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Symmetries of the Schrödinger Equation and Algebra/Superalgebra Duality*

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Abstract. Some key features of the symmetries of the Schrödinger equation that are common to a much broader class of dynamical systems (some under construction) are illustrated. I discuss the algebra/superalgebra duality involving first and second-order differential operators. It provides different viewpoints for the spectrum-generating subalgebras. The representation-dependent notion of on-shell symmetry is introduced. The difference in associating the time-derivative symmetry operator with either a root or a Cartan generator of the $sl(2)$ subalgebra is discussed. In application to one-dimensional Lagrangian superconformal sigma-models it implies superconformal actions which are either supersymmetric or non-supersymmetric.

1. Introduction

Focusing on simple examples, I illustrate some general features that apply to a vast class of theories including non-relativistic Schrödinger equations in $1 + d$ dimensions, the more general invariant equations associated with ℓ-conformal Galilei algebras, the $D = 1$ Lagrangian (super)conformal models (one-dimensional sigma-models), together with several extended supersymmetric versions of these theories.

This broad class of dynamical equations share the properties that are discussed here. For simplicity I review these features in application to the Schrödinger equation in $1 + 1$-dimension with the three choices of the potential (constant, linear and quadratic) which induce the Schrödinger algebra [1] as the symmetry algebra of first-order differential operators. On the sigma-model side, I illustrate the simplest $osp(1|2)$-invariant case [2] with one bosonic and one fermionic time-dependent field.

The key issues that I am pointing out are the following. We have that 2 of the 6 first-order Schrödinger operators, the ones generating the Heisenberg-Lie algebra, have a natural half-integer grading. By taking their anticommutators one can construct 3 second-order differential operators. The total number of 9 operators so constructed define a finite closed structure, which can be either Lie-algebraic (by taking their commutators) or super-Lie algebraic, in terms of the (anti)-commutators which respect the $\mathbf{Z}_2$ grading. The fact that two compatible structures can be defined for the same set of operators is referred to as "algebra/superalgebra" duality (it is worth pointing out that the use of higher-order differential operators entering a universal enveloping algebra, together with their relations with higher spin theories, was also advocated in [3]; however, an important feature is that finite Lie or super-Lie algebras are recovered when we include only second-order differential operators of special type).

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The operator Ω which defines the Schrödinger dynamics, being a second-order differential operator, does not belong to the Schrödinger algebra, but to the enlarged algebra (either its Lie-algebraic or its super-Lie algebraic version). The operator Ω does not commute with the first-order Schrödinger operators. On the other hand, *in the given representation*, the *on-shell* closure of the algebra is guaranteed once taking into account the $\Omega\Psi = 0$ dynamical equation (see Section **3**).

The algebra/superalgebra duality, applied to the spectrum generating subalgebras, is a reformulation of the celebrated result by Wigner in [4]. Essentially, for the harmonic oscillator, the Lie algebra side is based on the Heisenberg subalgebra and the construction of the eigenstates starts with the Fock's vacuum condition. In the dual super-Lie algebraic point of view, the same conditions are obtained from a highest weight representation of the $osp(1|2)$ spectrum-generating superalgebra.

For the free particle case, the operator Ω and the time-derivative operator belong to the grading 1 sector of the algebra (the time-derivative operator is a root of the $sl(2)$ subalgebra). In the harmonic oscillator case Ω and the time-derivative operator belong to the grading 0 sector of the algebra (the time-derivative operator is the Cartan of the $sl(2)$ subalgebra). This is the key difference which implies the continuum spectrum for the free particle and the discrete spectrum of the harmonic oscillator. In [2] a detailed analysis of one-dimensional (super)conformal models based on parabolic D-module reps (the time-derivative operator being associated with the positive $sl(2)$ root) versus hyperbolic/trigonometric D-module reps (the time-derivative operator being associated with the $sl(2)$ Cartan generator) was given. In the hyperbolic/trigonometric case extra potential terms are allowed. On the other hand, when invariance under superconformal algebras are considered, the difference is even more meaningful. In the parabolic case the dynamical system is both superconformal and supersymmetric. In the hyperbolic/trigonometric case the dynamical system is superconformal but not supersymmetric. This point is illustrated in Section **5**.

In the Conclusions I point out how the features here discussed enter more general dynamical systems. These features can be used to both identify and solve the invariant dynamics of this larger class of theories.

2. Schrödinger's equations in $d = 1$

We consider the differential operator Ω in $1+1$ dimensions:

$$\Omega \quad = \quad \partial_t + a\partial_x^2 - aV(x). \tag{1}$$

$V(x)$ is a potential term. If a is imaginary, the operator Ω defines the dynamics of the $1+1$ Schrödinger equation, written as

$$\Omega\Psi(x,t) \quad = \quad 0. \tag{2}$$

If a is real the above equation is the heat equation in the presence of the potential $V(x)$.

With standard methods, see [1], we can prove that, for three special cases of the potential, the invariance algebra of the equation (2), in terms of first-order differential operators, is given by the Schrödinger algebra ($l = \frac{1}{2}$ conformal Galilei algebra) in the presence of a central charge.

Indeed, the invariant condition $\Omega\delta\Psi(x,t) = 0$,
for $\delta\Psi(x,t) = f(x,t)\Psi_t(x,t) + g(x,t)\Psi_x(x,t) + h(x,t)\Psi(x,t)$,
leads to the set of equations

$$\begin{aligned} f_x &= 0, \\ f_t - 2g_x + af_{xx} &= 0, \end{aligned}$$

$$
\begin{aligned}
g_t + a\,(2aVf_x + 2h_x + g_{xx}) &= 0,\\
agV_x + h_t + 2a^2V_xf_x + aV(f_t + af_{xx}) + ah_{xx} &= 0.
\end{aligned}
\qquad (3)
$$

The three special cases correspond to the solutions with maximal number of generators:
i) the constant potential $V(x) = 0$ (free particle case),
ii) the linear potential $V(x) = \omega x$ and
iii) the quadratic potential $V(x) = \nu^2 x^2$ (harmonic oscillator case).

In all the above cases, without loss of generality, the potential can be shifted by a constant u, $V(x) \to V(x) + u$, via a similarity transformation $\Omega \to e^{aut}\Omega e^{-aut}$.

In the quadratic case a linear term in the potential can always be eliminated by shifting the space coordinate, so that $x \to x + b$.

For all the other potentials, the invariance algebra of eq. (2) is smaller than the Schrödinger algebra.

The quadratic and constant realizations can be mutually recovered, see [5], from similarity transformations coupled with change of space and time coordinates. The existence of this set of transformations, however, is not essential for the following discussion.

A compatible assignment of the dimensions is

$$
[t] = -1, \quad [x] = -\tfrac{1}{2}, \quad [\omega] = \tfrac{3}{2}, \quad [\nu] = 1, \quad ([\Omega] = 1). \qquad (4)
$$

The Schrödinger algebra is given by the 6 generators $z_{\pm 1}, z_0, w_\pm, c$. Their dimensions are

$$
[z_{\pm 1}] = \pm 1, \quad [w_\pm] = \pm\tfrac{1}{2}, \quad [z_0] = [c] = 0. \qquad (5)
$$

The generators $z_{\pm 1}, z_0$ close an $sl(2)$ subalgebra with z_0 as the Cartan element. The generator c is the central charge.

The three explicit D-module reps are given by

i) $V(x) = 0$, constant potential case,

$$
\begin{aligned}
z_{+1} &= \partial_t,\\
z_0 &= t\partial_t + \frac{1}{2}x\partial_x + \frac{1}{4},\\
z_{-1} &= t^2\partial_t + tx\partial_x - \frac{x^2}{4a} + \frac{1}{2}t,\\
w_+ &= \partial_x,\\
w_- &= t\partial_x - \frac{x}{2a},\\
c &= 1;
\end{aligned}
\qquad (6)
$$

ii) $V(x) = \omega x$, linear potential case,

$$
\begin{aligned}
z_{+1} &= t^2\partial_t + (a^2\omega t^3 + tx)\partial_x + (\frac{t}{2} - \frac{1}{4}a^3\omega^2 t^4 - \frac{3}{2}a\omega t^2 x - \frac{x^2}{4a}),\\
z_0 &= -t\partial_t - (\frac{3}{2}a^2\omega t^2 + \frac{x}{2})\partial_x + (\frac{1}{2}a^3\omega^3 t^3 + \frac{3}{2}a\omega t x - \frac{1}{4}),\\
z_{-1} &= \partial_t + 2a^2\omega t\partial_x - a^3\omega^2 t^2 - a\omega x,\\
w_+ &= -t\partial_x + \frac{x}{2a} + \frac{1}{2}a\omega t^2,\\
w_- &= \partial_x - a\omega t,\\
c &= 1;
\end{aligned}
\qquad (7)
$$

iii) $V(x) = \nu^2 x^2$, quadratic potential case

$$\begin{aligned} z_{+1} &= e^{4a\nu t}\left(\partial_t + 2a\nu x\partial_x + a\nu - 2a\nu^2 x^2\right), \\ z_0 &= \partial_t, \\ z_{-1} &= e^{-4a\nu t}\left(\partial_t - 2a\nu x\partial_x - a\nu - 2a\nu^2 x^2\right), \\ w_+ &= e^{2a\nu t}\left(\partial_x - \nu x\right), \\ w_- &= e^{-2a\nu t}\left(\partial_x + \nu x\right), \\ c &= 1. \end{aligned} \tag{8}$$

In the quadratic case the non-vanishing commutation relations are given by

$$\begin{aligned} [z_1, z_{-1}] &= -8a\nu z_0, \\ [z_0, z_{\pm 1}] &= \pm 4a\nu z_{\pm 1}, \\ [z_{\pm 1}, w_{\mp}] &= \mp 4a\nu w_{\pm}, \\ [z_0, w_{\pm}] &= \pm 2a\nu w_{\pm}, \\ [w_+, w_-] &= 2\nu c. \end{aligned} \tag{9}$$

In the constant and linear cases the commutation relations are obtained from the above formulas with the substitution $\nu = -\frac{1}{4a}$.

The above equations give the structure constants of the one-dimensional, centrally extended, Schrödinger algebra.

The generator z_0 defines the grading corresponding to the (5) dimensions.

One should note that the Hamiltonian (i.e., the time-derivative operator), corresponds to a grading 1 generator (a root generator of the $sl(2)$ subalgebra) in the free particle case and to a grading 0 generator (the Cartan generator of the $sl(2)$ subalgebra) for the harmonic oscillator case.

The generators $w_\pm$ have half-integer grading with respect to the grading defined by z_0.

3. The algebra/superalgebra symmetry with higher differential operators

The second-order differential operators w_1, w_0, w_{-1}, obtained by taking the anticommutators of $w_\pm$, can be constructed:

$$\begin{aligned} w_{+1} &= \{w_+, w_+\}, \\ w_0 &= \{w_+, w_-\}, \\ w_{-1} &= \{w_-, w_-\}. \end{aligned} \tag{10}$$

Their explicit form, in the three respective cases above, is given by

i) the constant case,

$$\begin{aligned} w_{+1} &= 2{\partial_x}^2, \\ w_0 &= 2t{\partial_x}^2 - \frac{x}{a}\partial_x - \frac{1}{2a}, \\ w_{-1} &= 2t^2{\partial_x}^2 - \frac{2tx}{a}\partial_x + \frac{x^2}{2a^2} - \frac{t}{a}; \end{aligned} \tag{11}$$

ii) the linear case,

$$\begin{aligned} w_1 &= \frac{1}{2a^2}\left(4a^2t^2{\partial_x}^2 - 4at(a^2t^2\omega + x)\partial_x + (a^4t^4\omega^2 - 2at + 2a^2t^2\omega x + x^2)\right), \\ w_0 &= \frac{1}{2a}\left(4at{\partial_x}^2 - (6a^2t^2\omega + 2x)\partial_x + (2a^3t^3\omega^2 + 2at\omega x - 1)\right), \\ w_{-1} &= 2{\partial_x}^2 - 4at\omega\partial_x + 2a^2t^2\omega^2; \end{aligned} \tag{12}$$

iii) the quadratic case,

$$
\begin{aligned}
w_{+1} &= e^{4at\nu}\left(2{\partial_x}^2 - 4\nu x\partial_x - 2\nu + 2\nu^2x^2\right), \\
w_0 &= 2{\partial_x}^2 - 2\nu^2x^2, \\
w_{-1} &= e^{-4at\nu}\left(2{\partial_x}^2 + 4\nu x\partial_x + 2\nu + 2\nu^2x^2\right).
\end{aligned} \tag{13}
$$

In all these cases we have two consistent closed structures which can be defined on the same set of differential operators, namely
1) the non-simple Lie algebra $eSch$ (the enlarged Schrödinger algebra), presented by the 9 generators $\{z_{\pm1}, z_0, w_\pm, w_{\pm1}, w_0, c\}$ and
2) the Lie superalgebra $sSch$ (the enlarged Schrödinger superalgebra) $sSch = S_0 \oplus S_1$, with 7 even generators $(z_{\pm1}, z_0, w_{\pm1}, w_0, c \in S_0)$ and 2 odd generators $(w_\pm \in S_1)$.

In all three cases (in the quadratic case for $\nu = -\frac{1}{4a}$), the extra non-vanishing structure constants besides (9) are given, for the $eSch$ algebra, by

$$
\begin{aligned}
[z_0, w_{\pm1}] &= \mp w_{\pm1}, \\
[z_{\pm1}, w_0] &= \pm w_{\pm1}, \\
[z_{\pm1}, w_{\mp1}] &= \pm 2w_0, \\
[w_\pm, w_0] &= \mp\frac{1}{a}w_\pm, \\
[w_\pm, w_{\mp1}] &= \mp\frac{2}{a}w_\mp, \\
[w_0, w_{\pm1}] &= \pm\frac{2}{a}w_{\pm1}, \\
[w_1, w_{-1}] &= -\frac{4}{a}w_0.
\end{aligned} \tag{14}
$$

For the $sSch$ superalgebra we have the anti-commutators

$$
\{w_+, w_+\} = w_{+1}, \quad \{w_+, w_-\} = w_0, \quad \{w_-, w_-\} = w_{-1} \tag{15}
$$

(one should note that the Heisenberg-Lie algebra $[w_+, w_-] = 2\nu c$ is not a subalgebra of the $sSch$ superalgebra).

In all three cases (constant, linear and quadratic), the second-order differential operator Ω is a generator belonging to the enlarged Schrödinger algebra (either $eSch$ or $sSch$). We have the following identifications:
i) constant case,

$$
\Omega = z_{+1} + \frac{1}{2}aw_{+1}, \tag{16}
$$

ii) linear case,

$$
\Omega = z_{-1} + \frac{a}{2}w_{-1}, \tag{17}
$$

iii) quadratic case,

$$
\Omega = z_0 + \frac{a}{2}w_0. \tag{18}
$$

In each case, either the $eSch$ algebra or the $sSch$ superalgebra, is the *on-shell* symmetry algebra of the evolution equation determined by Ω. We have indeed that
i) in the constant case, all commutators involving Ω are vanishing, apart from

$$\begin{aligned} [z_0, \Omega] &= -z_{+1} - \frac{a}{2} w_{+1}, \\ [z_{-1}, \Omega] &= -2z_0 - aw_0. \end{aligned} \tag{19}$$

In the given representation, on the other hand, the above commutators are identified with the representation-dependent formulas

$$\begin{aligned} [z_0, \Omega] &= -\Omega, \\ [z_{-1}, \Omega] &= -2t\Omega. \end{aligned} \tag{20}$$

ii) in the linear case all commutators involving Ω are vanishing, apart from

$$\begin{aligned} [z_{+1}, \Omega] &= 2z_0 + aw_0, \\ [z_0, \Omega] &= z_{-1} + \frac{a}{2} w_{-1}. \end{aligned} \tag{21}$$

In the given linear representation, on the other hand, we have

$$\begin{aligned} [z_{+1}, \Omega] &= 2t\Omega, \\ [z_0, \Omega] &= \Omega. \end{aligned} \tag{22}$$

iii) In the quadratic case all commutators involving Ω are vanishing, apart from

$$\begin{aligned} [z_{+1}, \Omega] &= z_{+1} + \frac{1}{2} w_{+1}, \\ [z_{-1}, \Omega] &= -z_{-1} - \frac{1}{2} aw_{-1}. \end{aligned} \tag{23}$$

In the given quadratic representation, on the other hand, we have

$$\begin{aligned} [z_{+1}, \Omega] &= e^{-t}\Omega, \\ [z_{-1}, \Omega] &= -e^{t}\Omega. \end{aligned} \tag{24}$$

Comment: the fact that we obtain the representation-dependent commutators

$$[g, \Omega] = f_g \cdot \Omega, \tag{25}$$

for any generator g of either the $eSch$ or the $sSch$ algebra, with f_g a given function, tells us that $eSch$ or $sSch$ is the on-shell symmetry (super)algebra for the $\Omega\Psi(t,x) = 0$ equation.

4. Duality for spectrum generating algebras/superalgebras

The famous Wigner's analysis in [4], which allows in particular to solve the harmonic oscillator without using the canonical commutation relations, can be understood from the results discussed in the previous Section. In particular from the notion of algebra/superalgebra duality for the first and second order differential operators closing the on-shell symmetry (super)algebra of the one-dimensional oscillator.

We recall that the given set of 9 differential operators close, on the Lie algebra side, the $eSch$ enlarged Schrödinger algebra, while on the super-Lie algebra side they induce the $sSch$ superalgebra.

On the Lie algebra side the spectrum generating algebra allowing to reconstruct the eigenfunctions and eigenvalues of the harmonic oscillator is the Heisenberg-Lie algebra generated by $w_\pm = e^{\pm 2a\nu t}(\partial_x \pm \nu x)$.

The (unnormalized) vacuum solution $\Psi_{vac}(x,t)$ of the $\Omega\Psi(x,t) = \Psi_t + a\Psi_{xx} - a\nu^2 x^2 \Psi = 0$ equation, on the Lie algebra side, satisfies the Fock's vacuum condition

$$w_+ \Psi_{vac}(x,t) \;=\; 0, \tag{26}$$

together with the equations

$$\begin{aligned} z_0 \Psi_{vac}(x,t) &= -a\nu \Psi_{vac}(x,t), \\ w_0 \Psi_{vac}(x,t) &= a\nu \Psi_{vac}(x,t). \end{aligned} \tag{27}$$

The explicit solution is given by $\Psi_{vac}(x,t) = Ce^{-a\nu t} e^{-\frac{\nu}{2}x^2}$.

The eigenstates $\Psi_n(x,t)$, corresponding to higher energy eigenvalues of the harmonic oscillator, are constructed through the positions

$$\Psi_n(x,t) \;=\; (w_-)^n \Psi_{vac}(x,t). \tag{28}$$

By construction, they satisfy the $\Omega\Psi_n(x,t) = 0$ equation.

From the dual, superalgebraic, point of view, we have a spectrum-generating superalgebra given by the simple Lie superalgebra $osp(1|2) \subset sSch$. Its generators are $w_0, w_{\pm 1}$ and $w_\pm$.

In the superalgebra picture the same conditions to reconstruct eigenstates and eigenvalues of the harmonic oscillator are read differently. The Equation (26) and the second equation in (27) define a highest weight representation of $osp(1|2)$, with $\Psi_{vac}(x,t)$ being its highest weight vector.

5. Other cases: supersymmetric versus non-supersymmetric superconformal mechanics

The symmetry operator of the Schrödinger equation expressed via the time derivative corresponds, in the free particle case, to the Cartan generator of $sl(2)$ and, in the oscillatorial case, to a root generator of the $s(2)$-invariant subalgebra.

This feature is also present in other different contexts. In particular, in the case of (super)conformal mechanics in $0+1$ dimensions, realized in the Lagrangian setting.

Based on some results in [6], it was shown in [2] that the D-module reps of the (super)conformal algebras admit parabolic as well as hyperbolic/trigonometric realizations. These transformations define superconformally invariant actions. In the hyperbolic/trigonometric case, extra potentials, not allowed in the parabolic case, are present. This one, on the other hand, is not the only difference concerning the various types of realizations. In the parabolic case, the time-derivative operator (i.e., the "Hamiltonian") is associated with a positive root of the $sl(2)$-invariant subalgebra, while in the trigonometric/hyperbolic case it is associated with the Cartan element. As a consequence, we obtain different classes of superconformally-invariant models. In the parabolic case, the Hamiltonian, being associated to a bosonic root, is the square of the fermionic symmetry operators related to the simple fermionic roots. The resulting theory, besides being superconformal, is also supersymmetric in the ordinary sense of the word "supersymmetry". A different picture emerges in the hyperbolic/trigonometric case. The Hamiltonian is still a symmetry operator. On the other hand, it cannot be expressed as a square of fermionic symmetry operators. The resulting theory is superconformally-invariant, but not supersymmetric. Alternatively (following [7], which proposed this term in a different context), we can introduce the notion of "weak supersymmetry" to refer to this feature.

Indeed, the $\mathcal{N}$-extended ordinary supersymmetry requires, for a given $\mathcal{N}$, that a set of $\mathcal{N}$ fermionic symmetry generators Q_i closes the supersymmetry algebra $\{Q_i, Q_j\} = 2\delta_{ij}H$, $[H, Q_i] = 0$ $(i, j = 1, \ldots, \mathcal{N})$, where H is the time-derivative operator (the "Hamiltonian").

In the hyperbolic/trigonometric cases, $\mathcal{N}$ fermionic symmetry generators can be found. They are the square roots of a symmetry generator (let's call it Z), which does not coincide with the Hamiltonian H. As a matter of fact, in the hyperbolic/trigonometric cases, two independent symmetry subalgebras $\{Q_i^\pm, Q_j^\pm\} = 2\delta_{ij}Z^\pm$, $[Z^\pm, Q_i^\pm] = 0$ (with $Z^+ \neq H$ and $Z^- \neq H$) are encountered. In the parabolic cases two independent symmetry subalgebras are also encountered and one of them can be identified with the ordinary supersymmetry ($Z^- = H$, $Z^+ \neq H$).

In the hyperbolic/trigonometric cases the Hamiltonian H continues to be a symmetry operator. It belongs, however, to the 0-grading sector of the superconformal algebra and is not the square of any fermionic symmetry operator (contrary to the operators $Z^\pm$, which belong to the ± 1 grading sectors, respectively).

These points are conveniently illustrated with the simplest example, the $\mathcal{N} = 1$ theory based on the $(1,1)$ supermultiplet (a single bosonic field φ and a single fermionic field ψ) admitting constant kinetic term and $osp(1|2)$ invariance. The action can be written as

$$\mathcal{S} \;=\; \int dt(\dot{\varphi}^2 - \psi\dot{\psi} + \epsilon\varphi^2). \qquad (29)$$

The potential term is absent ($\epsilon = 0$) in the parabolic realization of the superconformal invariance. It is present in the hyperbolic ($\epsilon = 1$) and in the trigonometric ($\epsilon = -1$) realizations. In the hyperbolic case the five invariant operators (closing the $osp(1|2)$ algebra) are given by

$$\begin{aligned} Q^\pm\varphi &= e^{\pm t}\psi, \qquad & Q^\pm\psi &= e^{\pm t}(\dot{\varphi} \mp \varphi), \\ Z^\pm\varphi &= e^{\pm 2t}(\dot{\varphi} \mp \varphi), \qquad & Z^\pm\psi &= e^{\pm 2t}\dot{\psi}, \\ H\varphi &= \dot{\varphi}, \qquad & H\psi &= \dot{\psi}. \end{aligned} \qquad (30)$$

One should note that $Z^\pm = (Q^\pm)^2$.

No change of time variable $t \mapsto \tau(t)$ allows to represent either Z^+ or Z^- as a time-derivative operator with respect to the new time τ.

6. Conclusions

The several key features discussed in this paper can be extended to investigate the dynamics of more complicated systems. The algebra/superalgebra duality involving a finite number of first-order and second-order differential operators can be constructed not only only for Schrödinger equations in $1+d$-dimensions, but also from ℓ-extended conformal Galilei algebras (a discussion of these first-order differential operator algebras can be found in [8], [9] and [10]), with half-integer ℓ ($\ell = \frac{1}{2}$ corresponds to the Schrödinger algebra). The generators associated with the half-integer grading can be naturally identified with the odd generators in the superalgebra picture. The representation-dependent notion of on-shell symmetry (as discussed in Section **3**) is applicable to construct the invariant dynamics associated with these conformal Galilei algebras. Invariant operators exist both at grading 1 (the generalization of the Schrödinger free case) and grading 0 (the generalization of the Schrödinger oscillatorial case). Unlike the Schrödinger case, where the invariant equation is given and the symmetry operators are derived with standard techniques, an inverse problem is defined. The algebra is now given; it is the invariant operator induced by the given representation that has to be computed. As a result we can identify new solvable differential equations. A joint paper with N. Aizawa and Z. Kuznetsova concerning this construction is currently under finalization.

Another feature which deserves to be noticed is that, if starting from a supersymmetric system, the construction leading to the algebra/superalgebra duality is replaced by a construction leading to superalgebra/ $(\mathbf{Z}_2 \times \mathbf{Z}_2)$-graded algebra duality, where the notion of $\mathbf{Z}_2 \times \mathbf{Z}_2$-graded superalgebra can be found, e.g., in [11]. Another paper about this construction is currently under finalization.

On the side of supersymmetric and non-supersymmetric superconformal one-dimensional sigma models, at present the quantization of these classical systems is under investigation.

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