_2^-(z,t)$ – see Eq. (55) – into the right-hand side of Eq. (13), and by integrating from $t=1$ to $t=\infty$, that is,

$$\mathcal{U}_1(z)\overset{(13)}{=}\int_1^{\infty}t^{-1-B_1}(t-1)^{-1-B_1}G_2^-(z,t)\;W(t)dt\overset{(55)}{=}e^{-i\omega z}[z(z-1)]^{1+B_1}\int_1^{\infty}e^{-i\omega t+2i\omega zt}\;W(t)dt$$

which gives

$$\mathcal{U}_1(z)\overset{(85)}{=}e^{-i\omega z}[z(z-1)]^{1+B_1}\sum_{n=0}^{\infty}b_n^1\int_1^{\infty}e^{2i\omega zt}(t-1)^{1+B_1}t^{-n-1}dt. \tag{86}$$

Thence, we obtain (83a) by using [5]

$$\int_1^{\infty}e^{-yt}(t-1)^{a-1}t^{c-a-1}dt=\Gamma(a)e^{-y}\Psi(a,c;y),\qquad [\mathrm{Re}\,a>0,\quad \mathrm{Re}\,y>0].$$

The integrability conditions on the right-hand side require that

$$\mathrm{Re}[2+B_1]>0 \text{ and } \mathrm{Re}[i\omega z]<0. \tag{87}$$

On the other side, the bilinear concomitant (16) reads

$$\begin{aligned}P(z,t)&=t^{-B_1}(t-1)^{-B_1}\left[W(t)\frac{\partial G_2^-(z,t)}{\partial t}-G_2^-(z,t)\frac{dW(t)}{dt}\right],\\&=e^{i\omega z(2t-1)}[z(z-1)]^{1+B_1}(t-1)^{2+B_1}\left\{\left[2i\omega(z-1)t+B_1\right]\sum_{n=0}^{\infty}b_n^1t^{-n-1}+\sum_{n=0}^{\infty}nb_n^1t^{-n-1}\right\}\end{aligned} \tag{88}$$

Since the series converge at $t=\infty$, the conditions (87) assure that $P(z,t=\infty)=0$. However, the concomitant is undetermined at $t=1$ because [for $\mathrm{Re}(2+B_1)>0$] $P(z,t)$ is given by the product of the vanishing factor $(t-1)^{2+B_1}$ by a divergent series. Despite this, we can check directly [38] that $\mathcal{U}_1(z)$ is indeed a solution of the spheroidal equation (9).

Now we use the ratio test to get the convergence of $\mathcal{U}_1$. Thus, when $n\to\infty$, we find that the minimal solution of (83b) satisfies [38]

$$\frac{b_{n+1}^1}{b_n^1}\sim 1+\frac{B_1}{n}\quad\Rightarrow\quad\frac{b_{n-1}^1}{b_n^1}\sim 1-\frac{B_1}{n}. \tag{89}$$

To get the ratio between successive Ψ, we use the relation [5]

$$(a+1-c)\Psi(a,c-1;y)+(c-1+y)\Psi(a,c;y)-y\Psi(a,c+1;y)=0.$$

Hence, by taking

$$a=2+B_1,\quad c=2+B_1-n,\quad y=-2i\omega z,\quad \Psi_n(y)=\Psi(2+B_1,2+B_1-n;-2i\omega z)$$

we obtain

$$(n+1)\,\frac{\Psi_{n+1}}{\Psi_n}-(n-1-B_1+2i\omega z)+2i\omega z\frac{\Psi_{n-1}}{\Psi_n}=0.$$

If z is bounded (that is, if $2i\omega z/n\to 0$), then when $n\to\infty$ this equation is satisfied by

$$\frac{\Psi_{n+1}}{\Psi_n}\sim 1-\frac{1}{n}\,(B_1+2)\quad\Leftrightarrow\quad\frac{\Psi_{n-1}}{\Psi_n}\sim 1+\frac{1}{n}\,(B_1+2)\;\text{ or} \tag{90}$$

$$\frac{\Psi_{n+1}}{\Psi_n}\sim\frac{2i\omega z}{n}\left(1+\frac{B_1}{n}\right)\quad\Leftrightarrow\quad\frac{\Psi_{n-1}}{\Psi_n}\sim\frac{n}{2i\omega z}\left[1-\frac{1}{n}\,(1+B_1)\right].$$

Only the first ratio is consistent with the fact that, if $|c| \to \infty$ while a and y remain fixed and bounded, then [37]

$$\Psi(a,c;y) \sim c^{-a}\left[(-1)^{-a} + \frac{\sqrt{2\pi}}{\Gamma(a)}\left(\frac{c}{ey}\right)^{c+a-\frac{3}{2}} y^{a-\frac{1}{2}}\, e^{y+a-\frac{3}{2}}\right]\left[1+O\left(\frac{1}{|c|}\right)\right],$$
$$[c \to \infty;\quad a \neq 0,-1,-2,\cdots;\quad |\arg(\pm c)| < \pi].$$

Thus, using (89) and (90), we find

$$\lim_{n\to\infty} \frac{b^1_{n+1}\Psi_{n+1}}{b^1_n \Psi_n} = 1 - \frac{2}{n} + O\left(\frac{1}{n^2}\right) \quad \text{for} \quad \mathcal{U}_1. \tag{91a}$$

Therefore, by the Raabe test the series may converge for any finite value of z (the ratios (90) are valid if z is finite); however, we must exclude the point $z=0$ because in general the function $\Psi(a,c;y)$ goes to infinity at $y=0$. On the other side, from $\lim_{y\to\infty}\Psi(a,c;y){=}y^{-a}$, we find

$$\lim_{z\to\infty}\mathcal{U}_1(z) = e^{i\omega z}\, z^{B_1}\sum_{n=0}^{\infty} b^1_n, \qquad \left|\frac{b^1_{n+1}}{b^1_n}\right| \overset{(89)}{=} 1 + \frac{\operatorname{Re} B_1}{n} + O\left(\frac{1}{n^2}\right), \quad n \to \infty. \tag{91b}$$

Thus, according to the Raabe test again, the series $\sum b^1_n$ converges at $z=\infty$ if $\operatorname{Re}(B_1) < -1$.

4.2. Solutions for the confluent Heun equation

Now the solution $\mathcal{U}_1(z)$ for the spheroidal equation, given in (83a), is extended for any CHE. In fact, we can construct a group of solutions $\mathcal{U}_i(z)$ whose series coefficient b^i_n satisfy the relations (79). To this end, in the right-hand side of (83a) we perform the substitutions

$$z^{1+B_1}(z-1)^{1+B_1} \;\mapsto\; z^{1+\frac{B_1}{z_0}}(z-z_0)^{1-B_2-\frac{B_1}{z_0}}, \qquad \Psi(2+B_1, 2+B_1-n; -2i\omega z) \;\mapsto\; \Psi(\alpha, \beta-n; -2i\omega z),$$

where, in the substitutions of the first line we have used $1+B_1/z_0$ and $1-B_2-B_1/z_0$ because these are indicial exponents at $z=0$ and $z=z_0$, respectively. By using the properties of $\Psi(a,c;y)$ we find that $\alpha = 2+i\eta-B_2/2$ and $\beta = 2+B_1/z_0$ [38]. Thus, $\mathcal{U}_1$ is given

$$\mathcal{U}_1(z) = e^{i\omega z} z^{1+\frac{B_1}{z_0}}[z-z_0]^{1-B_2-\frac{B_1}{z_0}}\sum_{n=0}^{\infty} b^1_n\;\Psi\left(2+i\eta-\tfrac{B_2}{2}, 2+\tfrac{B_1}{z_0}-n; -2i\omega z\right), \qquad [i\eta - B_2/2 \neq -2,-3,\cdots] \tag{92a}$$

where the coefficients b^1_n satisfy the recurrence relations (79) with [38]

$$\alpha^1_n = -2i\omega z_0(n+1), \qquad \beta^1_n = n\left[n+1-B_2-\tfrac{2B_1}{z_0}+2i\omega z_0\right] + \left[i\omega z_0 - 1 - \tfrac{B_1}{z_0}\right]\left[2-B_2-\tfrac{B_1}{z_0}\right] + 2 - B_2 + B_3,$$
$$\gamma^1_n = -\left[n+i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}\right]\left[n+1-B_2-\tfrac{B_1}{z_0}\right]. \tag{92b}$$

By the transformations (18), $\mathcal{U}_1$ produces a group constituted by 16 solutions, $\mathcal{U}_i$. Eight of these can be constructed as

$$\begin{aligned}
&\mathcal{U}_1(z), && \mathcal{U}_2(z) = T_1\mathcal{U}_1(z), && \mathcal{U}_3(z) = T_2\mathcal{U}_2(z), && \mathbb{U}_4(z) = T_1\mathcal{U}_3(z);\\
&\mathcal{U}_5(z) = T_4\mathcal{U}_1(z), && \mathcal{U}_6(z) = T_4\mathcal{U}_2(z), && \mathcal{U}_7(z) = T_4\mathcal{U}_3(z), && \mathcal{U}_8(z) = T_4\mathcal{U}_4(z),
\end{aligned} \tag{93}$$

while the others result by the transformation T_3 which changes (η,ω) by $(-\eta,-\omega)$ in the above solutions. Thus,

$$\mathcal{U}_2(z) = e^{i\omega z}[z-z_0]^{1-B_2-\frac{B_1}{z_0}}\sum_{n=0}^{\infty} b^2_n\Psi\left(1+i\eta-\tfrac{B_2}{2}-\tfrac{B_1}{z_0}, -\tfrac{B_1}{z_0}-n; -2i\omega z\right), \qquad \left[i\eta - \tfrac{B_2}{2} - \tfrac{B_1}{z_0} \neq -1,-2,\cdots\right] \tag{94a}$$

where, in the recurrence relations (79) for b^2_n,

$$\alpha^2_n = -2i\omega z_0(n+1), \qquad \beta^2_n = n\left[n+3-B_2+2i\omega z_0)\right] + i\omega z_0\left[2-B_2-\tfrac{B_1}{z_0}\right] + 2 - B_2 + B_3],$$
$$\gamma^2_n = -\left[n+i\eta+1-\tfrac{B_2}{2}\right]\left[n+1-B_2-\tfrac{B_1}{z_0}\right. \tag{94b}$$

The third solution reads

$$\mathcal{U}_3(z) = e^{i\omega z}\sum_{n=0}^{\infty} b^3_n\;\Psi\left(i\eta+\tfrac{B_2}{2}, -\tfrac{B_1}{z_0}-n; -2i\omega z\right), \qquad \left[i\eta + \tfrac{B_2}{2} \neq 0,-1,\cdots,\right] \tag{95a}$$

with

$$\alpha_n^3 = -2i\omega z_0(n+1), \qquad \beta_n^3 = n\left[n+1+B_2+\tfrac{2B_1}{z_0}+2i\omega z_0\right]+\left[B_2+\tfrac{B_1}{z_0}\right]\left[1+\tfrac{B_1}{z_0}+i\omega z_0\right]+B_3,$$
$$\gamma_n^3 = -\left[n+i\eta+\tfrac{B_2}{2}+\tfrac{B_1}{z_0}\right]\left[n-1+B_2+\tfrac{B_1}{z_0}\right]. \tag{95b}$$

At last, we write

$$\mathcal{U}_4(z) = e^{i\omega z}z^{1+\frac{B_1}{z_0}}\sum_{n=0}^{\infty} b_n^4\Psi\left(1+i\eta+\tfrac{B_2}{2}+\tfrac{B_1}{z_0}, 2+\tfrac{B_1}{z_0}-n; -2i\omega z\right), \qquad [i\eta+B_2/2+B_1/z_0\neq -1,-2,\cdots] \tag{96a}$$

with

$$\alpha_n^4 = -2i\omega z_0(n+1), \qquad \beta_n^4 = n\left[n-1+B_2+2i\omega z_0\right]+i\omega z_0\left[B_2+\tfrac{B_1}{z_0}\right]+B_3,$$
$$\gamma_n^4 = -\left[n+i\eta-1+\tfrac{B_2}{2}\right]\left[n-1+B_2+\tfrac{B_1}{z_0}\right]. \tag{96b}$$

The relation (91a) is valid also for the present case, whereas (91b) is replaced by [38]

$$\lim_{z\to\infty}\mathcal{U}_1(z) = e^{i\omega z}\ z^{-i\eta-\frac{B_2}{2}}\sum_{n=0}^{\infty}b_n^1, \qquad \text{with } \left|\frac{b_{n+1}^1}{b_n^1}\right| = 1+\frac{1}{n}\text{Re}\left(i\eta-\frac{B_2}{2}\right)+O\left(\frac{1}{n^2}\right) \text{ when } n\to\infty. \tag{97}$$

Then, $\mathcal{U}_1$ converges at $z=\infty$ if $\text{Re}(i\eta-B_2/2)<-1$. By using the transformations as in (93), we find that the $\mathcal{U}_i$ converge for finite values of z, excepting possibly the points $z=0$ (if $i=1,2,3,4$) and $z=z_0$ (if $i=5,6,7,8$). According to the Raabe test, these $\mathcal{U}_i$ converge also at $z=\infty$ if

$$\begin{aligned}
&\text{Re}\left[i\eta-\tfrac{B_2}{2}+1\right]<0:\ \mathcal{U}_1,\,\mathcal{U}_5; \qquad &&\text{Re}\left[i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}\right]<0:\ \mathcal{U}_2,\,\mathcal{U}_8;\\
&\text{Re}\left[i\eta+\tfrac{B_2}{2}-1\right]<0:\ \mathcal{U}_3,\,\mathcal{U}_7; \qquad &&\text{Re}\left[i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}\right]<0:\ \mathcal{U}_4,\,\mathcal{U}_6.
\end{aligned} \tag{98}$$

4.3. The radial part of the two-center problem

Now we consider the equations of the two-center problem of quantum mechanics, as the one describing the electron of the ionized hydrogen molecule. Using Leaver‘s conventions [9], the wave function ψ of the time-independent Schrödinger equation for an electron in the field of two Coulombian centers has the form

$$\psi = e^{im\varphi}\ \bar{R}(\lambda)\ \bar{S}(\mu), \qquad \lambda=\frac{r_1+r_2}{2a}, \qquad \mu=\frac{r_1-r_2}{2a}, \qquad m=0,\ \pm1,\ \pm2,\ \cdots, \tag{99}$$

where r_1 and r_2 are the distances from the electron to the two nuclei, and $2a$ the intercenter distance. By the definitions

$$\begin{aligned}
&\text{S}(z)=\bar{S}(\lambda)=z^{\frac{m}{2}}(2-z)^{\frac{m}{2}}U^-(z), \qquad z=\mu+1, \qquad [0\leq z\leq 2],\\
&\text{R}(z)=\bar{R}(\mu)=z^{\frac{m}{2}}(z-2)^{\frac{m}{2}}U^+(z), \qquad z=\lambda+1, \qquad [z\geq 2],
\end{aligned} \tag{100}$$

Leaver obtained CHEs in the form (1) for $U^{\pm}$, with the parameters $\left(\eta^{\pm}\text{ for }\eta\ ,\ B_3^{\pm}\text{ for }B_3\right)$

$$z_0=2, \qquad \omega^2=2a^2E, \qquad \omega\eta^{\pm}=-a(N_1\pm N_2), \qquad B_1=-2(m+1), \qquad B_2=2(m+1),$$
$$B_3^{\pm}=\omega^2+2a(N_1\pm N_2)+m(m+1)-A_{lm}. \tag{101}$$

where A_{lm} is a separation constant, and N_1 and N_2 are the charges on the two nuclei. Thus, there are two CHEs, one for the “angular” coordinate μ and one for the “radial” coordinate λ. Each CHE is associated with a characteristic equation (81) which determines the possible values of the constants A_{lm} and E.

Now we consider $\text{R}(z)$, the radial solution given in (100). For bound states ($E<0$) we take

$$i\omega=-a\sqrt{2|E|} \quad\Rightarrow\quad i\eta=i\eta^+=-(N_1+N_2)/\sqrt{2|E|} \tag{102}$$

in order to assure that the factor $\exp(i\omega z)$ remains finite when $z\to\infty$. Then, if $|E|$ is finite,

$$\text{Re}\left[i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}\right]=\text{Re}\left[i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}\right]=-\tfrac{N_1+N_2}{\sqrt{2|E|}}<0,$$

and, consequently, four of the solutions listed in (98) converge at $z=\infty$. To get wavefunctions bounded also at $z=2$, we select $\mathcal{U}_2$ if $m\leq 0$ and $\mathcal{U}_4$ if $m\geq 0$. Thus, we find

$$\mathrm{R}(z)=e^{-a\sqrt{2|E|}\,z}\,z^{-\frac{|m|}{2}}\,(z-2)^{\frac{|m|}{2}}\sum_{n=0}^{\infty}\,b_n^2\,\Psi\left(1-\tfrac{N_1+N_2}{\sqrt{2|E|}},1-|m|-n;a\sqrt{8|E|}\,z\right) \tag{103a}$$

where the coefficients b_n^2 satisfy the relations (79) with

$$\alpha_n^2=\sqrt{8|E|}\,a(n+1),\qquad \beta_n^2=n\left[n+1+2|m|-2a\sqrt{8|E|}\right]+\left[|m|+1\right]\left[|m|-a\sqrt{8|E|}\right]+$$
$$2a\left[N_1+N_2-a|E|\right]-A_{lm},\qquad \gamma_n^2=-\left[n+|m|\right]\left[n+|m|-\tfrac{N_1+N_2}{\sqrt{2|E|}}\right]. \tag{103b}$$

The expansion (103a) holds only if

$$(N_1+N_2)/\sqrt{2|E|}\neq l+1,\qquad [l=0,2,\cdots] \tag{104}$$

a condition which assures that $\Psi(a,b;y)$ is not a polynomial of degree l in y.

The condition (104) is also required by the Jaffé expansions. In effect, by using the solutions U_1^J (if $m\geq 0$) and U_2^J (if $m\leq 0$) given in Eqs (70a) and (73a), respectively, we find

$$\mathrm{R}^J(z)=e^{-a\sqrt{2|E|}\,z}\,z^{-1-\frac{|m|}{2}+\frac{N_1+N_2}{\sqrt{2|E|}}}\,(z-2)^{\frac{|m|}{2}}\sum_{n=0}^{\infty}a_n^1\left(\frac{z-2}{z}\right)^n, \tag{105a}$$

where the recurrence relations for a_n^1 have the form (79) with

$$\alpha_n^1=(n+1)(n+|m|+1),\qquad \beta_n^1=-2n\left[n+1+|m|+a\sqrt{8|E|}-\tfrac{N_1+N_2}{\sqrt{2|E|}}\right]+\left[|m|+1\right]\left[\tfrac{N_1+N_2}{\sqrt{2|E|}}-a\sqrt{8|E|}-1\right]+$$
$$2a\left[N_1+N_2-a|E|\right]-A_{lm},\qquad \gamma_n^1=-\left[n+|m|-\tfrac{N_1+N_2}{\sqrt{2|E|}}\right]\left[n-\tfrac{N_1+N_2}{\sqrt{2|E|}}\right]. \tag{105b}$$

Thus, $\gamma_{l+1}=0$ if $(N_1+N_2)/\sqrt{2|E|}=l+1$ and, then, $\mathrm{R}^J(z)$ becomes a finite-series solution with $0\leq n\leq l$, as stated after Eq. (81). In this case, the constant A_{lm} would be determined from the characteristic equation associated with the recurrence relations for a_n^1. However, if E and A_{lm} are both determined from the radial solution, we cannot satisfy the characteristic equation corresponding to the angular solutions (these are usually given by series where the summation begins at $n=0$ and, so, present recurrence relations having the form (80)). Therefore, also for the Jaffé solutions it is necessary that $(N_1+N_2)/\sqrt{2|E|}\neq l+1$. The same is true respecting Hylleraas' expansions in series of Laguerre polynomials [9, 39].

The convergence of solution (103a) follows immediately from the Raabe test. As to the Jaffé solution (105a), we have to examine its behavior at $z=\infty$. By using (101) together with (102), the expressions (74) imply that

$$\lim_{z\to\infty}\mathrm{R}^J(z)=e^{-a\sqrt{2|E|}\,z}\,z^{\frac{N_1+N_2}{\sqrt{2|E|}}-1}\sum_{n=0}^{\infty}a_n^1,\quad \frac{a_{n+1}^1}{a_n^1}=1-\frac{1}{n}\left[\tfrac{3}{4}+2\sqrt{an\sqrt{2|E|}}-2a\sqrt{2|E|}+\tfrac{N_1+N_2}{\sqrt{2|E|}}\right]+O\left(\tfrac{1}{n^2}\right). \tag{106}$$

Then, by a convenient choice of n, the constant A which appears in (82b) becomes less than -1 and so, by the Raabe test, the solution converges at $z=\infty$.

5. Final remarks: conclusion and open problems

By inserting a suitable weight function $w(z,t)$ into the integral relation (13) we have found the kernel equation (15) where the differential operators L_z and L_t have the same functional dependence as the operator of the CHE (11), a fact which allows to get transformations of the kernels by examining the known transformations of the solutions for the CHE. As mentioned, this is an extension of a similar correspondence found in 2011 for the general Heun equation (HE) [1].

Actually, in 1942 Erdélyi used the appropriate weight function for the HE but he could not infer how to transform the kernels because the transformations of the HE were fully established only in 2007 [3]. On the other side, transformations of confluent Heun equations are known since 1978 [25, 26] but have not been applied to transform kernels – see, for example, references [4, 6, 7, 27]. In the present study we have considered transformations of kernels of the CHE, where the initial kernels (to be transformed) come from kernels of the HE by a process of confluence [1]; however, for the sake of completeness, in section 2 we have reobtained them by solving the kernel equation.

By separation of variables we have found two groups of kernels presenting an arbitrary constant of separation. One group, with products of two confluent hypergeometric functions, includes some particular kernels already known in the literature [27]; the other group, with products of confluent hypergeometric functions and Gauss hypergeometric functions, is new as far as we known. For particular values of the constant of separation we have obtained three groups given by produtc of elementary functions with one special function: this is represented by confluent hypergeometric functions (two groups) and by Gauss hypergeometric functions (one group).

In section 3 we have found some integral transformations among known solutions of the confluent Heun equations. We have used two singularities as endpoints of integration and have supposed that the solutions to be transformed are convergent at both endpoints (this assures that the bilinear concomitants vanish there). If the solutions are modified by the rules (18), the kernels must be modified by the rules (20). This emphasizes that the correspondence between the transformations of the Heun equations and of the respective kernels are connected parts of the transformation theory.

The applications of section 3 simply relate known solutions without affording new ones. In contrast, in section 4, by means of an integral transformation we have obtained a new solution for the spheroidal wave equation, which in turn leads to a group of new solutions for the CHE. We have seen that these solutions may be used to compute the radial part of the wavefunctions for bound states of hydrogen moleculelike ions and, by this reason, can play the role of the expansions in series of Laguerre polynomials proposed by Hylleraas in 1931 [39] and the Jaffé power-series solutions which have been used from 1934 [22] up to now [41].

It is possible to realize further properties of the solutions by considering other problems, as the Lorentzian model of a quantum two-state system given by Ishkhanyan and Gregoryan [11]. This is ruled by a CHE with $z = (1+it)/2$ and $z_0 = 1$, where t denotes the time. According to the authors, for certain values of a parameter R, the problem admits finite-series solutions which are bounded for any admissible value of t and assure that the system returns to the initial state after the interaction. By using the solutions of section 4.2, we have verified that the previous statement is correct; it seems that no other known solution of the CHE permits to prove the statement. In addition, since in this case $|z| \geq 1/2$, it would be interesting to check if the solutions of section 4.2 lead to *infinite-series* solutions suitable for some range of the parameter R (the Hylleraas and Jaffé solutions do not converge for $|z| < 1$).

We have omitted details concerning the derivation of the new solutions of the CHE. In addition, the solutions must be improved as follows: (i) by considering also expansions in series of regular confluent hypergeometric functions to get solution valid in the neighborhood of $z = 0$ [38], (ii) by using the Whittaker-Ince limit as in Ref. [18] to obtain solutions for the RCHE (2), (iii) by inserting a "characteristic" parameter ν and letting that the series summation runs from minus to plus infinite [38] (two-sided series) in order to obtain solutions for a CHE without free parameters.

As mentioned in the first section, along with the RCHE we have disregarded two other equations which are associated with the CHE by formal limits. These are the double-confluent Heun equation (DHE) and the reduced DHE (RDHE) which appear when we allow that $z_0 \to 0$ in the CHE and RCHE, respectively, that is,

$$\text{DHE}: \qquad z^2\frac{d^2U}{dz^2} + (B_1 + B_2 z)\frac{dU}{dz} + \left(B_3 - 2\eta\omega z + \omega^2 z^2\right)U = 0, \qquad [B_1 \neq 0,\ \omega \neq 0]$$

$$\text{RDHE}: \qquad z^2\frac{d^2U}{dz^2} + (B_1 + B_2 z)\frac{dU}{dz} + (B_3 + qz)\,U = 0, \qquad [q \neq 0,\ B_1 \neq 0]$$

where now $z = 0$ and $z = \infty$ are irregular singularities. At $z = \infty$ the behaviour is again given by Eq. (5), that is,

$$\lim_{z\to\infty} U(z) \sim e^{\pm i\omega z}\, z^{\mp i\eta-(B_2/2)} \ \text{ for the DHE (1)}, \qquad \lim_{z\to\infty} U(z) \sim e^{\pm 2i\sqrt{qz}}\, z^{(1/4)-(B_2/2)} \ \text{for the RDHE (2)},$$

while at $z = 0$ the normal Thomé solutions affords [18]

$$\lim_{z\to 0} U(z) \sim 1 \qquad \text{or} \qquad \lim_{z\to 0} U(z) \sim e^{B_1/z} z^{2-B_2} \qquad \text{for DHE and RDCE.}$$

We show elsewhere [42] that the application of the Whitakker-Ince limit (3) and the Leaver limit ($z_0 \to 0$) on kernels of the CHE lead to new kernels for the RCHE, DHE and RDHE. This is in accordance with a previous conjecture [1] but, by integrating the kernel equations, we also find kernels which are not connected with known kernels of the CHE: for the RCHE we obtain a group of kernels expressed by products of Bessel and hypergeometric functions, while for the DHE and RDHE we obtain kernels given in terms of elementary functions. Therefore, in addition to the use of limiting procedures, we can solve the kernel equations to generate other kernels for the RCHE, DHE and RDHE.

Appendix A: Hypergeometric functions

The regular and irregular confluent hypergeometric functions are denoted by $\Phi(a, c; u)$ and $\Psi(a, c; u)$, respectively. They satisfy the confluent hypergeometric equation [37]

$$u\frac{d^2\varphi(u)}{du^2} + (c-u)\frac{d\varphi(u)}{du} - a\varphi(u) = 0 \tag{A.1}$$

which admits the solutions

$$\varphi^1(u) = \Phi(a, c; u), \quad \varphi^2(u) = e^u u^{1-c}\Phi(1-a, 2-c; -u), \quad \varphi^3(u) = \Psi(a, c; u), \quad \varphi^4(u) = e^u u^{1-c}\Psi(1-a, 2-c; -u). \tag{A.2}$$

All of them are defined and distinct only if c is not an integer. Alternative forms for these solutions follow from the relations

$$\Phi(a,c;u)=e^{u}\Phi(c-a,c;-u),\qquad \Psi(a,c;u)=u^{1-c}\Psi(1+a-c,2-c;u). \tag{A.3}$$

On the other side, solutions for the (Gauss) hypergeometric equation [37],

$$u(1-u)\frac{d^2F}{du^2}+\left[c-(a+b+1)u\right]\frac{dF}{du}-abF=0, \tag{A.4}$$

are given by hypergeometric functions $F(a,b;c;u)=F(b,a;c;u)$. In fact, in the vicinity of the singular points 0, 1 and ∞, the formal solutions for the hypergeometric equation (A.4) are, respectively,

$$F^1(u)=F\left(a,b;c;u\right),\qquad F^2(u)=u^{1-c}F\left(a+1-c,b+1-c;2-c;u\right); \tag{A.5}$$

$$F^3(u)=F\left(a,b;a+b+1-c;1-u\right),\qquad F^4(u)=(1-u)^{c-a-b}\,F\left(c-a,c-b;1+c-a-b;1-u\right); \tag{A.6}$$

$$F^5u)=u^{-a}F\left(a,a+1-c;a+1-b;\tfrac{1}{u}\right),\qquad F^6(u)=u^{-\mathrm{b}}F\left(b+1-c,b;b+1-a;\tfrac{1}{u}\right). \tag{A.7}$$

Each of these may be written in four forms by using the relations

$$F(a,b;c;u)=(1-u)^{c-a-b}F(c-a,c-b;c;u),\qquad F(a,b;c;u)=(1-u)^{-a}F\left[a,c-b;c;u/(u-1)\right]. \tag{A.8}$$

On the other side, the usual form for the Bessel equation is [5]

$$y^2\frac{d^2Z(y)}{dy^2}+y\frac{dZ(y)}{dy}+\left[y^2-\alpha^2\right]Z(y)=0. \tag{A.9}$$

The solutions for this equation are denoted by $Z_{\alpha}^{(j)}(y)$ according as [5, 34]

$$Z_{\alpha}^{(1)}(y)=J_{\alpha}(y),\qquad Z_{\alpha}^{(2)}(y)=Y_{\alpha}(y),\qquad Z_{\alpha}^{(3)}(y)=H_{\alpha}^{(1)}(y),\qquad Z_{\alpha}^{(4)}(y)=H_{\alpha}^{(2)}(y) \tag{A.10}$$

where $J_{\alpha}(y)$ and $Y_{\alpha}(y)$ are the Bessel functions of the first and second kind, respectively; $H_{\alpha}^{(1)}(y)$ and $H_{\alpha}^{(2)}(y)$ are the first and the second Hankel functions. There are formulas connecting these functions ([5]). For example,

$$Y_{\alpha}=\frac{1}{2i}\left[H_{\alpha}^{(1)}-H_{\alpha}^{(2)}\right]=\frac{\cos(\alpha\pi)J_{\alpha}-J_{-\alpha}}{\sin(\alpha\pi)}. \tag{A.11}$$

Bessel and confluent hypergeometric functions are connected by [37]

$$\Phi\left(\alpha+\tfrac{1}{2},2\alpha+1;-2iy\right)=\Gamma(\alpha+1)\,e^{-iy}\left(\tfrac{y}{2}\right)^{-\alpha}J_{\alpha}(y),\qquad \Psi\left(\alpha+\tfrac{1}{2},2\alpha+1;-2iy\right)=\tfrac{i\sqrt{\pi}}{2}e^{-i(y-\alpha\pi)}\left(2y\right)^{-\alpha}H_{\alpha}^{(1)}(y),$$
$$\Psi\left(\alpha+\tfrac{1}{2},2\alpha+1;2iy\right)=-\tfrac{i\sqrt{\pi}}{2}e^{i(y-\alpha\pi)}\left(2y\right)^{-\alpha}H_{\alpha}^{(2)}(y). \tag{A.12}$$

Appendix B: Wilson's asymptotic expansions for the CHE

Such solutions were considered in 1928 by Wilson [10]. Actually they are given by 8 asymptotic Thomé expansions [17] which we denote by $\mathcal{W}_i(z)$ $(i=1,2,3,4)$ and $T_3\mathcal{W}_i(z)$. For the CHE in the form (1) we find

$$\mathcal{W}_1(z)=e^{i\omega z}\;z^{-i\eta-\frac{B_2}{2}}\sum_{n=0}^{\infty}a_n^1z^{-n} \tag{B.1}$$

where the coefficients a_n^1 satisfy the three-term recurrence relations $(a_{-1}^1=0)$

$$2i\omega(n+1)a_{n+1}^1-\Big[n\left(n+1+2i\eta+2i\omega z_0\right)+i\omega z_0\Big(B_2+\tfrac{B_1}{z_0}\Big)+B_3+\Big(\tfrac{B_2}{2}+i\eta\Big)\left(1+i\eta-\tfrac{B_2}{2}\right)\Big]a_n^1$$
$$+z_0\Big(n+i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}\Big)\left(n+i\eta+\tfrac{B_2}{2}-1\right)a_{n-1}^1=0. \tag{B.2}$$

For the other solutions we take

$$\mathcal{W}_2(z)=T_2\mathcal{W}_1(z),\qquad \mathcal{W}_3(z)=T_4\mathcal{W}_1(z),\qquad \mathcal{W}_4(z)=T_4\mathcal{W}_2(z)=T_1\mathcal{W}_3(z) \tag{B.3}$$

and $\mathcal{W}_{i+4} = T_3\mathcal{W}_i$ $(i = 1, \cdots 4)$. Thus, from the first solution we get

$$\mathcal{W}_2(z) = e^{i\omega z}\ (z-z_0)^{1-B_2-\frac{B_1}{z_0}}\ z^{-i\eta-1+\frac{B_1}{z_0}+\frac{B_2}{2}} \sum_{n=0}^{\infty} a_n^2 z^{-n}, \qquad \text{where} \tag{B.4}$$

$$2i\omega(n+1)a_{n+1}^2 - \Big[n\,(n+1+2i\eta+2i\omega z_0) + i\omega z_0\Big(2-B_2-\tfrac{B_1}{z_0}\Big) + B_3 + \Big(\tfrac{B_2}{2}+i\eta\Big)\Big(1+i\eta-\tfrac{B_2}{2}\Big)\Big]a_n^2$$
$$+z_0\Big(n+i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}\Big)\Big(n+i\eta+1-\tfrac{B_2}{2}\Big)a_{n-1}^2 = 0. \tag{B.5}$$

This $\mathcal{W}_2(z)$ is the only solution relevant for section 5. For this reason we omit the other solutions.

By the D'Alembert test the solutions $\mathcal{W}_1(z)$ and $\mathcal{W}_2(z)$ converge for $|z| > |z_0|$, whereas $\mathcal{W}_3(z)$ and $\mathcal{W}_4(z)$ converge for $|z-z_0| > |z_0|$. However, by the Raabe test they converge also at $|z| = |z_0|$ and $|z-z_0| = |z_0|$ provided that

$$|z| \geq |z_0| \text{ if } \begin{cases} \text{Re}\left[B_2+\frac{B_1}{z_0}\right] < 1 \text{ in } \mathcal{W}_1(z), \\ \text{Re}\left[B_2+\frac{B_1}{z_0}\right] > 1 \text{ in } \mathcal{W}_2(z); \end{cases} \qquad |z-z_0| \geq |z_0| \text{ if } \begin{cases} \text{Re}\left[\frac{B_1}{z_0}\right] > -1 \text{ in } \mathcal{W}_3(z), \\ \text{Re}\left[\frac{B_1}{z_0}\right] < -1 \text{ in } \mathcal{W}_4(z), \end{cases} \tag{B.6}$$

where the restrictions on parameters of the equation are necessary only to assure convergence at $|z| = |z_0|$ or $|z-z_0| = |z_0|$.

The above regions of convergence suppose the minimal solutions for the series coefficients [29, 30]. In the following we consider only the series which appears in $W_1(z)$, the convergence for the other solutions being obtained by using the transformations as indicated above. Thus, when $n \to \infty$ in $W_1(z)$ we have

$$2i\omega\frac{a_{n+1}^1}{a_n^1} - (n+1+2i\eta+2i\omega z_0) + z_0\left(n+2i\eta+B_2+\tfrac{B_1}{z_0}-1\right)\frac{a_{n-1}^1}{a_n^1} = 0$$

whose minimal solution for a_{n+1}^1/a_n^1 when $n \to \infty$ satisfies

$$\frac{a_{n+1}^1}{a_n^1} \sim z_0\left[1+\frac{1}{n}\left(B_2+\frac{B_1}{z_0}-2\right)\right] \quad \Rightarrow \quad \frac{a_{n-1}^1}{a_n^1} \sim \frac{1}{z_0}\left[1-\frac{1}{n}\left(B_2+\frac{B_1}{z_0}-2\right)\right].$$

Thence, when $n \to \infty$,

$$\frac{a_{n+1}^1 z^{-n-1}}{a_n^1 z^{-n}} \sim \frac{z_0}{z}\left[1+\frac{1}{n}\left(B_2+\frac{B_1}{z_0}-2\right)\right] \Rightarrow \left|\frac{a_{n+1}^1 z^{-n-1}}{a_n^1 z^{-n}}\right| \sim \frac{|z_0|}{|z|}\left[1+\frac{1}{n}\text{Re}\left(B_2+\frac{B_1}{z_0}-2\right)\right].$$

So, by the D'Alambert test the series converge absolutely for $|z| > |z_0|$. However, by the Raabe test, the series converge even for $|z| = |z_0|$ provided that $\text{Re}\,[B_2 + (B_1/z_0)] < 1$.

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